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Research Article

Self-Similar Blow-Up Solutions of the KPZ Equation

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Self-similar blow-up solutions for the generalized deterministic KPZ equation $u_t = u_{xx} + |u_x|^q$ with q > 2 are considered. The asymptotic behavior of self-similar solutions is studied.

1. Introduction

We consider the generalized deterministic KPZ equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \left| \frac{\partial u}{\partial x} \right|^q \quad \text{for } (x, t) \in S_T := \mathbb{R} \times (0, T), \quad (1)$$

where q>2 and T>0. Equation (1) was first considered in the case q=2 by Kardar et al. [1] in connection with the study of the growth of surfaces. When q=2, (1) has since been referred to as the deterministic KPZ equation. For $q\neq 2$ it also called the generalized deterministic KPZ equation or Krug-Spohn equation because it was introduced in [2]. We refer to the review article [3] for references and a detailed historical account of the KPZ equation.

The existence and uniqueness of a classical solution of the Cauchy problem for (1) with q=1 and initial function $u_0\in C_0^3(\mathbb{R}^n)$ were proven in [4]. This result was extended to $u_0\in C^2(\mathbb{R}^n)\cap W^{2,\infty}(\mathbb{R}^n)$ and $q\geq 1$ in [5] and to $u_0\in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and $q\geq 0$ in [6]. Several papers [7–11] were devoted to the investigation of the Cauchy problem for irregular initial data, namely, for $u_0\in L^p(\mathbb{R}^n)$, $1\leq p<\infty$, or for bounded measures. The existence and uniqueness of a solution to the Cauchy problem with unbounded initial datum are proved in [12]. To confirm the optimality of obtained existence conditions, the authors of [12] analyze the asymptotic behavior of self-similar blow-up solutions of (1) for q<2.

In this paper we investigate the asymptotic behavior of self-similar blow-up solutions of (1) with q>2 having the form

$$u(x,t) = (T-t)^{\alpha} f(\xi),$$
where $\xi = |x| (T-t)^{\beta}, \ 0 < t < T.$ (2)

After substitution of (2) into (1) we find that

$$\alpha = \frac{q-2}{2(q-1)},$$

$$\beta = -\frac{1}{2}$$
(3)

and *f* should satisfy the following equation:

$$f'' + |f'|^q - \frac{1}{2}\xi f' + \alpha f = 0$$
 on $(0, +\infty)$. (4)

We will add to (4) the following initial data:

$$f(0) = -f_0 < 0,$$

 $f'(0) = 0.$ (5)

Put

$$C = \left[\frac{1}{q-1} \left(\frac{q-1}{q}\right)^q\right]^{1/(q-1)}.$$
 (6)

Let us state the main result.

Theorem 1. Let u be a self-similar blow-up solution of (1) with q > 2 which is defined in (2)–(5). Then

$$\lim_{t \to T} u(x,t) (T-t)^{1/(q-1)} = C |x|^{q/(q-1)}.$$
 (7)

A simple computation shows that Theorem 1 is a consequence of the following statement.

Theorem 2. Let q > 2 and let f be a solution of problem (4), (5). Then

$$\lim_{\xi \to \infty} \frac{f(\xi)}{\xi q/(q-1)} = C. \tag{8}$$

The behavior of self-similar solutions for (1) of the type $u(x,t) = t^{\alpha}g(xt^{\beta})$ has been analyzed in [13].

2. The Proof of Theorem 2

We start with a simple result which is used later on.

Lemma 3. Let f be a solution of problem (4), (5) defined on $[0, \overline{\xi})$. Then

$$f'\left(\xi\right)>0,$$

$$f'''(\xi) > 0 \tag{9}$$

for
$$\xi \in (0, \overline{\xi})$$
.

Proof. Obviously, $f''(0) = \alpha f_0 > 0$. Therefore, by continuity, f'' > 0 and f' > 0 in some right-neighborhood of 0. Suppose that there exists ξ_0 such that $0 < \xi_0 < \overline{\xi}$, f'' > 0 on $[0, \xi_0)$ and $f''(\xi_0) = 0$. Then f' > 0 on $(0, \xi_0]$ and $f'''(\xi_0) \le 0$. From (4) we find that $f'''(\xi_0) = f'(\xi_0)/[2(q-1)] > 0$. This contradiction proves (9).

Now we will obtain the upper bound for f'.

Lemma 4. There exists $\xi_0 > 0$ such that

$$f'(\xi) < \left\{\frac{\xi}{2}\right\}^{1/(q-1)} \quad \text{for } \xi \ge \xi_0.$$
 (10)

Proof. Lemma 3 implies that $f(\xi) \to \infty$ as $\xi \to \overline{\xi}$ and that there exists unique point $\xi_0 \in (0, \overline{\xi})$ such that f < 0 on $(0, \xi_0)$ and f > 0 on $(\xi_0, \overline{\xi})$. Substituting f'' > 0 and $f \ge 0$ in (4) yields $f'(\xi) < \{\xi/2\}^{1/(q-1)}$ for $\xi \in [\xi_0, \overline{\xi})$. Thus, $\overline{\xi} = \infty$ and (10) holds.

Changing variables in (4)

$$f'(\xi) = \xi^{1/(q-1)} q(t), \quad \xi = \exp t,$$
 (11)

we get the new equation

$$g'' + \frac{3 - q}{q - 1}g' - \frac{q - 2}{(q - 1)^2}g$$

$$= \left\{\frac{1}{2}g' - (g^q)' + \frac{1}{q - 1}g - \frac{q}{q - 1}g^q\right\} \exp(2t).$$
(12)

By (9), (10), and (11), there hold

$$g(t) > 0$$
 for any $t \in \mathbb{R}$, (13)

$$g(t) < \left\{\frac{1}{2}\right\}^{1/(q-1)},$$

$$g'(t) > -\frac{g}{g-1}$$
(14)

for large values of t. Put

$$C_0 = \left\{ \frac{1}{q} \right\}^{1/(q-1)},$$

$$C_1 = \left\{ \frac{1}{2q} \right\}^{1/(q-1)}.$$
(15)

It is obvious that $C_0 > C_1$. Now we will establish the asymptotic behavior of g(t) as $t \to +\infty$.

Lemma 5. Assume that g is defined in (11). Then

$$\lim_{t \to +\infty} g(t) = C_0. \tag{16}$$

Proof. From a careful inspection of (12) we conclude that a local maximum of g(t) can happen only when $g(t) > C_0$.

At first we suppose that g(t) does not tend to C_0 as $t \to +\infty$ and g(t) is monotonic solution of (12) for large values of t. Then there exists $\overline{C} \neq C_0$ such that $\lim_{t \to \infty} g(t) = \overline{C}$. It is not difficult to show that for any $\varepsilon > 0$ there exist A > 0 and a sequence $\{t_k\}_{k=1}^{\infty}$ with the properties:

$$\lim_{k \to \infty} t_k = +\infty,$$

$$\left| g''(t_k) \right| \le A,$$

$$\left| g'(t_k) \right| \le \varepsilon.$$
(17)

Indeed, let $g' \ge 0$ for the definiteness. We suppose that g'(t) is not monotonic function for large values of t since otherwise (17) is obvious. Denote by $\{\tau_k\}_{k=1}^{\infty}$ a sequence of local minima for g'. Then (17) holds for some subsequence of $\{\tau_k\}_{k=1}^{\infty}$.

Passing to the limit in (12) as $t = t_k \to +\infty$ and choosing ε in a suitable way we get that the left-hand side is bounded, while the right-hand side tends to infinity if $\overline{C} \neq 0$. Let $\overline{C} = 0$. Using (13) and (14) we conclude from (12) that

$$g'' + \frac{3-q}{q-1}g' \ge \frac{g}{3(q-1)}\exp(2t)$$
 (18)

for large values of t. Then for large values of k (17) and (18) imply

$$g(t_k) \le \gamma \exp\left(-2t_k\right),\tag{19}$$

where positive constant γ does not depend on k. Setting $\xi_k = \exp t_k$, from (11) and (19), we get

$$f'\left(\xi_{k}\right) \le \gamma \xi_{k}^{(3-2q)/(q-1)} \tag{20}$$

that contradicts (9).

Now until the end of the proof we assume that g(t) is not monotonic solution of (12) for large values of t. Suppose that $\liminf_{t\to\infty}g(t)< C_0$. Then there exist positive unbounded increasing sequences $\{s_k\}_{k=1}^\infty$ and $\{t_k\}_{k=1}^\infty$ such that $t_k>s_k$,

$$g'(t) \le 0 \quad \text{for } t \in [s_k, t_k],$$
 (21)

and $g(s_k) = C_0$, $g(t_k) = C_{\star}$, where $C_1 < C_{\star} < C_0$. Then

$$\frac{1}{2}g' - (g^{q})' = -q(g^{q-1} - C_1^{q-1})g'
\geq -q(C_{\star}^{q-1} - C_1^{q-1})g' \geq 0 \quad \text{on } [s_k, t_k].$$
(22)

So, (12) and (22) imply that

$$g''(t) + \frac{3 - q}{q - 1}g'(t)$$

$$\geq -q\left(C_{\star}^{q - 1} - C_{1}^{q - 1}\right)g'(t)\exp\left(2s_{k}\right)$$
for $t \in [s_{k}, t_{k}]$.
(23)

Hence, integrating with respect to t from s_k to t_k , we get

$$\left\{ g'(t) + \frac{3-q}{q-1}g(t) \right\} \Big|_{s_k}^{t_k} \\
\geq q \left(C_{\star}^{q-1} - C_1^{q-1} \right) \left(C_0 - C_{\star} \right) \exp(2s_k) . \tag{24}$$

This leads to a contradiction, since (13), (14), and (21) imply that the left-hand side of the last inequality is bounded, while the right-hand side becomes unbounded as $k \to \infty$.

Let us prove that $\liminf_{t\to\infty} g(t) = C_0$. Indeed, otherwise, there exist $\varepsilon > 0$ and a sequence $\{\tau_k\}_{k=1}^{\infty}$ of local minima for g with the properties $\tau_k \to +\infty$ as $k \to +\infty$ and

$$g(\tau_k) \ge C_0 + \varepsilon.$$
 (25)

Passing in (12) to the limit as $t = \tau_k \rightarrow +\infty$ we get a contradiction.

To end the proof we show that $\limsup_{t\to\infty} g(t) = C_0$. Otherwise, $\limsup_{t\to\infty} g(t) > C_0$. Then there exist unbounded increasing sequences $\{s_k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ such that $t_k > s_k > 2$,

$$g'(s_k) = 0,$$

$$g'(t_k) = 0,$$

$$g'(t) \ge 0$$

$$g(t_k) > C_0 + \delta,$$

$$|g(s_k) - C_0| < \varepsilon,$$
(26)

for $t \in [s_k, t_k]$,

where $\delta > 0$ and

$$\varepsilon = \min\left\{\frac{\delta}{2}, \frac{q-1}{4C_0}\delta^2, \left[1 - \left(\frac{7}{8}\right)^{1/(q-1)}\right]C_0\right\}.$$
 (27)

Without loss of a generality we can suppose

$$C_0 - \varepsilon < g\left(s_k\right) < C_0 \tag{28}$$

or

$$C_0 \le g\left(s_k\right) < C_0 + \varepsilon. \tag{29}$$

Let (28) be valid. If (29) is realized, the arguments are similar and simpler. Denote by $\{\bar{t}_k\}_{k=1}^{\infty}$ a sequence such that

$$\begin{aligned}
\bar{t}_k &\in (s_k, t_k), \\
g(\bar{t}_k) &= C_0.
\end{aligned} \tag{30}$$

Applying Hölder's inequality we derive

$$\int_{\bar{t}_{k}}^{t_{k}} g'(\tau) d\tau \leq \left(\int_{\bar{t}_{k}}^{t_{k}} \left(g'(\tau) \right)^{2} \exp(2\tau) d\tau \right)^{1/2} \cdot \left(\int_{\bar{t}_{k}}^{t_{k}} \exp(-2\tau) d\tau \right)^{1/2}$$

$$(31)$$

and therefore

$$\int_{\bar{t}_k}^{t_k} \left(g'(\tau) \right)^2 \exp(2\tau) d\tau \ge 2\delta^2 \exp(2\bar{t}_k). \tag{32}$$

We multiply (12) by g'(t) and integrate after over $[s_k, t_k]$. Using (15), (26)–(28), (30), and (32) we obtain

$$-\frac{q-2}{2(q-1)^{2}}g^{2}(t_{k}) \leq \frac{q-3}{q-1} \int_{s_{k}}^{t_{k}} (g'(\tau))^{2} d\tau$$

$$+ \int_{s_{k}}^{t_{k}} (g'(\tau))^{2} \left[\frac{1}{2} - qg^{q-1}(\tau) \right] \exp(2\tau) d\tau$$

$$+ \frac{\exp(2\bar{t}_{k})}{q-1}$$

$$\cdot \int_{s_{k}}^{\bar{t}_{k}} \left[\frac{1}{2} (g^{2}(\tau))' - \frac{q}{q+1} (g^{q+1}(\tau))' \right] d\tau \leq -\frac{1}{4} \quad (33)$$

$$\cdot \int_{\bar{t}_{k}}^{t_{k}} (g'(\tau))^{2} \exp(2\tau) d\tau$$

$$+ \frac{\exp(2\bar{t}_{k})}{q-1} \left(\frac{g^{2}(\tau)}{2} - \frac{qg^{q+1}(\tau)}{q+1} \right) \Big|_{s_{k}}^{\bar{t}_{k}}$$

$$\leq \left[-\frac{\delta^{2}}{2} + \frac{\varepsilon C_{0}}{q-1} \right] \exp(2\bar{t}_{k}) \leq -\frac{\delta^{2}}{4} \exp(2\bar{t}_{k}).$$

Passing to the limit as $k \to \infty$ we get a contradiction with (14).

Now (8) is a simple consequence of Lemma 5 and the definition of g(t).

Remark 6. Note that Theorem 2 demonstrates the optimality of Theorem 2.3 in [12]. The arguments are the same as in Remark 4.6 of that paper.

Our next result shows that (4) with initial data

$$f(0) = f_0 > 0,$$

 $f'(0) = 0$ (34)

has no global solution.

Theorem 7. Let q > 2 and let f be a solution of problem (4), (34). Then there exists ξ_* such that $0 < \xi_* < +\infty$ and $f(\xi) \to -\infty$ as $\xi \uparrow \xi_*$.

Proof. Suppose that problem (4), (34) has a solution f that is infinitely extendible to the right. Using the arguments of Lemma 3 we show that f' < 0 and f'' < 0 on $(0, +\infty)$. From (4) we obtain

$$f'''(\xi) < -\left(\left|f'(\xi)\right|^q\right)'. \tag{35}$$

After the integration of (35) over $[0, \xi]$ we conclude that

$$f''(\xi) < -\left|f'(\xi)\right|^{q}. \tag{36}$$

Integrating (36) over $[\xi_1, \xi]$ (0 < ξ_1 < ξ) we infer

$$\frac{1}{(q-1)|f'(\xi_1)|^{q-1}} > \xi - \xi_1. \tag{37}$$

Passing to the limit as $\xi \to \infty$ we obtain a contradiction which proves the theorem. \Box

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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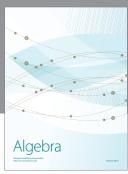
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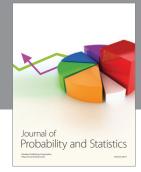
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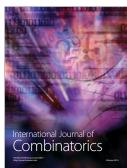














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