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Generalized fractional integral operators and the multivariable H-function

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Abstract

The main object of the present paper is to establish new fractional integral formulas (of Marichev-Saigo-Maeda type) involving the products of the multivariable H-functions and the first class of multivariable polynomials due to Srivastava and Garg. All the results derived here are of general character and can yield a number of (new and known) results in the theory of fractional calculus.

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1 Introduction

Nowadays, fractional calculus is used as a mathematical tool for modeling the processes in engineering, mathematical economics, physics, chemistry, biology and other branches of science (see [1–12]). Motivated by the above mentioned works, here we aim to establish two new fractional integral formulas (of Marichev-Saigo-Maeda type) involving the product of multivariable H-functions and the first class of multivariable polynomials $S_n^{m_1,\dots,m_s}(x_1,\dots,x_s)$. The so-called Marichev-Saigo-Maeda generalized fractional operators were introduced nearly 40 years ago by Marichev [13] and studied in some recent papers, including the papers by Saigo and Maeda [14] and by Saxena and Saigo [15]. Due to the usefulness and the importance of the Marichev-Saigo-Maeda fractional integral operators, many authors have presented a number of interesting integral formulas involving special functions by using the Marichev-Saigo-Maeda fractional integral operator (see [13, 16–20]). By virtue of the unified nature of Marichev-Saigo-Maeda fractional integral operators, a large number of new and known results involving Saigo, Riemann-Liouville and Erdélyi-Kober fractional integral operators follow as special cases of our main formulas.

Our main results consist in application of the Marichev-Saigo-Maeda generalized fractional operators:

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\eta}f\right)(x)$$
$$= \frac{x^{-\alpha}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\alpha'}F_3\left(\alpha,\alpha',\beta,\beta';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)\,dt \quad (\operatorname{Re}\eta>0) \qquad (1.1)$$

and

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\eta}f\right)(x)$$

$$= \frac{x^{-\alpha'}}{\Gamma(\eta)} \int_x^\infty (t-x)^{\eta-1} t^{-\alpha} F_3\left(\alpha,\alpha',\beta,\beta';\eta;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,dt \quad (\mathrm{Re}\,\eta > 0), \qquad (1.2)$$

to the multivariable H-functions (see, *e.g.*, [21, 22]) and the first class of multivariable polynomials $S_n^{m_1,\ldots,m_s}(x_1,\ldots,x_s)$ given by (1.15).

A variant of such operators (integral transforms) was introduced by Marichev [13] as Mellin type convolution operators with a special function $F_3(\cdot)$ in the kernel. These operators were rediscovered and studied by Saigo in [23] as a generalization of the so-called Saigo fractional integral operators, see [24]. The properties of these operators were studied by Saigo and Maeda [14]. In particular, relations of the operators with the Mellin transforms, hypergeometric operators (or Saigo fractional integral operators), their decompositions and acting properties in the McBride spaces $F_{p;\mu}$ (see [25]) were found.

In (1.1), (1.2) the symbol $F_3(\cdot)$ denotes the so-called 3rd Appell function (known also as Horn function) (see [26], p.413):

$$F_3\left(\alpha,\alpha';\beta,\beta';\eta;x;y\right) = \sum_{m,n=0}^\infty \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\eta)_{m+n}m!n!} x^m y^n \quad (\max\{|x|,|y|\} < 1). \qquad (1.3)$$

The properties of this function are discussed in [26], pp.412-415. In particular, its relation to the Gauss hypergeometric function is presented:

$$F_3(\alpha,\eta-\alpha;\beta,\eta-\beta;\eta;x;y) = {}_2F_1(\alpha,\beta;\eta;x+y-xy).$$

It is known that the 3rd Appell function cannot be expressed as a product of two ${}_2F_1$ functions and satisfy pairs of linear partial differential equations of the second order.

Operators (1.1), (1.2) can be reduced to the fractional integral operators introduced by Saigo [27] due to the following relations:

$$I_{0,x}^{\alpha,0,\beta,\beta',\gamma}f(x) = I_{0,x}^{\gamma,\alpha-\gamma,-\beta}f(x) \quad (\gamma \in \mathbb{C}) \qquad (1.4)$$

and

$$I_{x,\infty}^{\alpha,0,\beta,\beta',\gamma}f(x) = I_{x,\infty}^{\gamma,\alpha-\gamma,-\beta}f(x) \quad (\gamma \in \mathbb{C}). \qquad (1.5)$$

The multivariable H-function, introduced by Srivastava and Panda in a series of research papers (see [28], p.271, Eq. (4.1), [22], p.130, Eq. (1.3) and [29, 30]), can be defined and represented in the following manner (see, *e.g.*, [31], p.251, Eq. (C.1)):[a]

$$H[x_1,\ldots,x_r] = H_{p,q:\{p_r,q_r\}}^{0,n:\{m_r,n_r\}}\left[\begin{array}{c} x_1 \\ \vdots \\ x_r \end{array} \middle| \begin{array}{c} (a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,p} : \{(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\} \\ (b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q} : \{(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1}\cdots\int_{L_r} \phi(\xi_1,\ldots,\xi_r) \prod_{i=1}^r \left(\theta_i(\xi_i) x_i^{\xi_i}\,d\xi_i\right), \qquad (1.6)$$

where

$$\phi(\xi_1,\ldots,\xi_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i)}{\prod_{j=n+1}^{p} \Gamma(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i) \prod_{j=1}^{q} \Gamma(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} \xi_i)}, \tag{1.7}$$

$$\theta_i(\xi_i) = \frac{\prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i) \prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i)}{\prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i) \prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i)}, \quad \forall i \in \{1, 2, \ldots, r\}. \tag{1.8}$$

In these formulas, for any $i = 1, \ldots, r$, L_i represents the contours which start at the point $\tau_i - \omega\infty$ and go $\tau_i + \omega\infty$ with $\tau_i \in \mathbb{R} = (-\infty, \infty)$ such that all the poles of $\Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i)$, $j = 1, \ldots, m_i$; $i = 1, \ldots, r$, are separated from those of $\Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)$, $j = 1, \ldots, n_i$; $i = 1, \ldots, r$, and $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i)$, $j = 1, \ldots, n$. Here, the integers n, p, q, m_i, n_i, p_i, q_i satisfy the inequalities $0 \le n \le p$; $q \ge 0$, $1 \le m_i \le q_i$ and $1 \le n_i \le p_i$, $i = 1, \ldots, r$.

Further, suppose that the parameters a_j, $j = 1, \ldots, p$; $c_j^{(i)}$, $j = 1, \ldots, p_i$; $i = 1, \ldots, r$, b_j, $j = 1, \ldots, q$; $d_j^{(i)}$, $j = 1, \ldots, q_i$; $i = 1, \ldots, r$, are complex numbers, while the associated coefficients $\alpha_j^{(i)}$, $j = 1, \ldots, p$; $i = 1, \ldots, r$, $\gamma_j^{(i)}$, $j = 1, \ldots, p_i$; $i = 1, \ldots, r$, $\beta_j^{(i)}$, $j = 1, \ldots, q$; $i = 1, \ldots, r$; $\delta_j^{(i)}$, $j = 1, \ldots, q_i$; $i = 1, \ldots, r$, are positive real numbers such that

$$\Lambda_i = \sum_{j=1}^{p} \alpha_j^{(i)} + \sum_{j=1}^{p_i} \gamma_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} - \sum_{j=1}^{q_i} \delta_j^{(i)} \le 0, \tag{1.9}$$

$$\Omega_i = \sum_{j=1}^{n} \alpha_j^{(i)} - \sum_{j=n+1}^{p} \alpha_j^{(i)} + \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0. \tag{1.10}$$

It is assumed that the poles of integrand of (1.6) are simple. Under the conditions (1.10) the integrals in (1.6) converge absolutely (see [31], p.251) in the domain

$$\left| \arg(z_i) \right| < \frac{\pi}{2} \Omega_i, \quad i = 1, \ldots, r, \tag{1.11}$$

and points $z_i = 0$, $i = 1, \ldots, r$, and various exceptional parameter values being tacitly excluded.

From Srivastava and Panda [32], we have

$$H[z_1, \ldots, z_r] = o\left(|z_1|^{e_1}, \ldots, |z_r|^{e_r}\right), \qquad \max_{1 \le j \le r}\left[|z_j|\right] \to 0, \tag{1.12}$$

where $e_i = \min_{1 \le j \le m_i} \left[\frac{\Re(d_j^{(i)})}{\delta_j^{(i)}}\right]$, $i = 1, \ldots, r$.

If $n = 0$ in (1.6), then the following asymptotic expansion (Srivastava and Panda [32]) holds:

$$H[z_1, \ldots, z_r] = o\left(|z_1|^{g_1}, \ldots, |z_r|^{g_r}\right), \qquad \min_{1 \le j \le r}\left[|z_j|\right] \to \infty, \tag{1.13}$$

where $g_i = \max_{1 \le j \le m_i}\left[\frac{\Re(c_j^{(i)})}{\gamma_j^{(i)}}\right]$, $i = 1, \ldots, r$, provided that (1.9), (1.10) and (1.11) hold true.

Remark 1 When $n = 2$, the multivariable H-function defined by (1.6) reduces to the H-function of two variables studied by Mittal and Gupta [33]. The H-function of two

variables is also defined and studied by Munot and Kalla [34], Verma [35] and Hai and Yakubovich [36].

Remark 2 It is interesting to observe that for $n = p = q = 0$, the multivariable H-function breaks up into the product of r, H-functions; and consequently, there holds the following result (Saxena [21]):

$$
H^{0,0:\{m_r,n_r\}}_{0,0:\{p_r,q_r\}} \left[\begin{array}{c} x_1 \\ \vdots \\ x_r \end{array} \left| \begin{array}{c} - : \{(c^{(r)}_j, \gamma^{(r)}_j)_{1,p_r}\} \\ - : \{(d^{(r)}_j, \delta^{(r)}_j)_{1,q_r}\} \end{array} \right. \right] = \prod_{i=1}^{r} H^{m_i,n_i}_{p_i,q_i} \left[z_i \left| \begin{array}{c} \{(c^{(i)}_j, \gamma^{(i)}_j)_{1,p_i}\} \\ \{(d^{(i)}_j, \delta^{(i)}_j)_{1,q_i}\} \end{array} \right. \right]. \tag{1.14}
$$

Remark 3 The function defined by (1.6) was introduced and studied by Srivastava and Panda [37]. When $\alpha^{(1)}_j = \cdots = \alpha^{(r)}_j$, $j = 1, \ldots, p$; $\beta^{(1)}_j = \cdots = \beta^{(r)}_j$, $j = 1, \ldots, q$, in (1.6) the multivariable H-function defined and studied by Saxena [21, 38] is obtained. In case all the Greek letters are assumed to be unity, the H-function of several complex variables (1.6) reduces to the G-function of several complex variables studied by Khadia and Goyal [39, 40].

Remark 4 Fractional integrals involving multivariable H-functions are given in a series of papers by Saigo and Saxena [41–43], Srivastava and Hussain [44], Saigo *et al.* [45] and others.

In the sequel, Srivastava and Garg [46], p.686, Eq. (1.4), introduced the multivariable analogue of the polynomials $S^m_n(x)$. This first class of multivariable polynomials $S^{m_1,\ldots,m_s}_n(x_1,\ldots,x_s)$ is defined by

$$
S^{m_1,\ldots,m_s}_n(x_1,\ldots,x_s) = \sum_{k_1,\ldots,k_s=0}^{m_1 k_1 + \cdots + m_s k_s \leq n} (-n)_{m_1 k_1 + \cdots + m_s k_s} \Lambda(n; k_1, \ldots, k_s) \frac{x_1^{k_1} \cdots x_s^{k_s}}{k_1! \cdots k_s!}, \tag{1.15}
$$

where $m_1, \ldots, m_s$ are arbitrary positive integers and the coefficients $\Lambda(n; k_1, \ldots, k_s)$ ($n, k_i \geq 0$, $i = 1, \ldots, s$) are arbitrary constants, real or complex.

2 Main results

In this section, we prove two theorems on composition of the Marichev-Saigo-Maeda operators with the product of a multivariable H-function and the first class of multivariable polynomials. We start with presenting image formulas involving Marichev-Saigo-Maeda fractional integral operators.

Lemma 1 (see [14]) *Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$.*
If $\Re(\gamma) > 0$, $\Re(p) > \max[0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')]$, then

$$
I^{\alpha,\alpha',\beta,\beta',\gamma}_{0,x} x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)\Gamma(\rho+\beta')}. \tag{2.1}
$$

If $\Re(\gamma) > 0$, $\Re(p) < 1 + \min[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)]$, then

$$
I^{\alpha,\alpha',\beta,\beta',\gamma}_{x,\infty} x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1}
$$

$$
\times \frac{\Gamma(1+\alpha+\alpha'-\gamma-\rho)\Gamma(1+\alpha+\beta'-\gamma-\rho)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\Gamma(1+\alpha-\beta-\rho)}. \tag{2.2}
$$

Theorem 1 *Let* $\Re(\gamma) > 0$ *and* $\alpha, \alpha', \beta, \beta', \gamma, \mu, \eta, \delta_j, \nu_i, z_i, a, b, c_j \in \mathbb{C}$, $\lambda_j, \sigma_i > 0$ $(i \in \{1, \ldots, r\};$ $j \in \{1, \ldots, s\})$. *Let the following conditions be satisfied, too*:

(i) $|\arg z_i| < \frac{\pi}{2}\Omega_i$, $\Omega_i > 0$ $(i = 1, \ldots, r)$, *where*

$$
\Omega_i = \sum_{j=1}^{n} \alpha_j^{(i)} - \sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)}
$$

$$
- \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0. \tag{2.3}
$$

(ii) $\min_{1 \le j \le m_i, 1 \le i \le r} \left[-\frac{\sigma_i \Re(d_j^{(i)})}{\delta_j^{(i)}} \right] < \Re(\mu) + \min[\Re(\alpha'), \Re(\beta'), \Re(\gamma - \alpha - \beta)]$.

(iii) $|\frac{a}{b}x| < 1$, *also we have*

$$
\Re(\mu) + \sum_{i=1}^{r} \sigma_i \min_{1 \le j \le m_i} \left[\frac{\Re(d_j^{(i)})}{\delta_j^{(i)}} \right]
$$

$$
> \max\left[0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')\right] \quad (i = 1, \ldots, r),
$$

$$
\Re(\eta) + \sum_{i=1}^{r} \nu_i \min_{1 \le j \le m_i} \left[\frac{\Re(d_j^{(i)})}{\delta_j^{(i)}} \right]
$$

$$
> \max\left[0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')\right] \quad (i = 1, \ldots, r). \tag{2.4}
$$

Then the following result holds:

$$
\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu-1}(b - at)^{-\eta} S_n^{m_1,\ldots,m_s} \left[c_1 t^{\lambda_1}(b - at)^{-\delta_1}, \ldots, c_s t^{\lambda_s}(b - at)^{-\delta_s} \right] \right. \right.
$$

$$
\left. \left. \times H\left[z_1 t^{\sigma_1}(b - at)^{-\nu_1}, \ldots, z_r t^{\sigma_r}(b - at)^{-\nu_r} \right] \right) \right\}(x)
$$

$$
= b^{-\eta} x^{\mu - \alpha - \alpha' + \gamma - 1} \sum_{k_1,\ldots,k_s=0}^{m_1 k_1 + \cdots + m_s k_s \le n} \frac{(-n)_{m_1 k_1 + \cdots + m_s k_s}}{k_1! \cdots k_s!} \Lambda(n; k_1, \ldots, k_s) c_1^{k_1} \cdots c_s^{k_s}
$$

$$
\times b^{-\sum_{j=1}^{s} \delta_j k_j} x^{\sum_{j=1}^{s} \lambda_j k_j} H_{p+4,q+4:\{p_r,q_r\};0,1}^{0,n+4:\{m_r,n_r\};1,0}
$$

$$
\begin{bmatrix}
z_1 \frac{x^{\sigma_1}}{b^{\nu_1}} \\
\vdots \\
z_r \frac{x^{\sigma_r}}{b^{\nu_r}} \\
-\frac{a}{b}x
\end{bmatrix}
\left| \begin{array}{l}
(a_j; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)})_{1,p}, A : \{(c_j^{(r)}, \gamma_j^{(r)})_{1,p_r}\}; - \\
(b_j; \beta_j^{(1)}, \ldots, \beta_j^{(r)})_{1,q}, B : \{(d_j^{(r)}, \delta_j^{(r)})_{1,q_r}\}; (0,1)
\end{array} \right|, \tag{2.5}
$$

where

$$
A = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; \nu_1, \ldots, \nu_r, 1 \right), \left(1 - \mu - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1 \right),
$$

$$
\left(1 - \mu - \gamma + \alpha + \alpha' + \beta - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1 \right),
$$

$$
\left(1 - \mu + \alpha' - \beta' - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1 \right)
$$

and

$$B = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; \nu_1, \ldots, \nu_r, 0\right), \left(1 - \mu - \gamma + \alpha + \alpha' - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right)$$

$$\left(1 - \mu - \gamma + \alpha' + \beta - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right),$$

$$\left(1 - \mu - \beta' - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right).$$

Theorem 2 *Let* $\alpha, \alpha', \beta, \beta', \gamma, \mu, \eta, \delta_j, \nu_i, z_i, a, b, c_j \in \mathbb{C}$, $\lambda_j, \sigma_i > 0$ $(i \in \{1, \ldots, r\}; j \in \{1, \ldots, s\})$ *and* $\Re(\gamma) > 0$. *Let the following conditions be satisfied, too*:

$$\Re(\mu) - \sum_{i=1}^{r} \sigma_i \min_{1 \le j \le m_i} \left[\frac{\Re(d_j^{(i)})}{\delta_j^{(i)}}\right] < 1 + \max\left[\Re(-\beta), \Re(\alpha + \alpha' - \gamma), \Re(\alpha + \beta' - \gamma)\right],$$

$$\Re(\eta) - \sum_{i=1}^{r} \nu_i \min_{1 \le j \le m_i} \left[\frac{\Re(d_j^{(i)})}{\delta_j^{(i)}}\right] < 1 + \max\left[\Re(-\beta), \Re(\alpha + \alpha' - \gamma), \Re(\alpha + \beta' - \gamma)\right],$$

and the conditions (i)-(iii) *in Theorem* 1 *are also satisfied.*

Then the following result holds:

$$\left\{I_{0-}^{\alpha, \alpha', \beta, \beta', \gamma}\left(t^{\mu-1}(b-at)^{-\eta} S_n^{m_1, \ldots, m_s}\left[c_1 t^{\lambda_1}(b-at)^{-\delta_1}, \ldots, c_s t^{\lambda_s}(b-at)^{-\delta_s}\right]\right.\right.$$

$$\left.\left. \times H\left[z_1 t^{\sigma_1}(b-at)^{-\nu_1}, \ldots, z_r t^{\sigma_r}(b-at)^{-\nu_r}\right]\right)\right\}(x)$$

$$= b^{-\eta} x^{\mu - \alpha - \alpha' + \gamma - 1} \sum_{k_1, \ldots, k_s = 0}^{m_1 k_1 + \cdots + m_s k_s \le n} \frac{(-n)_{m_1 k_1 + \cdots + m_s k_s}}{k_1! \cdots k_s!} \Lambda(n; k_1, \ldots, k_s) c_1^{k_1} \cdots c_s^{k_s}$$

$$\times b^{-\sum_{j=1}^{s} \delta_j k_j} x^{\sum_{j=1}^{s} \lambda_j k_j} H_{p+4, q+4:\{p_r, q_r\}; 0, 1}^{0, n+4:\{m_r, n_r\}; 1, 0}$$

$$\left[\begin{matrix} z_1 \frac{x^{\sigma_1}}{b^{\nu_1}} \\ \vdots \\ z_r \frac{x^{\sigma_r}}{b^{\nu_r}} \\ -\frac{a}{b}x \end{matrix} \middle| \begin{matrix} (a_j; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)})_{1,p}, C : \{(c_j^{(r)}, \gamma_j^{(r)})_{1, p_r}\}; - \\ (b_j; \beta_j^{(1)}, \ldots, \beta_j^{(r)})_{1,q}, D : \{(d_j^{(r)}, \delta_j^{(r)})_{1, q_r}\}; (0, 1) \end{matrix}\right], \tag{2.6}$$

where

$$C = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; \nu_1, \ldots, \nu_r, 1\right), \left(1 - \mu - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right),$$

$$\left(1 + \alpha + \alpha' + \beta - \mu - \gamma - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right),$$

$$\left(1 + \alpha - \beta - \mu - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right)$$

and

$$D = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; \nu_1, \ldots, \nu_r, 0\right), \left(1 - \mu - \gamma + \alpha + \alpha' - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right)$$

$$\left(1 - \mu - \gamma + \alpha + \beta' - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right),$$

$$\left(1 - \mu - \beta - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right).$$

Proof For convenience, let the left-hand side of (2.5) be denoted by $\mathcal{I}$. Applying the Marichev-Saigo-Maeda generalized fractional operators (1.1) to the LHS of (2.5) and making use of (1.15) and (1.6), we obtain

$$\mathcal{I} = \left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu-1} (b - at)^{-\gamma} \sum_{k_1,\ldots,k_s=0}^{m_1 k_1 + \cdots + m_s k_s \leq n} \frac{(-n)_{m_1 k_1 + \cdots + m_s k_s}}{k_1! \cdots k_s!} \Lambda(n; k_1, \ldots, k_s) \right. \right.$$

$$\times \left(c_1 t^{\lambda_1} (b - at)^{\delta_1} \right)^{k_1} \cdots \left(c_s t^{\lambda_s} (b - at)^{\delta_s} \right)^{k_s} \times \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \phi(\xi_1, \ldots, \xi_r)$$

$$\left. \left. \times \prod_{i=1}^{r} \theta_i(\xi_i) \left(z_i t^{\sigma_i} (b - at)^{\nu_i} \right)^{\xi_i} d\xi_1 \cdots d\xi_r \right) \right\}(x). \tag{2.7}$$

By changing the order of summation and integration, we have

$$\mathcal{I} = \sum_{k_1,\ldots,k_s=0}^{m_1 k_1 + \cdots + m_s k_s \leq n} \frac{(-n)_{m_1 k_1 + \cdots + m_s k_s}}{k_1! \cdots k_s!} \Lambda(n; k_1, \ldots, k_s) c_1^{k_1} \cdots c_s^{k_s}$$

$$\times \left[\frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \phi(\xi_1, \ldots, \xi_r) \prod_{i=1}^{r} \{\theta_i(\xi_i) z_i\}^{\xi_i} \left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu + \sum_{j=1}^{s} \lambda_j k_j + \sum_{i=1}^{r} \sigma_i \xi_i - 1} \right. \right. \right.$$

$$\left. \left. \left. \times (b - at)^{-\eta - \sum_{j=1}^{s} \lambda_j k_j - \sum_{i=1}^{r} \sigma_i \xi_i} \right) \right\}(x) d\xi_1 \cdots d\xi_r \right]. \tag{2.8}$$

By using the binomial expansion for $(b - at)^{-\eta - \sum_{j=1}^{s} \lambda_j k_j - \sum_{i=1}^{r} \sigma_i \xi_i}$ and applying the Mellin-Barnes counter integral, we get

$$\mathcal{I} = \sum_{k_1,\ldots,k_s=0}^{m_1 k_1 + \cdots + m_s k_s \leq n} \frac{(-n)_{m_1 k_1 + \cdots + m_s k_s}}{k_1! \cdots k_s!} \Lambda(n; k_1, \ldots, k_s) c_1^{k_1} \cdots c_s^{k_s} b^{-\sum_{j=1}^{s} \lambda_j k_j}$$

$$\times \left[\frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \phi(\xi_1, \ldots, \xi_r) \times \prod_{i=1}^{r} \{\theta_i(\xi_i) z_i b^{-\sum_{i=1}^{r} \sigma_i \xi_i}\}^{\xi_i} \right.$$

$$\times \int_{L_{r+1}} \frac{\Gamma(\eta + \sum_{j=1}^{s} \lambda_j k_j + \sum_{i=1}^{r} \sigma_i \xi_i + \xi_{r+1})}{\Gamma(\eta + \sum_{j=1}^{s} \lambda_j k_j + \sum_{i=1}^{r} \sigma_i \xi_i) \Gamma(\xi_{r+1} + 1)}$$

$$\left. \times \left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu + \sum_{j=1}^{s} \lambda_j k_j + \sum_{i=1}^{r} \sigma_i \xi_i + \xi_{r+1} - 1} \right) \right\}(x) d\xi_1 \cdots d\xi_r \right]. \tag{2.9}$$

Applying (2.1) to (2.9) and re-interpreting the Mellin-Barnes counter integral in terms of the multivariable H-function of $r + 1$ variables, we obtain the right-hand side of (2.5) after a few simplifications.

Assertion (2.6) of Theorem 2 can be proved in a similar manner by using (2.2) of Lemma 1. So, we omit the details of the proof of Theorem 2. □

Here, we derive certain, presumably, new formulas involving Marichev-Saigo-Maeda type fractional integral operators. By setting $n = p = q = 0$ in (2.5) and (2.6), respectively, we obtain two fractional integral formulas involving product of the r, H-functions stated in Corollaries 1 and 2 below.

Corollary 1 *Let* $\alpha, \alpha', \beta, \beta', \gamma, \mu, \eta, \delta_j, \nu_i, z_i, a, b, c_j \in \mathbb{C}$, $\lambda_j, \sigma_i > 0$ $(i \in \{1, \dots r\}; j \in \{1, \dots, s\})$ *and* $\Re(\gamma) > 0$. *Then the following relation holds true:*

$$
\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu-1}(b-at)^{-\eta} S_n^{m_1,\dots,m_s} \left[c_1 t^{\lambda_1}(b-at)^{-\delta_1}, \dots, c_s t^{\lambda_s}(b-at)^{-\delta_s} \right] \right. \right.
$$

$$
\left. \left. \times \prod_{i=1}^r H_{p_i,q_i}^{m_i,n_i} \left[z_i t^{\sigma_i}(b-at)^{-\nu_i} \, \middle| \, \begin{matrix} (c_j^i, C_j^i)_{1,p_i} \\ (d_j^i, D_j^i)_{1,q_i} \end{matrix} \right] \right) \right\}(x)
$$

$$
= b^{-\eta} x^{\mu-\alpha-\alpha'+\gamma-1} \sum_{k_1,\dots,k_s=0}^{m_1 k_1 + \cdots + m_s k_s \leq n} \frac{(-n)_{m_1 k_1 + \cdots + m_s k_s}}{k_1! \cdots k_s!} \Lambda(n; k_1, \dots, k_s) c_1^{k_1} \cdots c_s^{k_s}
$$

$$
\times b^{-\sum_{j=1}^s \delta_j k_j} x^{\sum_{j=1}^s \lambda_j k_j} H_{4,4:p_i,q_i;0,1}^{0,4:m_i,n_i;1,0} \left[\begin{matrix} z_i \frac{x^{\sigma_i}}{b^{\nu_i}} \\ \frac{-a}{b} x \end{matrix} \, \middle| \, \begin{matrix} A : (c_j^{(r)}, \gamma_j^{(r)})_{1,p_i}; - \\ B : (d_j^{(r)}, \delta_j^{(r)})_{1,q_i}; (0,1) \end{matrix} \right],
\tag{2.10}
$$

where A *and* B *are given in Theorem* 1 *and* $H_{p,q}^{m,n}(\cdot)$ *is the familiar Fox H-function.*

Corollary 2 *Let* $\alpha, \alpha', \beta, \beta', \gamma, \mu, \eta, \delta_j, \nu_i, z_i, a, b, c_j \in \mathbb{C}$, $\lambda_j, \sigma_i > 0$ $(i \in \{1, \dots, r\}; j \in \{1, \dots, s\})$ *and* $\Re(\gamma) > 0$. *Then the following result holds true:*

$$
\left\{ I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu-1}(b-at)^{-\eta} S_n^{m_1,\dots,m_s} \left[c_1 t^{\lambda_1}(b-at)^{-\delta_1}, \dots, c_s t^{\lambda_s}(b-at)^{-\delta_s} \right] \right. \right.
$$

$$
\left. \left. \times \prod_{i=1}^r H_{p_i,q_i}^{m_i,n_i} \left[z_i t^{\sigma_i}(b-at)^{-\nu_i} \, \middle| \, \begin{matrix} (c_j^i, C_j^i)_{1,p_i} \\ (d_j^i, D_j^i)_{1,q_i} \end{matrix} \right] \right) \right\}(x)
$$

$$
= b^{-\eta} x^{\mu-\alpha-\alpha'+\gamma-1} \sum_{k_1,\dots,k_s=0}^{m_1 k_1 + \cdots + m_s k_s \leq n} \frac{(-n)_{m_1 k_1 + \cdots + m_s k_s}}{k_1! \cdots k_s!} \Lambda(n; k_1, \dots, k_s) c_1^{k_1} \cdots c_s^{k_s}
$$

$$
\times b^{-\sum_{j=1}^s \delta_j k_j} x^{\sum_{j=1}^s \lambda_j k_j} H_{4,4:p_i,q_i;0,1}^{0,4:m_i,n_i;1,0} \left[\begin{matrix} z_i \frac{x^{\sigma_i}}{b^{\nu_i}} \\ \frac{-a}{b} x \end{matrix} \, \middle| \, \begin{matrix} C' : (c_j^{(r)}, \gamma_j^{(r)})_{1,p_i}; - \\ D' : (d_j^{(r)}, \delta_j^{(r)})_{1,q_i}; (0,1) \end{matrix} \right],
\tag{2.11}
$$

where

$$
C' = \left(1 - \eta - \sum_{j=1}^s \delta_j k_j; \nu_1, \dots, \nu_r, 1 \right),
$$

$$
\left(1 - \mu - \sum_{j=1}^s \lambda_j k_j; \sigma_1, \dots, \sigma_r, 1 \right),
$$

$$\left(1 - \mu - \gamma + \alpha + \alpha' + \beta' - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right),$$

$$\left(1 - \mu + \alpha - \beta - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right)$$

and

$$D' = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; \nu_1, \ldots, \nu_r, 0\right), \left(1 - \mu - \gamma + \alpha + \alpha' - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right),$$

$$\left(1 - \mu - \gamma + \alpha + \beta' - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right),$$

$$\left(1 - \mu - \beta - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \ldots, \sigma_r, 1\right),$$

where $H_{p,q}^{m,n}(\cdot)$ is the familiar Fox H-function.

Interestingly, in view of relation (1.4), we arrive at the following corollaries concerning Saigo fractional integral operator. It is similar to the results due to Agarwal [47], p.587 and p.590, Eqs. (18) and (25).

Corollary 3 *Let $\alpha, \beta, \gamma, \mu, \eta, \delta_j, \nu_i, z_i, a, b, c_j \in \mathbb{C}$, $\lambda_j, \sigma_i > 0$ ($i \in \{1, \ldots, r\}; j \in \{1, \ldots, s\}$), and $\Re(\alpha) > 0$. Let the following conditions be satisfied, too:*

(i) $|\arg z_i| < \frac{\pi}{2} \Omega_i$, $\Omega_i > 0$ $(i = 1, \ldots, r)$, *where*

$$\Omega_i = -\sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)}$$

$$+ \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0 \quad (i \in \{1, \ldots, r\}).$$

(ii)

$$\Re(\mu) + \sum_{i=1}^{r} \sigma_i \min_{1 \le j \le m_i} \left[\frac{\Re(d_j^{(i)})}{\delta_j^{(i)}}\right] > \max[0, \Re(\beta - \gamma)],$$

$$\Re(\eta) + \sum_{i=1}^{r} \nu_i \min_{1 \le j \le m_i} \left[\frac{\Re(d_j^{(i)})}{\delta_j^{(i)}}\right] > \max[0, \Re(\beta - \gamma)] \quad and \quad \left|\frac{a}{b} x\right| < 1.$$

Then the following result holds:

$$\left\{I_{0+}^{\alpha,\beta,\gamma}\left(t^{\mu-1}(b-at)^{-\eta} S_n^{m_1,\ldots,m_s}\left[c_1 t^{\lambda_1}(b-at)^{-\delta_1}, \ldots, c_s t^{\lambda_s}(b-at)^{-\delta_s}\right]\right.\right.$$

$$\left.\left. \times H\left[z_1 t^{\sigma_1}(b-at)^{-\nu_1}, \ldots, z_r t^{\sigma_r}(b-at)^{-\nu_r}\right]\right)\right\}(x)$$

$$= b^{-\eta} x^{\mu-\beta-1} \sum_{k_1,\ldots,k_s=0}^{m_1 k_1 + \cdots + m_s k_s \le n} \frac{(-n)_{m_1 k_1 + \cdots + m_s k_s}}{k_1! \cdots k_s!} \Lambda(n; k_1, \ldots, k_s) c_1^{k_1} \cdots c_s^{k_s}$$

$$
\times b^{-\sum_{j=1}^{s}\delta_j k_j}\, x^{\sum_{j=1}^{s}\lambda_j k_j}\, H_{p+3,q+3:\{p_r,q_r\};0,1}^{0,n+3:\{m_r,n_r\};1,0}
$$

$$
\left[
\begin{matrix}
z_1\frac{x^{\sigma_1}}{b^{v_1}} \\
\vdots \\
z_r\frac{x^{\sigma_r}}{b^{v_r}} \\
-\frac{a}{b}x
\end{matrix}
\;\middle|\;
\begin{matrix}
(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,p}, A':\{(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\};- \\[4pt]
(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q}, B':\{(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\};(0,1)
\end{matrix}
\right],
\tag{2.12}
$$

where

$$
A' = \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j; v_1,\ldots,v_r,1\right),\left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,1\right),
$$

$$
\left(1-\mu-\gamma+\beta-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,1\right)
$$

and

$$
B' = \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j; v_1,\ldots,v_r,0\right),\left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,1\right)
$$

$$
\left(1-\mu-\gamma-\alpha-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,1\right),
$$

and the conditions of existence of the above corollary follow easily from Theorem 1.

Corollary 4 *Let* $\alpha,\beta,\gamma,\mu,\eta,\delta_j,v_i,z_i,a,b,c_j\in\mathbb{C}$, $\lambda_j,\sigma_i>0$ $(i\in\{1,\ldots,r\}; j\in\{1,\ldots,s\})$, *and* $\Re(\alpha)>0$. *Let the following conditions be satisfied, too*:

(i) $|\arg z_i|<\frac{\pi}{2}\Omega_i$, $\Omega_i>0$ $(i=1,\ldots,r)$, *where*

$$
\Omega_i = -\sum_{j=n+1}^{p}\alpha_j^{(i)}-\sum_{j=1}^{q}\beta_j^{(i)}+\sum_{j=1}^{n_i}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}\delta_j^{(i)}>0.
$$

(ii)

$$
\Re(\mu)-\sum_{i=1}^{r}\sigma_i\min_{1\le j\le m_i}\left[\frac{\Re(d_j^{(i)})}{\delta_j^{(i)}}\right]<1+\max\left[\Re(\beta),\Re(\gamma)\right]\quad(i=1,\ldots,r),
$$

$$
\Re(\eta)-\sum_{i=1}^{r}v_i\min_{1\le j\le m_i}\left[\frac{\Re(d_j^{(i)})}{\delta_j^{(i)}}\right]<1+\max\left[\Re(\beta),\Re(\gamma)\right]\quad(i=1,\ldots,r).
$$

Then the following result holds:

$$
\left\{\mathrm{I}_{0-}^{\alpha,\beta,\gamma}\left(t^{\mu-1}(b-at)^{-\eta}S_n^{m_1,\ldots,m_s}\left[c_1 t^{\lambda_1}(b-at)^{-\delta_1},\ldots,c_s t^{\lambda_s}(b-at)^{-\delta_s}\right]\right.\right.
$$

$$
\left.\left.\times H\left[z_1 t^{\sigma_1}(b-at)^{-v_1},\ldots,z_r t^{\sigma_r}(b-at)^{-v_r}\right]\right)\right\}(x)
$$

$$
= b^{-\eta}x^{\mu-\beta-1}\sum_{\substack{k_1,\ldots,k_s=0}}^{m_1 k_1+\cdots+m_s k_s\le n}\frac{(-n)_{m_1 k_1+\cdots+m_s k_s}}{k_1!\cdots k_s!}\Lambda(n;k_1,\ldots,k_s)c_1^{k_1}\cdots c_s^{k_s}
$$

$$
\times\, b^{-\sum_{j=1}^{s} \delta_j k_j}\, x^{\sum_{j=1}^{s} \lambda_j k_j}\, H_{p+3,q+3:\{p_r,q_r\};0,1}^{0,n+3:\{m_r,n_r\};1,0}
$$

$$
\left[
\begin{array}{c}
z_1 \frac{x^{\sigma_1}}{b^{\nu_1}} \\
\vdots \\
z_r \frac{x^{\sigma_r}}{b^{\nu_r}} \\
-\frac{a}{b}x
\end{array}
\;\middle|\;
\begin{array}{l}
(a_j; \alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,p},\, C'' : \{(c_j^{(r)}, \gamma_j^{(r)})_{1,p_r}\};- \\[4pt]
(b_j; \beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q},\, D'' : \{(d_j^{(r)}, \delta_j^{(r)})_{1,q_r}\}; (0,1)
\end{array}
\right],
\tag{2.13}
$$

where

$$
C'' = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; \nu_1,\ldots,\nu_r, 1\right), \left(1 - \mu - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1,\ldots,\sigma_r, 1\right),
$$

$$
\left(1 + \alpha + \beta - \mu - \gamma - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1,\ldots,\sigma_r, 1\right)
$$

and

$$
D'' = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; \nu_1,\ldots,\nu_r, 0\right), \left(1 - \gamma - \mu - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1,\ldots,\sigma_r, 1\right)
$$

$$
\left(1 + \beta - \mu - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1,\ldots,\sigma_r, 1\right),
$$

and the conditions of existence of the above corollary follow easily from Theorem 2.

Remark 5 By putting $\beta = -\alpha$ and $\beta = 0$ in Corollaries 3 and 4, respectively, we can obtain very interesting results involving the Riemann-Liouville and Erdélyi-Kober fractional integral operators.

3 Special cases and concluding remarks

Here we consider another variation of the results derived in the preceding sections. The multivariable H-function occurring in these results can be suitably specialized to yield a wide variety of special functions (or product of such functions) of one or more variables (see [31], pp.18-19, 88-93, 253-254 and [48]). Again, by suitably specializing the coefficient of the first class of multivariable polynomials, it can be reduced to other multivariable hypergeometric polynomials and classical orthogonal polynomials of one or more variables.

(i) If we reduce the multivariable H-functions into the product of two Fox H-functions in Theorem 1 and then reduce one H-function to the exponential function by taking $\sigma_1 = 1$, $\nu_1 \to 0$, we obtain the following result after a little simplification:

$$
\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu-1}(b-at)^{-\eta} S_n^{m_1,\ldots,m_s}\left[c_1 t^{\lambda_1}(b-at)^{-\delta_1},\ldots,c_s t^{\lambda_s}(b-at)^{-\delta_s}\right] \right.\right.
$$

$$
\left.\left. \times\, e^{-z_1 t} H_{p_2,q_2}^{m_2,n_2}\left[z_2 t^{\sigma_2}(b-at)^{-\nu_2} \;\middle|\; \begin{array}{l}(c_j, C_j)_{1,p_2} \\ (d_j, D_j)_{1,q_2}\end{array}\right] \right) \right\}(x)
$$

$$
= b^{-\eta} x^{\mu-\alpha-\alpha'+\gamma-1} \sum_{k_1,\ldots,k_s=0}^{m_1 k_1 + \cdots + m_s k_s \leq n} \frac{(-n)_{m_1 k_1 + \cdots + m_s k_s}}{k_1! \cdots k_s!} \Lambda(n; k_1,\ldots,k_s) c_1^{k_1} \cdots c_s^{k_s}
$$

$$\times \, b^{-\sum_{j=1}^{s}\delta_j k_j}\, x^{\sum_{j=1}^{s}\lambda_j k_j}\, H^{0,4:1,0;m_2,n_2;1,0}_{4,4:0,1;p_2,q_2;0,1}$$

$$\left[\begin{array}{c} z_1 x \\ z_2 \frac{x^{\sigma_2}}{b^{\nu_2}} \\ -\frac{a}{b}x \end{array} \middle| \begin{array}{c} E : (c_j,\gamma_j)_{1,p_2}; - \\ F : (0,1); \{(d_j,\delta_j)_{1,q_2}\}; (0,1) \end{array} \right], \tag{3.1}$$

where

$$E = \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j; 1, \nu_2, 1\right), \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j; 1, \sigma_2, 1\right),$$

$$\left(1-\mu-\gamma+\alpha+\alpha'+\beta-\sum_{j=1}^{s}\lambda_j k_j; 1, \sigma_2, 1\right),$$

$$\left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_j k_j; 1, \sigma_2, 1\right),$$

$$F = \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j; 1, \nu_2, 0\right), \left(1-\mu-\gamma+\alpha+\alpha-\sum_{j=1}^{s}\lambda_j k_j; 1, \sigma_2, 1\right),$$

$$\left(1-\mu-\gamma+\alpha'+\beta-\sum_{j=1}^{s}\lambda_j k_j; 1, \sigma_2, 1\right), \left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_j k_j; 1, \sigma_2, 1\right).$$

The conditions of validity of the above result easily follow from Theorem 1.

Remark 6 If we reduce the Marichev-Saigo-Maeda type fractional integral operator into the Riemann-Liouville fractional integral operator and put $S_n^{m_1,\dots,m_s} = 1, \eta, \nu_2 = 0$, and make a suitable adjustment in the parameters in the above equation, then we obtain the known result by Kilbas and Saigo [49], p.52, Eq. (2.7.9); in addition, if we put $S_{n_j}^{m_j} = 1, \eta, \nu_2 = 0$, $z_2 = 1/4, \sigma_2 = 2$ and reduce the Fox H-function to the Bessel function of the first kind, then we obtain the known result given by Kilbas and Sebastian [50], p.873, Eqs. (25)-(29).

(ii) Next if we take $z_2, \sigma_2 = 1$ and $\nu_2 = 0$ in the result (3.1) and reduce the H-function of one variable to a generalized Mittag-Leffler function [51] (see also [52]), we can easily obtain the following result after a few simplifications:

$$\left\{ I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+} \left(t^{\mu-1}(b-at)^{-\eta} S_n^{m_1,\dots,m_s} \left[c_1 t^{\lambda_1}(b-at)^{-\delta_1}, \dots, c_s t^{\lambda_s}(b-at)^{-\delta_s} \right] \right. \right.$$

$$\left. \left. \times \, e^{-z_1 t} E^{\nu}_{\omega,\xi}[t] \right) \right\}(x)$$

$$= \frac{b^{-\eta} x^{\mu-\alpha-\alpha'+\gamma-1}}{\Gamma(\nu)} \sum_{k_1,\dots,k_s=0}^{m_1 k_1+\cdots+m_s k_s \le n} \frac{(-n)_{m_1 k_1+\cdots+m_s k_s}}{k_1!\cdots k_s!} \Lambda(n; k_1,\dots,k_s) c_1^{k_1}\cdots c_s^{k_s}$$

$$\times \, b^{-\sum_{j=1}^{s}\delta_j k_j}(-x)^{\sum_{j=1}^{s}\lambda_j k_j}\, H^{0,4:1,0;1,1;1,0}_{4,3:0,1;1,3;0,1}$$

$$\left[\begin{array}{c} z_1 x \\ x \\ -\frac{a}{b}x \end{array} \middle| \begin{array}{c} E : -; (1-\nu,1); - \\ F : (0,1); (0,1),(1-\eta;0);(1-\xi,\omega);(0,1) \end{array} \right], \tag{3.2}$$

where

$$E = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; 1, 0, 1\right), \left(1 - \mu - \sum_{j=1}^{s} \lambda_j k_j; 1, 1, 1\right),$$

$$\left(1 - \mu - \gamma + \alpha + \alpha' + \beta - \sum_{j=1}^{s} \lambda_j k_j; 1, 1, 1\right), \left(1 - \mu + \alpha' - \beta' - \sum_{j=1}^{s} \lambda_j k_j; 1, 1, 1\right),$$

$$F = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; 1, 0, 0\right), \left(1 - \mu - \gamma + \alpha + \alpha' - \sum_{j=1}^{s} \lambda_j k_j; 1, 1, 1\right),$$

$$\left(1 - \mu - \gamma + \alpha' + \beta - \sum_{j=1}^{s} \lambda_j k_j; 1, 1, 1\right), \left(1 - \mu - \beta' - \sum_{j=1}^{s} \lambda_j k_j; 1, 1, 1\right).$$

The conditions of validity of (3.2) can be easily followed directly by those given with (2.5).

Remark 7 If we reduce the Marichev-Saigo-Maeda type fractional integral operator to the Riemann-Liouville fractional integral operator and set $S_n^{m_1,\ldots,m_s} = 1$, $\eta, \nu_2 = 0$, and make a suitable adjustment in the parameters in equation (3.2), then we arrive at the known result given by Saxena *et al.* [53], p.168, Eq. (2.1).

(iii) Again, if we reduce the H-function of one variable to the generalized Wright hypergeometric function [31], p.19, Eq. (2.6.11), in the result (3.1), we arrive at the following new and interesting result:

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu-1}(b-at)^{-\eta} S_n^{m_1,\ldots,m_s}\left[c_1 t^{\lambda_1}(b-at)^{-\delta_1}, \ldots, c_s t^{\lambda_s}(b-at)^{-\delta_s} \right] \right. \right.$$

$$\left. \left. \times e^{-z_1 t} \, {}_{p_2}\psi_{q_2}\left[-z_2 t^{\sigma_2}(b-at)^{-\nu_2} \, \Bigg| \, \begin{array}{c} (1-c_j, C_j)_{1,p_2} \\ (0,1),(1-d_j, D_j)_{1,q_2} \end{array} \right] \right) \right\}(x)$$

$$= b^{-\eta} x^{\mu-\alpha-\alpha'+\gamma-1} \sum_{\substack{k_1,\ldots,k_s=0}}^{m_1 k_1 + \cdots + m_s k_s \leq n} \frac{(-n)_{m_1 k_1 + \cdots + m_s k_s}}{k_1! \cdots k_s!} \Lambda(n; k_1, \ldots, k_s) c_1^{k_1} \cdots c_s^{k_s}$$

$$\times b^{-\sum_{j=1}^{s} \delta_j k_j} x^{\sum_{j=1}^{s} \lambda_j k_j} H_{4,4:0,1;p_2,q_2;0,1}^{0,4:1,0;1,p_2;1,0}$$

$$\left[\begin{array}{c} z_1 x \\ -z_2 \frac{x^{\sigma_2}}{b^{\nu_2}} \\ -\frac{a}{b} x \end{array} \, \Bigg| \, \begin{array}{c} E : (c_j, C_j)_{1,p_2}; - \\ F : (0,1); \{(d_j, D_j)_{1,q_2}\}; (0,1) \end{array} \right], \tag{3.3}$$

where

$$E = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; 1, \nu_2, 1\right), \left(1 - \mu - \sum_{j=1}^{s} \lambda_j k_j; 1, \sigma_2, 1\right),$$

$$\left(1 - \mu - \gamma + \alpha + \alpha' + \beta - \sum_{j=1}^{s} \lambda_j k_j; 1, \sigma_2, 1\right), \left(1 - \mu + \alpha' - \beta' - \sum_{j=1}^{s} \lambda_j k_j; 1, \sigma_2, 1\right),$$

$$F = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; 1, \nu_2, 0\right), \left(1 - \mu - \gamma + \alpha + \alpha' - \sum_{j=1}^{s} \lambda_j k_j; 1, \sigma_2, 1\right),$$

$$\left(1 - \mu - \gamma + \alpha' + \beta - \sum_{j=1}^{s} \lambda_j k_j; 1, \sigma_2, 1\right), \left(1 - \mu - \beta' - \sum_{j=1}^{s} \lambda_j k_j; 1, \sigma_2, 1\right).$$

(iv) Further Theorem 1, if we reduce the multivariable H-function to the product of r-different Whittaker functions [31], p.18, Eq. (2.6.7), and take $\nu_i \to 0$, $\sigma_i = 1$, then we arrive at the following result:

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu-1} (b - at)^{-\eta} S_n^{m_1,\dots,m_s} \left[c_1 t^{\lambda_1} (b - at)^{-\delta_1}, \dots, c_s t^{\lambda_s} (b - at)^{-\delta_s} \right] \right. \right.$$

$$\left. \left. \times \prod_{i=1}^{r} e^{\frac{-z_i t}{2}} W_{\lambda_i, \mu_i}(z_i t) \right) \right\} (x)$$

$$= b^{-\eta} x^{\mu - \alpha - \alpha' + \gamma - 1} \sum_{k_1,\dots,k_s=0}^{m_1 k_1 + \cdots + m_s k_s \leq n} \frac{(-n)_{m_1 k_1 + \cdots + m_s k_s}}{k_1! \cdots k_s!} \Lambda(n; k_1, \dots, k_s) c_1^{k_1} \cdots c_s^{k_s}$$

$$\times b^{-\sum_{j=1}^{s} \delta_j k_j} x^{\sum_{j=1}^{s} \lambda_j k_j} H_{4,4:1,2;\dots;1,2;0,1}^{0,4:2,0;\dots;2,0;1,0}$$

$$\begin{bmatrix} z_1 x \\ \vdots \\ z_r x \\ -\frac{a}{b} x \end{bmatrix} \left| \begin{array}{l} G : -; (1 - \lambda_1, 1); \dots; (1 - \lambda_r, 1); - \\ H : (1/2 \pm \mu_1, 1); \dots; (1/2 \pm \mu_r, 1)(0, 1) \end{array} \right], \tag{3.4}$$

where

$$G = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; \underbrace{1, \dots, 1}_{r \text{ times}}, 1\right), \left(1 - \mu - \sum_{j=1}^{s} \lambda_j k_j; \underbrace{1, \dots, 1}_{r \text{ times}}, 1\right),$$

$$\left(1 - \mu - \gamma + \alpha + \alpha' + \beta - \sum_{j=1}^{s} \lambda_j k_j; \underbrace{1, \dots, 1}_{r \text{ times}}, 1\right),$$

$$\left(1 - \mu + \alpha' - \beta' - \sum_{j=1}^{s} \lambda_j k_j; \underbrace{1, \dots, 1}_{r \text{ times}}, 1\right),$$

$$H = \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; \underbrace{1, \dots, 1}_{r \text{ times}}, 0\right), \left(1 - \mu - \gamma + \alpha + \alpha' - \sum_{j=1}^{s} \lambda_j k_j; \underbrace{1, \dots, 1}_{r \text{ times}}, 1\right),$$

$$\left(1 - \mu - \gamma + \alpha' + \beta - \sum_{j=1}^{s} \lambda_j k_j; \underbrace{1, \dots, 1}_{r \text{ times}}, 1\right), \left(1 - \mu - \beta' - \sum_{j=1}^{s} \lambda_j k_j; \underbrace{1, \dots, 1}_{r \text{ times}}, 1\right).$$

The conditions of validity of the above result easily follow from Theorem 1.

Remark 8 If we reduce the Marichev-Saigo-Maeda type fractional integral operator to the Riemann-Liouville fractional integral operator and set $S_n^{m_1,\dots,m_s} = 1$, $\eta, \nu_2 = 0$, and make a suitable adjustment in the parameters in equation (3.4), we arrive at the known result given by Kilbas [54], p.117, Eq. (11).

(v) Finally, if we reduce the multivariable H-function into the product of two Fox H-functions in Theorem 1 and then reduce one H-function to the exponential function by setting σ_1, λ_j, ν_1, $\delta_j = 0$, c_j, $s = 1$ and S_n^m to the Hermite polynomial [6, 55], and we set $S_n^2[x] = x^{n/2}H_n[\frac{1}{2\sqrt{x}}]$, in this case $m = 2$, $A_{n,k} = (-1)^k$, we obtain the following interesting result:

$$
\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}(b-at)^{-\eta}t^{n/2}H_n\left[\frac{1}{2\sqrt{t}}\right] \right.\right.
$$

$$
\left.\left. \times\, e^{-z_1 t}H_{p_2,q_2}^{m_2,n_2}\left[z_1 t^{\sigma_1}(b-at)^{-\nu_1} \,\middle|\, \begin{matrix}(c_j,C_j)_{1,p_2}\\(d_j,D_j)_{1,q_2}\end{matrix}\right]\right) \right\}(x)
$$

$$
= b^{-\eta}x^{\mu-\alpha-\alpha'+\gamma-1}\sum_{k_1,\ldots,k_s=0}^{m_1 k_1+\cdots+m_s k_s \le n}\frac{(-n)_{m_1 k_1+\cdots+m_s k_s}}{k_1!\cdots k_s!}\Lambda(n;k_1,\ldots,k_s)c_1^{k_1}\cdots c_s^{k_s}
$$

$$
\times\, b^{-\sum_{j=1}^s \delta_j k_j}x^{\sum_{j=1}^s \lambda_j k_j}H_{4,4:0,1;\{p_2,q_2\};0,1}^{0,4:1,0;\{m_2,n_2\};1,0}
$$

$$
\left[\begin{matrix} z_1 x \\ z_2\frac{x^{\sigma_2}}{b^{\nu_2}} \\ -\frac{a}{b}x \end{matrix}\middle| \begin{matrix} G:(c_j^{(r)},\gamma_j^{(r)})_{1,p_r};-\\ H:(0,1);\{(d_j,\delta_j)_{1,q_2}\};(0,1)\end{matrix}\right], \tag{3.5}
$$

where

$$
G = \left(1-\eta-\sum_{j=1}^s \delta_j k_j;1,\nu_2,1\right),\left(1-\mu-\sum_{j=1}^s \lambda_j k_j;1,\sigma_2,1\right),
$$

$$
\left(1-\mu-\gamma+\alpha+\alpha'+\beta-\sum_{j=1}^s \lambda_j k_j;1,\sigma_2,1\right),\left(1-\mu+\alpha'-\beta'-\sum_{j=1}^s \lambda_j k_j;1,\sigma_2,1\right),
$$

$$
H = \left(1-\eta-\sum_{j=1}^s \delta_j k_j;1,\nu_2,0\right),\left(1-\mu-\gamma+\alpha+\alpha-\sum_{j=1}^s \lambda_j k_j;1,\sigma_2,1\right),
$$

$$
\left(1-\mu-\gamma+\alpha'+\beta-\sum_{j=1}^s \lambda_j k_j;1,\sigma_2,1\right),\left(1-\mu-\beta'-\sum_{j=1}^s \lambda_j k_j;1,\sigma_2,1\right).
$$

The conditions of validity of the above result can be easily derived from Theorem 1.

A large number of other special cases of our main result can also be obtained, but we do not mention them here on account of the lack of space.

Now, we conclude our present investigation by remarking that the fractional integration (of Marichev-Saigo-Maeda type) of the products of multivariable H-functions and the first class of multivariable polynomials established in this paper will be useful for investigators in various disciplines of applied sciences and engineering physics. We are also trying to find certain possible applications of those results presented here to some other research areas, for example, obtaining a closed form solution of a fractional generalization of a free electron equation.

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Endnote

[a] Here and in what follows, we denote by ω an imaginary unit.

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