

Regularization of ultraviolet divergence for a particle interacting with a scalar quantum field

O. D. Skoromnik,^{1,*} I. D. Feranchuk,² D. V. Lu,² and C. H. Keitel¹

¹*Max Planck Institute for Nuclear Physics,
Saupfercheckweg 1, 69117 Heidelberg, Germany*

²*Belarusian State University, 4 Nezavisimosty Ave., 220030, Minsk, Belarus*

When a non-relativistic particle interacts with a scalar quantum field, the standard perturbation theory leads to a dependence of the energy of its ground state on an undefined parameter—“momentum cut-off”—due to the ultraviolet divergence. We show that the use of non-asymptotic states of the system results in a calculation scheme in which all observable quantities remain finite and continuously depend on the coupling constant without any additional parameters. It is furthermore demonstrated that the divergence of traditional perturbation series is caused by the energy being a function with a logarithmic singularity for small values of the coupling constant.

PACS numbers: 11.10.-z, 11.10.Gh, 11.15.Bt, 11.15.Tk, 63.20.kd

Keywords: quantum field theory; non-perturbative theory; renormalization; divergences

I. INTRODUCTION

A characteristic property of the majority of quantum field theories (QFT) is the divergence of integrals appearing in the perturbation theory for the calculation of physical quantities such as mass and charge of the interacting particles. The divergences appear in the integrations over momenta in intermediate states both on the lower limit (infrared divergence) and on the upper limit (ultraviolet divergence). In order to circumvent this difficulty, a renormalization procedure is used, which allows the redefinition of the initial parameters of the system through their observable values. The renormalization scheme was firstly developed for quantum electrodynamics (QED) [1] and later generalized to other QFT models [2]. These schemes can be used for so-called renormalizable theories, for which the reconstructed perturbation theory can be built in a way that the infinite values are included in the definition of “physical” charge and mass and, therefore, do not appear in other observables of the system. However, even the founders of QED anticipated “that the renormalization theory is simply a way to sweep the difficulties of the divergences in electrodynamics under the rug.” (R. P. Feynman [3]). In many papers P. A. M. Dirac wrote that this approach was in contradiction with logical principles of quantum mechanics [4, 5].

Accordingly, the question arises whether these divergences are an intrinsic property of quantum field models or they are caused by the application of perturbation theory for the calculation of physical quantities, which are non-analytical functions of the coupling constant such as in the theory of superconductivity [6]. A large number of works is devoted to this problem, nevertheless a solution still has not been found up to now. However, this question is of great importance for a correct mathematical formulation of fundamental physical theories and is essential for examining the applicability of non-renormalizable theories [7] for the description of real physical systems.

Let us recall that in standard perturbation theory the Hamiltonian of non-interacting fields is used as a zeroth-order approximation, while Fock states of the free fields are employed for

* Corresponding author: olegskor@gmail.com

the calculation of the transition matrix elements in the subsequent corrections to observable characteristics of the system. This approach is based on the assumption of an asymptotic switch off of the interaction between the fields [8]. However, in a series of works it has been shown [9–12] that the infrared divergence arises just because of the use of asymptotically free field states. As follows from reference [9] and the subsequent publication [10], the infrared divergence disappears in all orders of perturbation theory in QED if, in the zeroth-order approximation, the coherent states of the electromagnetic field bound to the particle are used and the parameters of these states are appropriately chosen.

At first glance, this may contradict representation theory in quantum mechanics, in accordance to which the result of the calculation of the observable characteristics of the system should not depend on the choice of basis states, provided that those form a full basis in a Hilbert space as for free field states. However, this statement is correct only for the exact solution of the problem, whereas individual terms of the perturbation series can change with a different choice of basis in zeroth-order approximation. As was demonstrated in reference [13, 14] the transition from one basis to another corresponds to the partial summation of a divergent series within standard perturbation theory and allows for the non-perturbative calculation of subsequent corrections in the form of a convergent sequence. A good example of how the basis choice influences the approximate calculation of the characteristics of the quantum system with a continuous spectrum is given by the scattering at a Coulomb potential [15]. In this well known case, the wave function of the system has no singularities. In contrast, Born’s scattering amplitude, approximately calculated via an asymptotically free basis, displays a singularity for scattering at small angles. As was demonstrated in references [15], this singularity does not appear with the use of non-asymptotic wave functions.

The main goal of our work is to investigate whether a proper choice of basis in zeroth-order approximation allows to construct a calculation scheme free of ultraviolet divergences. In order not to overload the proposed approach with details related to the internal degrees of freedom and to render all calculations as transparent as possible, we investigate as a representative example a model system, which consists of a non-relativistic particle without spin interacting with a scalar quantum field. A standard-perturbation-theory series, in this case, does not exhibit infrared divergence, contains however ultraviolet divergence. This results in a dependence of the energy of the ground state on the undefined momentum cut-off, which is required for the calculation of high-order corrections. Consequently, our task is to prove that the energy of the ground state of the considered model system can be calculated without any additional parameters such as a momentum cut-off. At the same time, it is important to show that the energy of the system is a non-analytical function of the coupling constant and consequently can not be represented as a series in the framework of conventional perturbation theory.

With the inclusion of a field polarization our employed model coincides with non-relativistic QED [16] or if the field is scalar it has a physical realization in solids [17], where however, due to the discrete structure of a crystal, a natural momentum cut-off intrinsically appears, defined via the Brillouin-zone boundary. In free space this regularization is not present and has to be artificially included, e.g., via lattice models [18], where the boundary momentum is defined through an artificial lattice period. In contrast, in our formulation we will consider a system in free space without neither natural nor artificial cut-off.

In addition, in a series of works [19–22] an analogous model of a particle interacting with a scalar quantum field with the momentum cut-off was used for the investigation of the fundamental mathematical problem of the existence of the solutions of the Schrödinger equation.

The article is organized in the following way. In section II the model of a non-relativistic particle without spin interacting with a scalar quantum field is described and its parameters are calculated in the framework of conventional perturbation theory. In section III the basis of non-asymptotic states is investigated and the iteration scheme of the calculations is presented. The zeroth-order approximation, which is found to be free of ultraviolet divergence for the

energy and effective mass is then worked out using this basis. In section IV the proposed iteration scheme is employed for computing the correction to the zeroth-order approximation of the energy. The convergence of all integrals is demonstrated and the character of the singularity of the energy as a non-analytic function of the coupling constant is determined in the weak coupling limit. In addition, the details of all calculations are presented in appendices.

II. MODEL DESCRIPTION

Let us examine the Hamiltonian of the system consisting of a non-relativistic particle interacting with a scalar quantum field

$$H = H_0 + H_{\text{int}}, \quad (1)$$

$$H_0 = \frac{\mathbf{p}^2}{2} + \sum_{\mathbf{k}} \omega_{\mathbf{k}} \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{a}_{\mathbf{k}}, \quad (2)$$

$$H_{\text{int}} = \frac{g}{\sqrt{2\Omega}} \sum_{\mathbf{k}} A_{\mathbf{k}} \left(e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{a}_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{r}} \mathbf{a}_{\mathbf{k}}^\dagger \right). \quad (3)$$

Here, we select the system of units in which $m = 1$, $\hbar = c = 1$, the momentum operator $\mathbf{p} = -i\nabla$, normalization volume Ω , vertex function $A_{\mathbf{k}} = 1/\sqrt{\omega_{\mathbf{k}}}$, creation (annihilation) operators $\mathbf{a}_{\mathbf{k}}^\dagger$ ($\mathbf{a}_{\mathbf{k}}$) of the field mode with the frequency $\omega_{\mathbf{k}} = k = |\mathbf{k}|$, and the coupling constant g . The real physical system, which is described via Hamiltonian (1) corresponds to an electron interacting with acoustic phonons in a continuous model of a crystal [17]. If we choose $\omega_{\mathbf{k}} = 1$, $A_{\mathbf{k}} = 1/k$, and $g = \sqrt{8\pi\alpha}$, operator (1) corresponds to the Fröhlich Hamiltonian [23], which describes the interaction of an electron with optical phonons in a crystal, i.e., the so-called “polaron” problem [24–26].

The total momentum operator

$$\mathbf{P} = -i\nabla + \sum_{\mathbf{k}} \mathbf{k} \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{a}_{\mathbf{k}} \quad (4)$$

commutes with the Hamiltonian of the system (1) and consequently the eigenvalues $E(\mathbf{P})$ and eigenfunctions $|\Psi_{\mathbf{P}}\rangle$ are defined as solutions of the following system of equations

$$H|\Psi_{\mathbf{P}}\rangle = E(\mathbf{P})|\Psi_{\mathbf{P}}\rangle, \quad (5)$$

$$\mathbf{P}|\Psi_{\mathbf{P}}\rangle = \mathbf{P}|\Psi_{\mathbf{P}}\rangle. \quad (6)$$

In the conventional perturbation expansion over the coupling constant in the zeroth-order approximation the solution of the stationary Schrödinger equation with Hamiltonian (2) is simply determined and corresponds to the free particle with momentum \mathbf{p} and Fock states of the phonon field with the set of occupation numbers $\{n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots\} \equiv \{n_{\mathbf{k}}\}$:

$$|\Psi_{\mathbf{p}, \{n_{\mathbf{k}}\}}^{(0)}\rangle = \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{\sqrt{\Omega}} |\{n_{\mathbf{k}}\}\rangle, \quad \sum_{\mathbf{k}} \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{a}_{\mathbf{k}} |\{n_{\mathbf{k}}\}\rangle = \sum_{\mathbf{k}} n_{\mathbf{k}} |\{n_{\mathbf{k}}\}\rangle, \quad (7)$$

$$E^{(0)}(\mathbf{P}, \{n_{\mathbf{k}}\}) = \frac{1}{2} \left(\mathbf{P} - \sum_{\mathbf{k}} \mathbf{k} n_{\mathbf{k}} \right)^2 + \sum_{\mathbf{k}} \omega_{\mathbf{k}} n_{\mathbf{k}}, \quad \mathbf{P} = \mathbf{p} + \sum_{\mathbf{k}} \mathbf{k} n_{\mathbf{k}}. \quad (8)$$

Let us suppose that the system is in the ground state of the phonon field $\{n_{\mathbf{k}}\} = 0$, which leads

to the following eigenfunction and eigenvalue

$$|\Psi_{\mathbf{P},0}^{(0)}\rangle = \frac{e^{i\mathbf{P}\cdot\mathbf{r}}}{\sqrt{\Omega}}|0\rangle \quad (9)$$

$$E^{(0)}(\mathbf{P}, 0) = \frac{P^2}{2}, \quad \mathbf{P} = \mathbf{p}. \quad (10)$$

The first non-vanishing correction to the system energy arises in the second order of perturbation theory (single-phonon intermediate transitions) and corresponds to the self-energy diagram, which defines the mass operator $\Sigma(\mathbf{P})$ and is determined as

$$\Sigma(\mathbf{P}) = \Delta E^{(2)}(\mathbf{P}, 0) = -\frac{g^2}{2\Omega} \sum_{\mathbf{k}} \frac{1}{\omega_{\mathbf{k}}} \frac{1}{k^2/2 - \mathbf{P} \cdot \mathbf{k} + \omega_{\mathbf{k}}} = -\frac{g^2}{16\pi^3} \int \frac{d\mathbf{k}}{k[k^2/2 - \mathbf{P} \cdot \mathbf{k} + k]}. \quad (11)$$

In order to select bound state energy $E_b = E^{(2)}(0, 0)$ and effective mass m^* of a particle we expand the energy in a series over \mathbf{P} up to second order

$$E^{(0)}(\mathbf{P}, 0) + \Delta E^{(2)}(\mathbf{P}, 0) \approx E_b + \frac{P^2}{2m^*} \equiv -\frac{g^2}{16\pi^3} \int \frac{d\mathbf{k}}{k[k^2/2 + k]} + \frac{P^2}{2} - \frac{g^2}{16\pi^3} \int \frac{d\mathbf{k}}{k[k^2/2 + k]^3} (\mathbf{P} \cdot \mathbf{k})^2. \quad (12)$$

The first integral in equation (12) logarithmically diverges, that is the bound state energy depends on the momentum cut-off K

$$E_b = -\frac{g^2}{2\pi^2} \ln\left(\frac{K}{2} + 1\right), \quad (13)$$

and becomes infinite when $K \rightarrow \infty$, such that the correction to the energy is undefined in the framework of the perturbation theory for our model [27].

At the same time, the corrected mass is well defined and equal to

$$\frac{1}{m^*} \simeq 1 - \frac{g^2}{6\pi^2}; \quad m^* \simeq 1 + \frac{g^2}{6\pi^2}. \quad (14)$$

In contrast to this in the polaron problem all integrals are convergent because they contain in the denominator the additional power of k . The corresponding quantities for the polaron problem read as [24–26]

$$E_b \simeq -\alpha; \quad m^* \simeq 1 + \frac{\alpha}{6}. \quad (15)$$

It is important to stress here that in our model, the interaction energy between particle and field is observable and consequently, the infinite energy (12) can not be included in the mass renormalization. Thus, we can conclude that the use of perturbation theory for two physically close quantum-field models leads to qualitatively different results. Therefore, a modification of the calculation method of subsequent corrections to the energy for our model is required and appears achievable.

III. ITERATION SCHEME, BASIS CHOICE AND ZERO-ORDER APPROXIMATION OF THE SYSTEM'S ENERGY

In order to build an iteration scheme not in the framework of perturbation theory we will employ the operator method (OM) for the solution of the Schrödinger equation, which was

introduced in reference [28] and its detailed explanation is given in the monograph [13, 14]. Let us quickly revise here the basics of this method. Suppose, the eigenvalues E_μ and eigenvectors $|\Psi_\mu\rangle$ with a set of quantum numbers μ of the stationary Schrödinger equation need to be found:

$$\mathbf{H}|\Psi_\mu\rangle = E_\mu|\Psi_\mu\rangle. \quad (16)$$

In contrast to perturbation theory, where the Hamiltonian \mathbf{H} of the system is split into the zeroth-order approximation and perturbation parts, according to OM the total Hamiltonian is taken into account as is, while however, the state vector is probed via an approximate state:

$$|\Psi_\mu\rangle \approx |\psi_\mu(\omega_\mu)\rangle,$$

which depends on a set of variational parameters ω_μ . Then, the exact solution can be represented as a series

$$|\Psi_\mu\rangle = |\psi_\mu(\omega_\mu)\rangle + \sum_{v \neq \mu} C_{\mu v} |\psi_v(\omega_\mu)\rangle. \quad (17)$$

Here we want to pay attention to the fact that for a given set of quantum numbers μ , the set ω_μ is fixed. By plugging the expansion (17) into Schrödinger's equation (16) and projecting on different states $|\psi_\mu(\omega_\mu)\rangle$ and $|\psi_v(\omega_\mu)\rangle$ one obtains the equations for the energies E_μ and coefficients $C_{\mu v}$:

$$E_\mu = \left[1 + \sum_{v \neq \mu} C_{\mu v} I_{\mu v} \right]^{-1} \left[H_{\mu\mu} + \sum_{v \neq \mu} C_{\mu v} H_{\mu v} \right]; \quad (18)$$

$$C_{\mu\gamma} = \left[E_\mu - H_{\gamma\gamma} \right]^{-1} \left[H_{\gamma\mu} - E_\mu I_{\gamma\mu} + \sum_{v \neq \mu \neq \gamma} C_{\mu v} (H_{\gamma v} - E_\mu I_{\gamma v}) \right]; \quad (19)$$

$$H_{\mu v} \equiv \langle \psi_\mu(\omega_\mu) | \mathbf{H} | \psi_v(\omega_\mu) \rangle; \quad I_{\mu v} \equiv \langle \psi_\mu(\omega_\mu) | \psi_v(\omega_\mu) \rangle.$$

It is important to stress here that all matrix elements are calculated with the *full* Hamiltonian of the system and the set of vectors $|\psi_\mu(\omega_\mu)\rangle$ can be normalized, while not necessarily being mutually orthogonal. The system of equations (18), (19) is the exact representation of the Schrödinger equation. For the approximate solution of this system, in accordance with OM the following concept is used: the closer the zeroth-order approximation of the state vector is to the exact solution, the closer the matrix $H_{\mu v}$ becomes to the diagonal one. Therefore, an iteration scheme for the solution of the system (18), (19) can be built, for which convergence is determined with the ratios of non-diagonal elements $H_{\mu v}$ to the diagonal ones $H_{\mu\mu}$ in the representation of the state vectors $|\psi_\mu(\omega_\mu)\rangle$. A sufficiently detailed discussion of the convergence of the iteration scheme for different physical systems is given in the monograph [13]. Consequently, we find the system of recurrent equations

$$E_\mu^{(s)} = \left[1 + \sum_{v \neq \mu} C_{\mu v}^{(s-1)} I_{\mu v} \right]^{-1} \left[H_{\mu\mu} + \sum_{v \neq \mu} C_{\mu v}^{(s-1)} H_{\mu v} \right]; \quad (20)$$

$$C_{\mu\gamma}^{(s)} = \left[E_\mu^{(s-1)} - H_{\gamma\gamma} \right]^{-1} \left[H_{\gamma\mu} - E_\mu^{(s-1)} I_{\gamma\mu} + \sum_{v \neq \mu \neq \gamma} C_{\mu v}^{(s-1)} (H_{\gamma v} - E_\mu^{(s-1)} I_{\gamma v}) \right]; \quad (21)$$

$$C_{\mu v}^{(-1)} = C_{\mu v}^{(0)} = 0; \quad E_\mu^{(0)} = H_{\mu\mu}. \quad (22)$$

As opposed to conventional perturbation theory, where the exact solution is defined as a sum of corrections of all orders, in OM the exact value of the energy of the system is given as a limit

of a sequence

$$E_\mu = \lim_{s \rightarrow \infty} E_\mu^{(s)}; \quad s = 0, 1, \dots \quad (23)$$

In particular, for the first two iterations one can find

$$E_\mu^{(1)} = E_\mu^{(0)} = H_{\mu\mu}; \quad (24)$$

$$E_\mu^{(2)} = \left[1 + \sum_{\nu \neq \mu} \frac{(H_{\nu\mu} - E_\mu^{(0)} I_{\nu\mu}) I_{\mu\nu}}{E_\mu^{(0)} - H_{\nu\nu}} \right]^{-1} \left[H_{\mu\mu} + \sum_{\nu \neq \mu} \frac{(H_{\nu\mu} - E_\mu^{(0)} I_{\nu\mu}) H_{\mu\nu}}{E_\mu^{(0)} - H_{\nu\nu}} \right]. \quad (25)$$

The last equation looks analogously to the second-order correction of perturbation theory, while the main difference is related to the denominators of equation (25), where the matrix element $H_{\nu\nu}$ is calculated with the full Hamiltonian of the system, whereas the perturbation theory relations merely involve the diagonal element of the unperturbed Hamiltonian. As will be shown below, this is exactly the reason, which determines the convergence of integrals over intermediate states.

Before proceeding with the application of the iteration scheme, let us discuss the choice of the parameters $\{\omega_\mu\} = \{\omega_\mu^1, \dots, \omega_\mu^n, \dots\}$ in more detail. For this we note that the representation (18-19) is exactly equivalent to the Schrödinger equation (16), provided that the set of states $\{|\psi_\mu(\{\omega_\mu\})\rangle\}$ is a complete one. It is evident that if the state vectors $\{|\psi_\mu(\{\omega_\mu\})\rangle\}$ coincide with the exact eigenstates $|\Psi_\mu\rangle$ of the full Hamiltonian, the matrix $H_{\mu\nu}$ is a diagonal one, i.e. $H_{\mu\nu} = E_\mu \delta_{\mu\nu}$ and the coefficients $C_{\mu\nu} = 0$. The eigenvalues E_μ are determined exactly and are independent of the set of parameters $\{\omega_\mu\}$. Therefore, the relation

$$\frac{\partial E_\mu}{\partial \omega_\mu^n} \equiv 0, \quad n = \{1, 2, \dots\} \quad (26)$$

holds identically.

According to our initial assumption we choose the trial parameters $\{\omega_\mu\}$ in the basis states $|\psi_\mu(\{\omega_\mu\})\rangle$ such that they determine the best possible approximation for the exact solution $|\Psi_\mu\rangle$ in the chosen class of functions. This is equivalent to the supposition that the off-diagonal elements of the matrix $H_{\mu\nu}$ are small numbers such that the ratios $H_{\mu\nu}/H_{\mu\mu}$ are proportional to some effective small parameter ϵ . Therefore, the zeroth-order approximation of the operator method is chosen as

$$E_\mu^{(0)}(\{\omega_\mu\}) = H_{\mu\mu}(\{\omega_\mu\}), \quad C_{\mu\nu}^{(0)} = 0, \quad |\Psi_\mu^{(0)}\rangle = |\psi_\mu(\{\omega_\mu\})\rangle. \quad (27)$$

However, the matrix $H_{\mu\nu}$ contains small off-diagonal elements, which need to be taken into account. Hence, the subsequent approximations read

$$E_\mu = H_{\mu\mu}(\{\omega_\mu\}) + \sum_{s=1}^{\infty} \epsilon^s E_\mu^{(s)}(\{\omega_\mu\}), \quad (28)$$

$$C_{\mu\nu}(\{\omega_\mu\}) = \sum_{s=1}^{\infty} \epsilon^s C_{\mu\nu}^{(s)}(\{\omega_\mu\}), \quad \mu \neq \nu. \quad (29)$$

As the left-hand side of equation (28) does not depend on the parameters $\{\omega_\mu\}$ it is natural to require that in each order in ϵ the right-hand side also does not depend on $\{\omega_\mu\}$:

$$\frac{\partial E_\mu}{\partial \omega_\mu^n} = 0, \quad s = \{0, 1, \dots\} \quad (30)$$

for each ω_μ^n . In the monograph [13] it was demonstrated that the recalculation of the parameters $\{\omega_\mu\}$ in every order in ϵ speeds up the convergence of the iteration scheme, however does not change the qualitative behaviour of the energy levels of the system. For this reason, in all calculations below we will fix the parameters $\{\omega_\mu\}$ via the zeroth-order approximation:

$$\frac{\partial E_\mu^{(0)}}{\partial \omega_\mu^n} = 0. \quad (31)$$

In what follows we apply the above approach to the description of our model. In accordance with OM we choose a variational state vector, which incorporates the qualitative peculiarities of the system. From a physical point of view a field can be considered as a system of an infinite number of harmonic oscillators. Due to the interaction with a particle the equilibrium positions of these harmonic oscillators are modified. In a representation of creation and annihilation operators the shift of equilibrium positions corresponds to the displacement of a classical component u_k on these operators [29], i.e. $a_k^\dagger \rightarrow a_k^\dagger + u_k^*$ and $a_k \rightarrow a_k + u_k$, such that we choose a basis of field oscillators consisting of coherent states. As a result a so-called localized state of a particle in the field of these classical components arises. This means that during its existence the particle becomes “dressed”, i.e. somewhat smeared out while still localized. Moreover, this “dressed” state should be an eigenstate of the total momentum operator \mathbf{P} , since \mathbf{P} commutes with H . Concluding, we formulate the following conditions for the state vectors: i) representation in the basis of coherent states; ii) imposing the localization of the particle state; iii) use of variational state vectors as eigenstates of \mathbf{P} (4).

In order to incorporate the first two conditions in the state vector, we choose it as the product of the square integrable wave function of a particle, localized near an arbitrary point \mathbf{R} in space, and a coherent state of the field, analogous to the polaron problem [24, 30]:

$$|\Psi(\mathbf{r}, \mathbf{R})\rangle = \phi(\mathbf{r} - \mathbf{R}) \exp\left(\sum_k (u_k^* e^{-ik \cdot \mathbf{R}} a_k^\dagger - u_k e^{ik \cdot \mathbf{R}} a_k)\right) |0\rangle. \quad (32)$$

In the state (32) the classical component of the field u_k and the wave function $\phi(\mathbf{r} - \mathbf{R})$ can be considered as the variational parameters $\{\omega_\mu\}$ of OM. In accordance with the above described procedure of the choice of the parameters $\{\omega_\mu\}$, the functional derivative over these parameters from the functional $\langle \Psi(\mathbf{r}, \mathbf{R}) | H | \Psi(\mathbf{r}, \mathbf{R}) \rangle$ should be equal to zero. This yields an equation for the classical components of the field u_k and the wave function $\phi(\mathbf{r} - \mathbf{R})$:

$$\frac{\delta}{\delta u_k} [\langle \Psi(\mathbf{r}, \mathbf{R}) | H | \Psi(\mathbf{r}, \mathbf{R}) \rangle] = \frac{\delta}{\delta \phi(\mathbf{r} - \mathbf{R})} [\langle \Psi(\mathbf{r}, \mathbf{R}) | H | \Psi(\mathbf{r}, \mathbf{R}) \rangle] = 0. \quad (33)$$

By calculating the functional with the Hamiltonian (1) and corresponding derivatives one obtains the connection between the classical components of the field and the wave function of the particle:

$$u_k = -\frac{g}{\sqrt{2\Omega\omega_k^3}} \int d\mathbf{r} |\phi(\mathbf{r})|^2 e^{-ik \cdot \mathbf{r}}. \quad (34)$$

In the general case, the second equation in (33) leads to the integral equation for the function $\phi_P(\mathbf{r})$. However, according to reference [13], the convergence of the iteration scheme of OM does not depend on the particular choice of variational parameters, under the condition that the approximate state vector takes into account qualitative characteristics of the system. Therefore, for the analytical investigation of the energy $E_L^{(0)}(\mathbf{P}, g)$ we replace the exact numerical solution with a trial wave function, which depends on the single parameter λ and is equal to

$$\phi(\mathbf{r}) = \frac{\lambda^{\frac{3}{4}}}{\pi^{\frac{3}{4}}} e^{-\frac{\lambda^2 r^2}{2}}. \quad (35)$$

We notice that in the polaron problem the application of OM with the trial wave function (35) yields an accuracy of the order of 1% in the calculation of the bound state energy and the effective mass [30]. With this choice of wave function, we proceed to calculate the classical component of the field $u_{\mathbf{k}}$ and the Fourier transform of the wave function (35), which will be required below:

$$u_{\mathbf{k}} = -\frac{g}{\sqrt{2\Omega}} \frac{1}{\sqrt{k^3}} \int d\mathbf{r} |\phi(\mathbf{r})|^2 e^{-i\mathbf{k}\cdot\mathbf{r}} = -\frac{g}{\sqrt{2\Omega}} \frac{e^{-\frac{k^2}{4\lambda^2}}}{\sqrt{k^3}}; \quad (36)$$

$$\phi_{\mathbf{k}} = \int d\mathbf{r} \phi(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} = 2\sqrt{2} \frac{\pi^{\frac{3}{4}}}{\lambda^{\frac{3}{2}}} e^{-\frac{k^2}{2\lambda^2}} = \phi_0 e^{-\frac{k^2}{2\lambda^2}}. \quad (37)$$

Furthermore, the states (32) are not the eigenstates of the total momentum operator \mathbf{P} of the system, i.e. they are not translationary invariant. Moreover, these states are degenerate, as they do not depend on the localization point \mathbf{R} of the particle in space. The choice of the correct linear combination of these states allows one to build a set of states which are not degenerate and are eigenstates of the total momentum operator \mathbf{P} :

$$|\Psi_{P_1, n_{\mathbf{k}}}^{(0)}\rangle = \frac{1}{N_{P_1, n_{\mathbf{k}}} \sqrt{\Omega}} \int d\mathbf{R} \phi_{P_1}(\mathbf{r} - \mathbf{R}) \exp\{i(\mathbf{P}_1 - \mathbf{k}n_{\mathbf{k}}) \cdot \mathbf{R}\} \exp\left\{\sum_{\mathbf{q}} (u_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{R}} \mathbf{a}_{\mathbf{q}}^\dagger - u_{\mathbf{q}}^* e^{i\mathbf{q}\cdot\mathbf{R}} \mathbf{a}_{\mathbf{q}})\right\} |n_{\mathbf{k}}\rangle, \quad (38)$$

$$\mathbf{P} |\Psi_{P_1, n_{\mathbf{k}}}^{(0)}\rangle = \mathbf{P}_1 |\Psi_{P_1, n_{\mathbf{k}}}^{(0)}\rangle.$$

Here Ω is the normalization volume and \mathbf{P}_1 the total momentum of the system, $|n_{\mathbf{k}}\rangle$ are Fock field states with occupation number $n_{\mathbf{k}}$, $\phi_{P_1}(\mathbf{r} - \mathbf{R})$ is the wave function of the particle localized at point $\mathbf{r} = \mathbf{R}$ and the classical component of the field $u_{\mathbf{k}}$ is defined via equation (34). The normalization constant for the state $|\Psi_{P_1, 1_{\mathbf{k}}}^{(0)}\rangle$ is defined as

$$|N_{P_1, 1_{\mathbf{k}}}|^2 = \int d\mathbf{R}_1 d\boldsymbol{\rho} \phi_{P_1}^*(\boldsymbol{\rho}) \phi_{P_1}(\boldsymbol{\rho} - \mathbf{R}_1) e^{i(\mathbf{P}_1 - \mathbf{k}) \cdot \mathbf{R}_1 + \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 (e^{-i\mathbf{k}\cdot\mathbf{R}_1} - 1)} (2|u_{\mathbf{k}}|^2 (\cos \mathbf{k} \cdot \mathbf{R}_1 - 1) + 1). \quad (39)$$

In addition the set of states (38) forms a complete and an orthonormal basis in Hilbert space. The completeness of these states follows from the fact that they are eigenstates of a Hermitian operator \mathbf{P} , with an explicit proof given in Appendix A. Concluding, the set of states (38) takes into account physical peculiarities of the system, forms the complete set of states in Hilbert space for arbitrary functions $\phi_P(\mathbf{r})$ and $u_{\mathbf{k}}$, which are an analog to the parameters $\{\omega_{\mu}\}$, and, therefore, can be usable in the iteration scheme (20-22).

The zeroth-order approximation for the ground state vector following above procedure then reads as

$$|\Psi_P^{(L)}\rangle = \frac{1}{N_P \sqrt{\Omega}} \int d\mathbf{R} \phi_P(\mathbf{r} - \mathbf{R}) \exp\left(i\mathbf{P} \cdot \mathbf{R} + \sum_{\mathbf{k}} (u_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{R}} \mathbf{a}_{\mathbf{k}}^\dagger - u_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} \mathbf{a}_{\mathbf{k}})\right) |0\rangle, \quad (40)$$

whereas equations (24), (25) look like

$$E^{(2)} = \frac{E_L^{(0)} + \sum_{P_1, \{n_{\mathbf{k}} \neq 0\}} C_{P_1, \{n_{\mathbf{k}}\}}^{(1)} \langle \Psi_P^{(L)} | \mathbf{H} | \Psi_{P_1, \{n_{\mathbf{k}}\}} \rangle}{1 + \sum_{P_1, \{n_{\mathbf{k}} \neq 0\}} C_{P_1, \{n_{\mathbf{k}}\}}^{(1)} \langle \Psi_P^{(L)} | \Psi_{P_1, \{n_{\mathbf{k}}\}} \rangle}; \quad (41)$$

$$C_{P_1, \{n_{\mathbf{k}}\}}^{(1)} = \frac{E_L^{(0)} \langle \Psi_{P_1, \{n_{\mathbf{k}}\}} | \Psi_P^{(L)} \rangle - \langle \Psi_{P_1, \{n_{\mathbf{k}}\}} | \mathbf{H} | \Psi_P^{(L)} \rangle}{H_{P_1, \{n_{\mathbf{k}}\}; P_1, \{n_{\mathbf{k}}\}} - E_L^{(0)}}; \quad (42)$$

$$H_{P_1, \{n_{\mathbf{k}}\}; P_2, \{n_{1\mathbf{k}}\}} = \langle \Psi_{P_1, \{n_{\mathbf{k}}\}} | \mathbf{H} | \Psi_{P_2, \{n_{1\mathbf{k}}\}} \rangle, \quad E_L^{(0)} = \langle \Psi_P^{(L)} | \mathbf{H} | \Psi_P^{(L)} \rangle.$$

We want to emphasize once more that all matrix elements are calculated with the full Hamiltonian of the system

$$H = \frac{1}{2} \left(P^2 - 2 \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \mathbf{k} \cdot \mathbf{P} + \left(\sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \mathbf{k} \right)^2 \right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{g}{\sqrt{\Omega}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(e^{i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}}^\dagger \right). \quad (43)$$

Let us calculate the ground state energy $E_L^{(0)}$ of the system in this basis. The details of the calculations can be found in appendix C. The ground state energy reads accordingly

$$E_L^{(0)}(\mathbf{P}, g) = \frac{P^2}{2} - \mathbf{P} \cdot \mathbf{Q} + G + E_f(\mathbf{P}) + E_{\text{int}}(\mathbf{P}), \quad (44)$$

with

$$\begin{aligned} \mathbf{Q} &= \frac{1}{|N_P|^2} \sum_{\mathbf{k}} \mathbf{k} |u_{\mathbf{k}}|^2 \int d\mathbf{R} d\mathbf{r} \phi_P^*(\mathbf{r}) \phi_P(\mathbf{r} - \mathbf{R}) e^{\Phi(\mathbf{R}) + i(\mathbf{P} - \mathbf{k}) \cdot \mathbf{R}}, \\ G &= \frac{1}{2} \frac{1}{|N_P|^2} \sum_{\mathbf{m}, \mathbf{l}} \mathbf{m} \cdot \mathbf{l} |u_{\mathbf{m}}|^2 |u_{\mathbf{l}}|^2 \int d\mathbf{r} d\mathbf{R} \phi^*(\mathbf{r}) \phi(\mathbf{r} - \mathbf{R}) e^{i\mathbf{P} \cdot \mathbf{R} + \Phi(\mathbf{R}) - i(\mathbf{m} + \mathbf{l}) \cdot \mathbf{R}}, \\ E_f(\mathbf{P}) &= \frac{1}{|N_P|^2} \sum_{\mathbf{k}} \left(k + \frac{k^2}{2} \right) |u_{\mathbf{k}}|^2 \int d\mathbf{R} d\mathbf{r} \phi_P^*(\mathbf{r}) \phi_P(\mathbf{r} - \mathbf{R}) e^{\Phi(\mathbf{R}) + i(\mathbf{P} - \mathbf{k}) \cdot \mathbf{R}}, \\ E_{\text{int}}(\mathbf{P}) &= \frac{g}{|N_P|^2} \sum_{\mathbf{k}} \frac{u_{\mathbf{k}}}{\sqrt{2k\Omega}} \int d\mathbf{R} d\mathbf{r} \left(\phi_P^*(\mathbf{r} + \mathbf{R}) \phi_P(\mathbf{r}) + \phi_P^*(\mathbf{r}) \phi_P(\mathbf{r} - \mathbf{R}) \right) e^{\Phi(\mathbf{R}) + i(\mathbf{P} \cdot \mathbf{R} + \mathbf{k} \cdot \mathbf{r})}, \\ \Phi(\mathbf{R}) &= \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 (e^{-i\mathbf{k} \cdot \mathbf{R}} - 1); \\ |N_P|^2 &= \int d\mathbf{R} d\mathbf{r} \phi_P^*(\mathbf{r}) \phi_P(\mathbf{r} - \mathbf{R}) e^{\Phi(\mathbf{R}) + i\mathbf{P} \cdot \mathbf{R}}. \end{aligned}$$

Actually, the iteration scheme (20), (21), (22) can be used for arbitrary coupling constants [13]. However, as was described above, in the framework of our model we are interested in the behavior of the ground state energy $E_L^{(0)}$ in the weak coupling limit. In this limit we can neglect the function

$$\Phi(\mathbf{R}) = \sum_{\mathbf{m}} |u_{\mathbf{m}}|^2 (e^{-i\mathbf{m} \cdot \mathbf{R}} - 1) \sim g^2,$$

in the exponent of all integrals in equation (44) as $g \ll 1$.

First of all, we investigate the situation of a particle at rest, i.e. $\mathbf{P} = 0$. In this case for the weak coupling limit the integrals in equation (44) can be expressed through the Fourier transforms of the wave function $\phi(\mathbf{r})$:

$$\int d\mathbf{R}_1 d\mathbf{r} \phi^*(\mathbf{r}) \phi(\mathbf{r} - \mathbf{R}_1) e^{-i\mathbf{k} \cdot \mathbf{R}_1} = \int d\mathbf{r} \phi^*(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} \int d\mathbf{R} \phi(\mathbf{R}) e^{i\mathbf{k} \cdot \mathbf{R}} = \phi_{\mathbf{k}}^* \phi_{-\mathbf{k}} = \phi_{\mathbf{k}}^2, \quad (45)$$

$$\int d\mathbf{R}_1 d\mathbf{r} \phi^*(\mathbf{r}) \phi(\mathbf{r} - \mathbf{R}_1) = |\phi_0|^2 = \phi_0^2, \quad (46)$$

$$\int d\mathbf{R}_1 d\mathbf{r} \phi^*(\mathbf{r}) \phi(\mathbf{r} - \mathbf{R}_1) e^{-i\mathbf{k} \cdot \mathbf{r}} = \phi_{\mathbf{k}}^* \phi_0 = \phi_{\mathbf{k}} \phi_0. \quad (47)$$

With the use of equations (44), (45-47) we can rewrite the energy of the ground state in a form

$$E_L^{(0)}(0, g) = \frac{1}{2} \sum_{m, l} \mathbf{m} \cdot \mathbf{l} |u_m|^2 |u_l|^2 \frac{\phi_{l+m}^2}{\phi_0^2} + \sum_k \left(k + \frac{k^2}{2} \right) |u_k|^2 \frac{\phi_k^2}{\phi_0^2} + \frac{2g}{\sqrt{2\Omega}} \sum_k \frac{u_k}{\sqrt{k}} \frac{\phi_k}{\phi_0}, \quad (48)$$

which up to fourth order in g yields

$$E_L^{(0)}(0, g) = \frac{g^2}{24\pi^2} \left(\lambda(-4 + \sqrt{2}) \sqrt{3\pi} + \lambda^2 \right) + O(g^4). \quad (49)$$

By minimizing the energy with respect to λ one finds

$$E_L^{(0)}(0, g) = -g^2 \frac{(-4 + \sqrt{2})^2}{32\pi}; \quad \lambda = \frac{\sqrt{3\pi}}{2} (4 - \sqrt{2}). \quad (50)$$

In the weak-coupling limit it is also possible to obtain a renormalization for the mass of the particle. This is accomplished by expanding the energy (44) in a series over \mathbf{P} up to second order. The details of the calculation can be found in appendix D. The result reads

$$E_L^{(0)}(\mathbf{P}, g) \approx E_L^{(0)}(0, g) + \frac{P^2}{2} \left[1 - \frac{g^2}{9\pi^2} \frac{17 - \sqrt{2}}{21} \right]; \quad m^{(0)*} = 1 + \frac{g^2}{9\pi^2} \frac{17 - \sqrt{2}}{21}. \quad (51)$$

From this equation we can conclude that the factor, which determines the corrected mass is half the one via the leading second-order term from perturbation theory, see e.g. equation (14).

IV. SECOND ORDER ITERATION FOR THE ENERGY AND CONVERGENCE

In the previous section we have found the energy of the ground state and the renormalized mass, which are proportional to the square of the coupling constant in zeroth-order approximation. However, the correction to the energy coming from single-phonon intermediate transitions is of the same order with respect to the coupling constant. Consequently, its contribution should also be taken into account, thus requiring the calculation of the energy of the system in the second iteration (41).

In order to calculate the second order iteration for the energy we notice (appendix E) that the matrix elements $\langle \Psi_{\mathbf{P}_1, \{n_k\}} | \Psi_{\mathbf{P}}^{(L)} \rangle$ and $\langle \Psi_{\mathbf{P}_1, \{n_k\}} | \mathbf{H} | \Psi_{\mathbf{P}}^{(L)} \rangle$, which are found in equations (41), (42) are proportional to the delta function of the total momentum of the system $\delta(\mathbf{P}_1 - \mathbf{P})$. Therefore, during the evaluation of the sum over \mathbf{P}_1 in equation (41) for the energy we have used the usual procedure [8]: one of the delta functions in its square was replaced through the normalization volume Ω , and the integration over the remaining one yields $\mathbf{P} = \mathbf{P}_1$, thus expressing the conservation of momentum.

Firstly, we consider the case, when a particle is at rest, i. e. $\mathbf{P} = 0$. The results, which are expressed through the Fourier components of the particle wave function in the weak coupling limit read:

$$E^{(2)}(0, g) = \frac{A}{B}, \quad (52)$$

where

$$\begin{aligned}
A = E_L^{(0)} + \sum_{\mathbf{k}} \frac{1}{\phi_{\mathbf{k}}^2 \phi_0^2} & \left[-u_{\mathbf{k}} \phi_{\mathbf{k}}^2 \left(\frac{k^2}{2} + k \right) - \frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{k}} \phi_0}{\sqrt{k}} - u_{\mathbf{k}} \phi_{\mathbf{k}}^2 (g^2 I_{\mathbf{k}} + g^4 J_{\mathbf{k}} - E_L^{(0)}) \right] \\
& \times \left[u_{\mathbf{k}} \phi_{\mathbf{k}}^2 \left(\frac{k^2}{2} + k \right) + \frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{k}} \phi_0}{\sqrt{k}} + u_{\mathbf{k}} \phi_{\mathbf{k}}^2 (g^2 I_{\mathbf{k}} + g^4 J_{\mathbf{k}}) - E_L^{(0)} u_{\mathbf{k}} \phi_0^2 \right] \\
& \times \left[\left(\frac{k^2}{2} + k \right) + g^2 I_{\mathbf{k}} + g^4 J_{\mathbf{k}} - E_L^{(0)} \right]^{(-1)}, \quad (53)
\end{aligned}$$

and

$$B = 1 + \sum_{\mathbf{k}} \frac{1}{\phi_{\mathbf{k}}^2 \phi_0^2} \frac{\left[-u_{\mathbf{k}} \phi_{\mathbf{k}}^2 \left(\frac{k^2}{2} + k \right) - \frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{k}} \phi_0}{\sqrt{k}} - u_{\mathbf{k}} \phi_{\mathbf{k}}^2 (g^2 I_{\mathbf{k}} + g^4 J_{\mathbf{k}} - E_L^{(0)}) \right] u_{\mathbf{k}} (\phi_{\mathbf{k}}^2 - \phi_0^2)}{\left(\frac{k^2}{2} + k \right) + g^2 I_{\mathbf{k}} + g^4 J_{\mathbf{k}} - E_L^{(0)}}. \quad (54)$$

In equations (53- 54) we have introduced the following notations

$$\sum_{\mathbf{m}} m |u_{\mathbf{m}}|^2 \phi_{\mathbf{m}+\mathbf{k}}^2 \equiv g^2 \phi_{\mathbf{k}}^2 I_{\mathbf{k}}^{(1)}; \quad I_{\mathbf{k}}^{(1)} = \frac{\mathbf{k}}{k^2} \frac{\lambda^2}{32\pi^2} \frac{4k - e^{\frac{2}{3} \frac{k^2}{\lambda^2}} \sqrt{6\pi\lambda} \text{Erf} \frac{\sqrt{\frac{3}{2}} k}{\lambda}}{k}; \quad (55)$$

$$\sum_{\mathbf{m}} \left(\frac{m^2}{2} + m \right) |u_{\mathbf{m}}|^2 \phi_{\mathbf{m}+\mathbf{k}}^2 \equiv g^2 \phi_{\mathbf{k}}^2 I_{\mathbf{k}}^{(2)}; \quad I_{\mathbf{k}}^{(2)} = \frac{\lambda^2}{96\pi^2} \frac{\sqrt{6\pi\lambda} e^{\frac{2}{3} \frac{k^2}{\lambda^2}} \text{Erf} \frac{\sqrt{\frac{3}{2}} k}{\lambda} + 6\pi \text{Erfi} \frac{\sqrt{\frac{3}{2}} k}{\lambda}}{k}; \quad (56)$$

$$\frac{g\phi_{\mathbf{k}}}{\sqrt{2\Omega}} \sum_{\mathbf{m}} \frac{u_{\mathbf{m}}}{\sqrt{m}} (\phi_{\mathbf{m}+\mathbf{k}} + \phi_{\mathbf{m}-\mathbf{k}}) \equiv g^2 \phi_{\mathbf{k}}^2 I_{\mathbf{k}}^{(3)}; \quad I_{\mathbf{k}}^{(3)} = -\frac{\lambda^2}{4\pi} \frac{\text{Erfi} \frac{k}{\sqrt{3}\lambda}}{k}; \quad (57)$$

$$I_{\mathbf{k}} = \mathbf{k} \cdot \mathbf{I}_{\mathbf{k}}^{(1)} + I_{\mathbf{k}}^{(2)} + I_{\mathbf{k}}^{(3)}; \quad (58)$$

$$\frac{1}{2} \sum_{\mathbf{l}, \mathbf{m}} \mathbf{l} \cdot \mathbf{m} |u_{\mathbf{l}}|^2 |u_{\mathbf{m}}|^2 \phi_{\mathbf{l}+\mathbf{m}+\mathbf{k}}^2 \equiv g^4 \phi_{\mathbf{k}}^2 J_{\mathbf{k}}; \quad J_{\mathbf{k}} \approx \frac{5^{1/2} \lambda^2}{4(2\pi)^3 3^5} e^{\frac{4}{3} \frac{k^2}{\lambda^2}} \frac{\frac{2}{15} \frac{k^2}{\lambda^2} - 1}{(1 + \frac{4}{45} \frac{k^2}{\lambda^2})^3}, \quad (59)$$

where $\text{Erf}(x) = 2/\sqrt{\pi} \int_0^x e^{-z^2} dz$ and $\text{Erfi}(x) = -i\text{Erf}(ix)$ are the error function and the imaginary error functions, respectively. When we calculated the energy of the ground state, we dropped all terms with power in g larger than g^2 . Consequently, we can neglect the term $g^4 J_{\mathbf{k}}$ in comparison with $g^2 I_{\mathbf{k}}$, which can be confirmed by the direct numerical calculation of the integral.

Prior to the numerical evaluation of the integrals (53) and (54), let us understand their structure through the approximate analytical calculation. We investigate the behavior of the numerator and denominator of the quantities A and B . We start from breaking the integration region into two parts, namely $[0, k_0]$ and $[k_0, \infty)$. The value k_0 will be fixed below. Let us work out the behavior of the quantity $I_{\mathbf{k}}$ for small and large values of k . First of all we notice that $g^2 I_0$ gives exactly the ground state energy $E_L^{(0)}$. For small values of k , with the increase of k the value of $g^2 I_{\mathbf{k}} \sim -g^2 k^2/(18\pi^2) + E_L^{(0)}$, i.e, it grows quadratically in absolute value, while being negative. Therefore, due to the presence of g^2 , this term is small in comparison with $k^2/2 + k$ for small values of k , so that, in the denominator of quantity A , the leading term is $k^2/2 + k$. For large values of k , the quantity $g^2 I_{\mathbf{k}}$ exponentially grows as $I_{\mathbf{k}} \sim e^{\frac{2}{3} \frac{k^2}{\lambda^2}}/k$ and becomes the leading contribution in comparison with $k^2/2 + k$, despite the higher power of g .

In analogy, we can analyze the numerator of the quantity A . For small values of k we can neglect in every square bracket in equation (53) the large powers of g , i.e. terms with exponents larger than 1. Consequently, for small values of k , the integrand within A looks like

$$-\frac{\left[u_k \phi_k^2 \left(\frac{k^2}{2} + k\right) + \frac{g}{\sqrt{2\Omega}} \frac{\phi_k \phi_0}{\sqrt{k}}\right]^2}{\phi_k^2 \phi_0^2 \left(\frac{k^2}{2} + k\right)}. \quad (60)$$

For large values of k , the numerator is exponentially decreasing, with the leading term being $(-\frac{g}{\sqrt{2\Omega}} \frac{\phi_k \phi_0}{\sqrt{k}})(-E_L^{(0)} u_k)$. This follows from the fact that $u_k \sim e^{-\frac{k^2}{4\lambda^2}}$ and $\phi_k \sim e^{-\frac{k^2}{2\lambda^2}}$. Consequently, the integrand for large values of k can be presented as

$$\frac{(-\frac{g}{\sqrt{2\Omega}} \frac{\phi_k \phi_0}{\sqrt{k}})(-E_L^{(0)} u_k)}{g^2 \phi_k^2 I_k}. \quad (61)$$

Combining all together, we find that the quantity A can be approximately calculated as

$$A \approx E_L^{(0)} + \sum_{k < k_0} \frac{-\left(u_k \frac{\phi_k}{\phi_0} \left(\frac{k^2}{2} + k\right) + \frac{g}{\sqrt{2\Omega}} \frac{1}{\sqrt{k}}\right)^2}{\left(\frac{k^2}{2} + k\right)} + \sum_{k > k_0} \frac{(-\frac{g}{\sqrt{2\Omega}} \frac{\phi_k \phi_0}{\sqrt{k}})(-E_L^{(0)} u_k)}{\phi_k^2 g^2 I_k}. \quad (62)$$

In this expression, both sums are well defined and remain finite. The sum over the region $k > k_0$ is finite and convergent, while the ratio of numerator and denominator in the integrand is exponentially decreasing as $e^{-\frac{5k^2}{12\lambda^2}}$.

In equation (62) the point k_0 is determined as a solution of the equation

$$\frac{k^2}{2} + k + g^2 I_k - E_L^{(0)} = 0, \quad (63)$$

or employing the asymptotic behavior for the function I_k (appendix F)

$$\frac{k_0^2}{2} + k_0 = g^2 \frac{\lambda^3 \sqrt{6\pi}}{48\pi^2} \frac{e^{\frac{2}{3} \frac{k_0^2}{\lambda^2}}}{k_0}, \quad (64)$$

and by finding the following logarithm

$$\ln \frac{(\frac{k_0^2}{2} + k_0)k_0}{a} = -2|\ln g| + \frac{2}{3} \frac{k_0^2}{\lambda^2}, \quad (65)$$

with

$$a = \frac{\lambda^3 \sqrt{6\pi}}{48\pi^2}.$$

In the limit of extremely small g , we can build the solution of equation (65) via iterations, thus yielding

$$k_0 \sim \lambda \sqrt{3|\ln g|}. \quad (66)$$

The estimation of quantity B can be performed in a similar fashion and one finds

$$B \approx 1 + \sum_{k < k_0} \frac{-\left(u_k \frac{\phi_k^2}{\phi_0^2} \left(\frac{k^2}{2} + k\right) + \frac{g}{\sqrt{2\Omega}} \frac{\phi_k}{\phi_0 \sqrt{k}}\right) u_k (\phi_k^2 - \phi_0^2)}{\phi_k^2 \left(\frac{k^2}{2} + k\right)} + \sum_{k > k_0} \frac{(-\frac{g}{\sqrt{2\Omega}} \frac{\phi_k \phi_0}{\sqrt{k}}) u_k (\frac{\phi_k^2}{\phi_0^2} - 1)}{\phi_k^2 g^2 I_k}. \quad (67)$$

At first sight, it may appear that the quantity B features an infrared divergence, because the term $\frac{g}{\sqrt{2\Omega}} \frac{\phi_k u_k}{\phi_0 \sqrt{k}} / (\phi_k^2 (\frac{k^2}{2} + k)) \sim 1/k^3$ as $k \rightarrow 0$. However, this additional power of k in the denominator is cancelled through the difference $\phi_k^2 - \phi_0^2 \sim k^2/\lambda^2$. The convergence at infinity is manifested with the exponential decrease of the integrand $\sim e^{-\frac{5k^2}{12\lambda^2}}$.

By plugging into equations (62) and (67) the values of ϕ_k and u_k , which are defined in equations (36-37) and calculating the integrals (appendix F) we find the approximate analytical formula for the second iteration for the ground state energy

$$A \approx E_L^{(0)} - \left[\frac{g^2 \lambda}{24\pi^2} \left(\sqrt{6\pi} \text{Erf} \left(\frac{\sqrt{\frac{3}{2}} k_0}{\lambda} \right) + \lambda - \lambda e^{-\frac{3k_0^2}{2\lambda^2}} \right) - \frac{g^2 \lambda}{2\sqrt{3}\pi^{3/2}} \text{Erf} \left(\frac{\sqrt{3} k_0}{2\lambda} \right) \right] - \frac{g^2}{2\pi^2} \ln \left(\frac{k_0}{2} + 1 \right) + E_L^{(0)} \frac{12\sqrt{6\pi}}{5\lambda\pi} e^{-\frac{5k_0^2}{12\lambda^2}}, \quad (68)$$

$$B \approx 1 + \frac{g^2}{12\pi^2} (1 - e^{-\frac{3}{2} \frac{k_0^2}{\lambda^2}}) - g^2 f \left(\frac{k_0}{\lambda} \right) - \frac{144\sqrt{6\pi}}{25\lambda\pi} \left(1 + \frac{5}{12} \frac{k_0^2}{\lambda^2} \right) e^{-\frac{5k_0^2}{12\lambda^2}}, \quad (69)$$

$$f(x) = \frac{1}{4\pi^2} \int_0^x \frac{tdt}{1+t/2} e^{-\frac{3}{4}t^2}.$$

Within the accuracy of the approximate formulas, we can set $B \approx 1$. Therefore, one finally obtains

$$E^{(2)}(0, g) \approx A. \quad (70)$$

The use of our simple analytical expressions allows to establish the behavior of the energy as a function of the coupling constant and consequently to determine the character of the singularity. In order to select the singularity, we investigate the limit

$$\lim_{g \rightarrow 0} E^{(2)}(0, g).$$

In this limit, the value of k_0 logarithmically grows. Consequently, we can approximately set $k_0 \rightarrow \infty$ both in the expression in square brackets and in the last term of equation (68). This way, the square bracket becomes equal to the energy of the ground state (appendix F) and cancels $E_L^{(0)}$. The last term also does not contribute to the energy as being exponentially small. Consequently, the only term remains, which exactly determines the character of the singularity and is equal to

$$E^{(2)}(0, g) \xrightarrow{g \rightarrow 0} -\frac{g^2}{2\pi^2} \ln \left(\frac{k_0}{2} + 1 \right); \quad (71)$$

$$k_0 \approx \lambda \sqrt{3|\ln g|}. \quad (72)$$

We observe that this term exactly coincides with the result via perturbation theory, i.e. equation (13), however, here with a well specified “cut-off”. Moreover, the most contributions to the integral in the energy arise from the region $k < k_0$ and this is exactly the reason for the natural “cut-off”, which is determined self consistently and is directly related to the only parameter of the Hamiltonian, namely the coupling constant. Let us mention here that the corrections to the energy of the system (71) arise in the subsequent iteration and are related to the transitions into intermediate states with two phonons. These contributions are proportional to g^4 .

In addition we note here that the absence of the ultraviolet divergence in the energy of the ground state in equations (52, 53, 54) is due to the fact that as in the zeroth-order approximation and in the second-order iteration in the resolvent $[E_\mu^{(0)} - H_{\nu\nu}]^{-1}$ of equation (25) the dressed

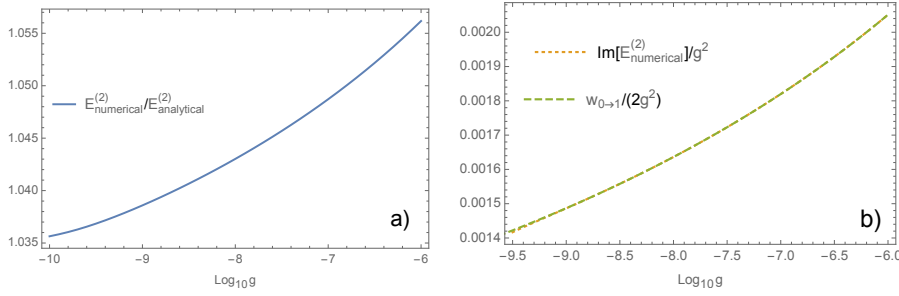


Figure 1. (Color online) (a) The dependence on the coupling constant of the ratio of the exact numerical evaluation to the approximate analytical formula of the second iteration for the energy. The value k_0 in the analytical approximation is equal to $k_0 = \lambda \sqrt{3} |\ln g|$. (b) The dependence of the imaginary part of the energy of the system on the coupling constant. The imaginary part corresponds to the finite lifetime of the state.

wave functions (38) were used. This leads to the effective momentum cut-off $k_0(g)$, which is determined as the solution of equation (63). This cut-off is a function of the coupling constant and is not a phenomenological parameter, which needs to be introduced for the removal of the ultraviolet divergence. Moreover, as follows from equation (71), the energy of the ground state has a logarithmic singularity as $g \rightarrow 0$. It is clearly seen that this dependence can not be sorted out in the framework of perturbation theory, which yields a power series over the coupling constant g .

In order to ensure that our interpretation is correct, we have evaluated the integrals numerically and have found in the limit of extremely small g the ratio of the results via exact numerical versus analytical evaluations. This ratio is almost constant and is approximately equal to one, as presented in Figure 1. Therefore, we can conclude that the main reason why conventional perturbation theory fails is related to the fact that the energy of the system is a non-analytical function of the coupling constant and consequently can not be expanded in a series over g near a singular point.

The second interesting consequence of the numerical evaluation of the integral is related to the fact that the energy of the system contains a small imaginary part, which means that the state has a finite lifetime and is quasi-stationary. To prove this, we have calculated the transition probability to the state $|\Psi_{P_1, 1k}\rangle$ for the case when a particle is at rest, i.e.

$$\begin{aligned} \frac{w_{0 \rightarrow 1}}{2} &= \pi \int |\langle \Psi_{P_1, \{n_k\}} | H | \Psi_P^{(L)} \rangle|^2 \delta(H_{P_1, 1k; P_1, 1k} - E_L^{(0)}) \frac{\Omega dk}{(2\pi)^3} \\ &= \frac{\Omega}{2\pi} \frac{k^2}{|k + 1 + g^2 I_k'|} \frac{\left[u_k \phi_k^2 \left(\frac{k^2}{2} + k \right) + \frac{g}{\sqrt{2}\Omega} \frac{\phi_k \phi_0}{\sqrt{k}} + u_k \phi_k^2 g^2 I_k - E_L^{(0)} u_k \phi_0^2 \right]^2}{\phi_0^2 \phi_k^2} \bigg|_{\frac{k^2}{2} + k + g^2 I_k - E_L^{(0)} = 0}. \end{aligned} \quad (73)$$

The result of evaluation is presented in Figure 1. As can be seen from the figure the two curves coincide exactly. This can be interpreted via the diagram technique [8]. The second order iteration for the energy of the particle can be presented via the diagram depicted in Figure 2. If the diagram is split by the dashed line, the imaginary part will correspond to the transition probability to the state $|\Psi_{P_1, 1k}\rangle$.

Here, we need to stress that contrary to standard perturbation theory in our formulation the conservation of energy is governed not by the free Hamiltonian H_0 , but through the expectation value of the total Hamiltonian $H_{P_1, 1k; P_1, 1k}$. Therefore, for certain values of k and for certain

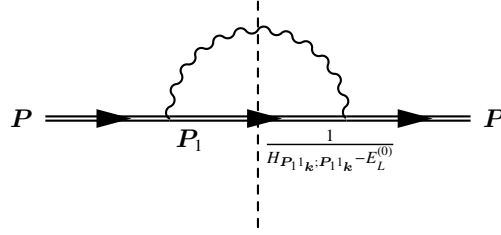


Figure 2. (Color online) Feynman diagram of the process. If the diagram is split by the dashed line, the imaginary part will correspond to the transition probability to the state $|\Psi_{P_1, 1_k}\rangle$. However, in the intermediate states the resolvent of the full Hamiltonian instead of a free one is used.

coupling constants g this energy level might appear to be below the energy $E_L^{(0)}$, featuring a so called quasi-intersection of energy levels. If the transition probability to the state $|\Psi_{P_1, 1_k}\rangle$ were large, the description of the system with the state vectors (38) would not be applicable and the reconstruction of the states would need to be performed, which takes into account the degeneracy between the energies $E_L^{(0)}$ and $H_{P_1 1_k; P_1 1_k}$. In our case, however, the transition probability is small and consequently the description with a complex energy, with small imaginary part, is valid, in analogy to the theory of a natural line width of the atomic states or anharmonic oscillator $p^2/2 + x^2/2 - \mu x^4$, with $\mu > 0$.

In order to conclude our formulation we have calculated the renormalized mass in the second iteration. In terms of the introduced abbreviations the second order iteration for the particle energy can be written as

$$E^{(2)}(\mathbf{P}, g) = \frac{\frac{P^2}{2} + \tilde{E}_L^{(0)}(\mathbf{P}, g) + A_P}{B_P}, \quad (74)$$

with

$$\tilde{E}_L^{(0)}(\mathbf{P}, g) = -\mathbf{P} \cdot \sum_{\mathbf{m}} \mathbf{m} |u_{\mathbf{m}}|^2 \frac{\phi_{\mathbf{P}-\mathbf{m}}^2}{\phi_{\mathbf{P}}^2} + \sum_{\mathbf{m}} \left(\frac{m^2}{2} + m \right) |u_{\mathbf{m}}|^2 \frac{\phi_{\mathbf{P}-\mathbf{m}}^2}{\phi_{\mathbf{P}}^2} + \frac{2g}{\sqrt{2\Omega}} \sum_{\mathbf{m}} \frac{u_{\mathbf{m}}}{\sqrt{m}} \frac{\phi_{\mathbf{P}-\mathbf{m}}}{\phi_{\mathbf{P}}}. \quad (75)$$

The quantity A_P reads as

$$\begin{aligned} A_P = \sum_{\mathbf{k}} \left[\frac{P^2}{2} u_{\mathbf{k}} (\phi_{\mathbf{P}-\mathbf{k}}^2 - \phi_{\mathbf{P}}^2) + \left(\frac{k^2}{2} + \mathbf{k} - \mathbf{P} \cdot \mathbf{k} \right) u_{\mathbf{k}} \phi_{\mathbf{P}-\mathbf{k}}^2 + \frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{P}-\mathbf{k}} \phi_{\mathbf{P}}}{\sqrt{k}} + g^2 u_{\mathbf{k}} \phi_{\mathbf{P}-\mathbf{k}}^2 I_{\mathbf{P}-\mathbf{k}} - \tilde{E}_L^{(0)}(\mathbf{P}, g) u_{\mathbf{k}} \phi_{\mathbf{P}}^2 \right] \\ \times \left[\tilde{E}_L^{(0)}(\mathbf{P}, g) u_{\mathbf{k}} \phi_{\mathbf{P}-\mathbf{k}}^2 - \left(\left(\frac{k^2}{2} + \mathbf{k} - \mathbf{P} \cdot \mathbf{k} \right) u_{\mathbf{k}} \phi_{\mathbf{P}-\mathbf{k}}^2 + \frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{P}-\mathbf{k}} \phi_{\mathbf{P}}}{\sqrt{k}} + g^2 u_{\mathbf{k}} \phi_{\mathbf{P}-\mathbf{k}}^2 I_{\mathbf{P}-\mathbf{k}} \right) \right] \frac{1}{\phi_{\mathbf{P}-\mathbf{k}}^2 \phi_{\mathbf{P}}^2} \\ \times \left[\left(\frac{k^2}{2} + \mathbf{k} - \mathbf{P} \cdot \mathbf{k} \right) + g^2 I_{\mathbf{P}-\mathbf{k}} - \tilde{E}_L^{(0)}(\mathbf{P}, g) \right]^{-1}, \end{aligned} \quad (76)$$

and the quantity B_P is equal to

$$\begin{aligned} B_P = 1 + \sum_{\mathbf{k}} \left[\tilde{E}_L^{(0)}(\mathbf{P}, g) u_{\mathbf{k}} \phi_{\mathbf{P}-\mathbf{k}}^2 - \left(\left(\frac{k^2}{2} + \mathbf{k} - \mathbf{P} \cdot \mathbf{k} \right) u_{\mathbf{k}} \phi_{\mathbf{P}-\mathbf{k}}^2 + \frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{P}-\mathbf{k}} \phi_{\mathbf{P}}}{\sqrt{k}} + g^2 u_{\mathbf{k}} \phi_{\mathbf{P}-\mathbf{k}}^2 I_{\mathbf{P}-\mathbf{k}} \right) \right] \frac{u_{\mathbf{k}} (\phi_{\mathbf{P}-\mathbf{k}}^2 - \phi_{\mathbf{P}}^2)}{\phi_{\mathbf{P}-\mathbf{k}}^2 \phi_{\mathbf{P}}^2} \\ \times \left[\left(\frac{k^2}{2} + \mathbf{k} - \mathbf{P} \cdot \mathbf{k} \right) + g^2 I_{\mathbf{P}-\mathbf{k}} - \tilde{E}_L^{(0)}(\mathbf{P}, g) \right]^{-1}. \end{aligned} \quad (77)$$

To proceed, we again break the limit of the integration into two parts, however, now we know that the main contribution to the energy of the system comes from the region $[0, k_0]$. In this region, we again drop all terms, with power of g larger than one. We recall here that the classical component of the field $u_{\mathbf{k}}$ is proportional to g and the energy $E_L^{(0)}(\mathbf{P}, g) \sim g^2$. In addition, the limit $\mathbf{P} \ll 1$ is considered. Moreover, as the quantity $B_{\mathbf{P}}$, after expansion over momentum \mathbf{P} , will have a form $B_{\mathbf{P}} = 1 - g^2 F(\mathbf{P}^2)$, we can set $\mathbf{P} = 0$ in $B_{\mathbf{P}}$, in order to preserve the same accuracy.

In this approximation, the quantity $A_{\mathbf{P}}$ takes the form

$$A_{\mathbf{P}} = -\frac{P^2}{2} \sum_{\mathbf{k} < k_0} \left\{ u_{\mathbf{k}} \frac{(\phi_{\mathbf{P}-\mathbf{k}}^2 - \phi_{\mathbf{P}}^2)}{\phi_{\mathbf{P}-\mathbf{k}} \phi_{\mathbf{P}}} \left[u_{\mathbf{k}} \frac{\phi_{\mathbf{P}-\mathbf{k}}}{\phi_{\mathbf{P}}} + \frac{g}{\sqrt{2\Omega}} \frac{1}{\sqrt{k}} \left(\frac{k^2}{2} + k - \mathbf{P} \cdot \mathbf{k} \right)^{-1} \right] \right\} \\ - \sum_{\mathbf{k} < k_0} \left[\left(\frac{k^2}{2} + k - \mathbf{P} \cdot \mathbf{k} \right) u_{\mathbf{k}} \frac{\phi_{\mathbf{P}-\mathbf{k}}}{\phi_{\mathbf{P}}} + \frac{g}{\sqrt{2\Omega}} \frac{1}{\sqrt{k}} \right]^2 \left(\frac{k^2}{2} + k - \mathbf{P} \cdot \mathbf{k} \right)^{-1} \quad (78)$$

and

$$B_{\mathbf{P}} = B_0 = 1 - \sum_{\mathbf{k} < k_0} \frac{\left(u_{\mathbf{k}} \frac{\phi_{\mathbf{k}}^2}{\phi_0^2} \left(\frac{k^2}{2} + k \right) + \frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{k}}}{\phi_0 \sqrt{k}} \right) u_{\mathbf{k}} (\phi_{\mathbf{k}}^2 - \phi_0^2)}{\phi_{\mathbf{k}}^2 \left(\frac{k^2}{2} + k \right)}. \quad (79)$$

If the definition of $E_L^{(0)}(\mathbf{P}, g)$ (75), together with equations (78) and (79), is used in equation (74), the second iteration for the energy of a moving particle is obtained

$$E^{(2)}(\mathbf{P}, g) = E^{(2)}(0, g) - \frac{g^2}{2\Omega} \sum_{\mathbf{k} < k_0} \frac{(\mathbf{P} \cdot \mathbf{k})^2}{k(k^2/2 + k)^3} \\ + \frac{P^2}{2} \left\{ 1 - \sum_{\mathbf{k} < k_0} \left[u_{\mathbf{k}}^2 \left(\frac{\phi_{\mathbf{k}}^2}{\phi_0^2} - 1 \right) + \frac{g}{\sqrt{2\Omega}} \frac{u_{\mathbf{k}}}{\sqrt{k}(k^2/2 + k)} \frac{(\phi_{\mathbf{k}}^2 - \phi_0^2)}{\phi_{\mathbf{k}} \phi_0} \right] \right\} \\ \times \left(1 - \sum_{\mathbf{k} < k_0} \left[u_{\mathbf{k}}^2 \left(\frac{\phi_{\mathbf{k}}^2}{\phi_0^2} - 1 \right) + \frac{g}{\sqrt{2\Omega}} \frac{u_{\mathbf{k}}}{\sqrt{k}(k^2/2 + k)} \frac{(\phi_{\mathbf{k}}^2 - \phi_0^2)}{\phi_{\mathbf{k}} \phi_0} \right] \right)^{-1}, \quad (80)$$

or after simplification, taking into account the fact that the sum in the denominator is proportional to g^2 , one finally obtains

$$E^{(2)}(\mathbf{P}, g) = E^{(2)}(0, g) + \frac{P^2}{2} - \frac{g^2}{2\Omega} \sum_{\mathbf{k} < k_0} \frac{(\mathbf{P} \cdot \mathbf{k})^2}{k(k^2/2 + k)^3}. \quad (81)$$

From here, we see that the second iteration for the renormalized mass

$$m^{(2)*} \approx 1 + \frac{g^2}{6\pi^2} \quad (82)$$

coincides with the one via perturbation theory.

V. CONCLUSION

In current methods of renormalization in QFT, the momentum cut-off plays an important role [31], which in fact is an additional and undefined parameter of the theory. Usually, the inclusion

of such parameter for a concrete model is justified with the argument that the theory becomes incorrect on a small scale, where a more general theory must be used instead. For example, in the case of QED it is widely accepted that on a small scale the Standard Model, with its own characteristic length, should be rather used. However, in the Standard Model, as in its possible generalizations, for the renormalization of perturbation theory the cut-off is required. Consequently, we come to the requirement of the inclusion of some “fundamental length” or unobservable parameter of any QFT.

However, the Fröhlich Hamiltonian demonstrates the absence of a cut-off in the polaron theory. In this QFT all corrections are determined through convergent integrals and, consequently, the cut-off is not required. Here we considered a more general QFT than the one associated with the polaron problem, for which standard perturbation theory gives rise to divergences. The main result of the present work consists in the construction of a calculation scheme for this more general QFT that only leads to convergent integrals. In addition to that, the regularization of all integrals is related to the effective-cut-off momentum, which is defined through the parameters of the system itself. Moreover, the divergences of standard perturbation theory are explained through the energy being a non-analytical function of the coupling constant, of a form $\ln(\sqrt{|\ln g|/2} + 1)$, around zero, and, therefore, can not be represented as a power series around this singular point. It is also important that the character of the singularity, defined in equation (71) in the weak coupling limit does not depend on the particular choice of the wave functions $\phi_{\mathbf{P}}(\mathbf{r})$ of the zeroth-order approximation.

From a formal point of view, the convergence of all integrals is explained as follows: i) the use of the decomposition (17), i.e. the special state vectors, which are the product of the wave function of a localized particle and a coherent state of the field and ii) the calculation of the energy of the system with the iteration scheme (25), in which the resolvent of the operator $[H_{kk} - E^{(0)}]^{-1}$ contains the matrix elements of the full Hamiltonian of the system. In standard perturbation theory the Hamiltonian of non-interacting fields is used in the analogous expressions.

From a physical point of view the argument i) corresponds to avoiding an adiabatic switch off of the interaction. This means that a particle during its existence time is considered as “dressed”, i.e. to be in a localized state which is created due to the interaction between the particle and the field. The argument ii) leads to the “cutting” of all integrals for a large momentum due to the reconstruction of a localized state in intermediate states, caused by the quasi-intersection of the ground and the single-phonon states.

Our approach should not be considered and does not pretend to be the full solution of the renormalization problem in QFT, specifically, because of the use of a simple, non-relativistically covariant model. Nevertheless, it demonstrates an alternative, succeeding without introducing any phenomenological momentum cut-off.

ACKNOWLEDGMENTS

The authors are grateful to S. Cavaletto for useful discussions.

APPENDIX A: PROOF THAT THE STATES (38) ARE EIGENSTATES OF THE TOTAL MOMENTUM OPERATOR

In this appendix we present an explicit proof that the total momentum operator

$$\mathbf{P} = -i\nabla_{\mathbf{r}} + \sum_{\mathbf{k}} \mathbf{k} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}$$

commutes with the Hamiltonian

$$H = -\frac{1}{2}\Delta + \sum_{\mathbf{k}} k a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \frac{g}{\sqrt{2\Omega}} \sum_{\mathbf{k}} A_{\mathbf{k}} (e^{i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}}^{\dagger})$$

of the system and that the states (38) are eigenstates of \mathbf{P}

$$\mathbf{P}|\Psi_{P_1, n_k}^{(0)}\rangle = P_1|\Psi_{P_1, n_k}^{(0)}\rangle, \quad (83)$$

consequently forming a complete set in the Hilbert space.

Let us begin with the commutator:

$$\begin{aligned} [H, \mathbf{P}] &= \left[\frac{g}{\sqrt{2\Omega}} \sum_{\mathbf{k}} A_{\mathbf{k}} (e^{i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}}^{\dagger}), -i\nabla \right] \\ &\quad + \left[\frac{g}{\sqrt{2\Omega}} \sum_{\mathbf{k}} A_{\mathbf{k}} (e^{i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}}^{\dagger}), \sum_{\mathbf{k}} k a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \right] \\ &= \frac{g}{\sqrt{2\Omega}} \sum_{\mathbf{k}} A_{\mathbf{k}} (a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} (-\mathbf{k}) + a_{\mathbf{k}}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{r}} \mathbf{k}) \\ &\quad + \frac{g}{\sqrt{2\Omega}} \sum_{\mathbf{k}} A_{\mathbf{k}} (a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{k} + a_{\mathbf{k}}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{r}} (-\mathbf{k})) = 0, \end{aligned} \quad (84)$$

which was to be proven.

Now we will demonstrate that the relation (83) holds. Also, we will introduce the notations

$$D(\mathbf{R}) = \exp \left\{ \sum_{\mathbf{q}} (u_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{R}} a_{\mathbf{q}}^{\dagger} - u_{\mathbf{q}}^* e^{i\mathbf{q}\cdot\mathbf{R}} a_{\mathbf{q}}) \right\}, \text{ with} \quad (85)$$

$$D^{\dagger}(\mathbf{R})D(\mathbf{R}) = D(\mathbf{R})D^{\dagger}(\mathbf{R}) = 1, \quad (86)$$

$$D^{\dagger}(\mathbf{R})a_{\mathbf{k}}D(\mathbf{R}) = a_{\mathbf{k}} + u_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{R}}, \quad (87)$$

$$D^{\dagger}(\mathbf{R})a_{\mathbf{k}}^{\dagger}D(\mathbf{R}) = a_{\mathbf{k}}^{\dagger} + u_{\mathbf{k}}^*e^{i\mathbf{k}\cdot\mathbf{R}}, \quad (88)$$

$$i\frac{\partial D(\mathbf{R})}{\partial \mathbf{R}} = D(\mathbf{R}) \sum_{\mathbf{q}} \mathbf{q} (u_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{R}} a_{\mathbf{q}}^{\dagger} + u_{\mathbf{q}}^* e^{i\mathbf{q}\cdot\mathbf{R}} a_{\mathbf{q}}) \quad (89)$$

Consequently, with the help of equations (85-89) we may write

$$\begin{aligned} \mathbf{P}|\Psi_{P_1, n_k}^{(0)}\rangle &= \left(-i\nabla_{\mathbf{r}} + \sum_{\mathbf{q}} \mathbf{q} a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}} \right) \frac{1}{N_{P_1, n_k} \sqrt{\Omega}} \int d\mathbf{R} \phi_{P_1}(\mathbf{r} - \mathbf{R}) \exp \{i(\mathbf{P}_1 - \mathbf{k}n_k) \cdot \mathbf{R}\} D(\mathbf{R})|n_k\rangle \\ &= \frac{1}{N_{P_1, n_k} \sqrt{\Omega}} \int d\mathbf{R} (-i\nabla_{\mathbf{r}})(\phi_{P_1}(\mathbf{r} - \mathbf{R})) \exp \{i(\mathbf{P}_1 - \mathbf{k}n_k) \cdot \mathbf{R}\} D(\mathbf{R})|n_k\rangle \\ &\quad + \frac{1}{N_{P_1, n_k} \sqrt{\Omega}} \int d\mathbf{R} \phi_{P_1}(\mathbf{r} - \mathbf{R}) \exp \{i(\mathbf{P}_1 - \mathbf{k}n_k) \cdot \mathbf{R}\} \sum_{\mathbf{q}} \mathbf{q} a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}} D(\mathbf{R})|n_k\rangle. \end{aligned} \quad (90)$$

By noticing that $-i\nabla_{\mathbf{r}}(\phi_{P_1}(\mathbf{r} - \mathbf{R})) = i\nabla_{\mathbf{R}}(\phi_{P_1}(\mathbf{r} - \mathbf{R}))$ and transforming $\sum_{\mathbf{q}} \mathbf{q} a_{\mathbf{q}}^{\dagger} a_{\mathbf{q}} D(\mathbf{R}) =$

$D(\mathbf{R})D^\dagger(\mathbf{R}) \sum_q q a_q^\dagger a_q D(\mathbf{R})$ one obtains

$$\begin{aligned} \mathbf{P}|\Psi_{P_1, n_k}^{(0)}\rangle &= \frac{1}{N_{P_1, n_k} \sqrt{\Omega}} \int d\mathbf{R} (i\nabla_{\mathbf{R}}) [\phi_{P_1}(\mathbf{r} - \mathbf{R}) \exp\{i(P_1 - kn_k) \cdot \mathbf{R}\} D(\mathbf{R})] |n_k\rangle \\ &\quad - \frac{1}{N_{P_1, n_k} \sqrt{\Omega}} \int d\mathbf{R} \phi_{P_1}(\mathbf{r} - \mathbf{R}) (i\nabla_{\mathbf{R}}) [\exp\{i(P_1 - kn_k) \cdot \mathbf{R}\} D(\mathbf{R})] |n_k\rangle \\ &\quad + \frac{1}{N_{P_1, n_k} \sqrt{\Omega}} \int d\mathbf{R} \phi_{P_1}(\mathbf{r} - \mathbf{R}) \exp\{i(P_1 - kn_k) \cdot \mathbf{R}\} D(\mathbf{R}) \\ &\quad \times \sum_q q (a_q^\dagger + u_q^* e^{iq \cdot \mathbf{R}}) (a_q + u_q e^{-iq \cdot \mathbf{R}}) |n_k\rangle. \end{aligned} \quad (91)$$

The first term in equation (91) vanishes due to the square-integrability of the function $\phi_{P_1}(\mathbf{r} - \mathbf{R})$. The derivative in the second term is equal to

$$\begin{aligned} (i\nabla_{\mathbf{R}}) [\exp\{i(P_1 - kn_k) \cdot \mathbf{R}\} D(\mathbf{R})] &= -(P_1 - kn_k) \exp\{i(P_1 - kn_k) \cdot \mathbf{R}\} D(\mathbf{R}) \\ &\quad + \exp\{i(P_1 - kn_k) \cdot \mathbf{R}\} D(\mathbf{R}) \sum_q q (u_q e^{-iq \cdot \mathbf{R}} a_q^\dagger + u_q^* e^{iq \cdot \mathbf{R}} a_q) \end{aligned} \quad (92)$$

and, therefore, the terms which are not proportional to P_1 in equation (92) cancel the last term in equation (91). As a result, equation (91) transforms into

$$\mathbf{P}|\Psi_{P_1, n_k}^{(0)}\rangle = P_1 |\Psi_{P_1, n_k}^{(0)}\rangle, \quad (93)$$

which was to be proven.

According to reference [32], the eigenstates of a Hermitian operator form a complete and orthogonal set of functions in the Hilbert space. As the functions (38) are eigenstates of the Hermitian operator \mathbf{P} , they form a complete orthogonal set for arbitrary generalized parameters $\phi_{P_1}(\mathbf{r} - \mathbf{R})$ and u_k .

APPENDIX B: MATRIX ELEMENTS CALCULATION

In all subsequent calculations, the matrix elements of a type

$$\begin{aligned} \langle n_j | \exp \left(- \sum_m a_m^\dagger u_m e^{-im \cdot \mathbf{R}'} - a_m u_m^* e^{im \cdot \mathbf{R}'} \right) \sum_l f(a_l, a_l^\dagger) \\ \times \exp \left(\sum_m a_m^\dagger u_m e^{-im \cdot \mathbf{R}} - a_m u_m^* e^{im \cdot \mathbf{R}} \right) | n_k \rangle \end{aligned} \quad (94)$$

need to be evaluated. By using the identities

$$D = e^{\beta a^\dagger - \beta^* a} = e^{-|\beta|^2/2} e^{\beta a^\dagger} e^{-\beta^* a} = e^{|\beta|^2/2} e^{-\beta^* a} e^{\beta a^\dagger}, \quad (95)$$

$$D^{-1} a D = a + \beta, \quad D^{-1} a^\dagger D = a^\dagger + \beta^*, \quad (96)$$

equation (94) can be transformed into the form

$$\begin{aligned} & \frac{\exp(\sum_m |u_m|^2 (e^{-i\mathbf{m} \cdot (\mathbf{R} - \mathbf{R}')}) - 1))}{\sqrt{n_{\mathbf{k}}! n_{\mathbf{j}}!}} \\ & \times \langle 0 | (\mathbf{a}_{\mathbf{j}} - u_{\mathbf{j}} e^{-i\mathbf{j} \cdot \mathbf{R}'} + u_{\mathbf{j}} e^{-i\mathbf{j} \cdot \mathbf{R}})^{n_{\mathbf{j}}} \sum_{\mathbf{l}} f(\mathbf{a}_{\mathbf{l}} + u_{\mathbf{l}} e^{-i\mathbf{l} \cdot \mathbf{R}}, \mathbf{a}_{\mathbf{l}}^\dagger + u_{\mathbf{l}}^* e^{i\mathbf{l} \cdot \mathbf{R}'}) \\ & \times (\mathbf{a}_{\mathbf{k}}^\dagger + u_{\mathbf{k}}^* e^{i\mathbf{k} \cdot \mathbf{R}'} - u_{\mathbf{k}}^* e^{i\mathbf{k} \cdot \mathbf{R}})^{n_{\mathbf{k}}} |0\rangle. \end{aligned} \quad (97)$$

The evaluation of equation (97) is performed in the usual manner, i.e, by noticing that $\mathbf{a}|0\rangle = \langle 0|\mathbf{a}^\dagger = 0$ and the vacuum average is not equal to zero only if the number of creation operators is equal to the one of annihilation operators and is an even number.

APPENDIX C: GROUND STATE ENERGY

According to equation (42) of the manuscript, the ground state energy is defined as

$$E_L^{(0)} = \langle \Psi_{\mathbf{P}}^{(L)} | \mathbf{H} | \Psi_{\mathbf{P}}^{(L)} \rangle \quad (98)$$

with the wave function

$$|\Psi_{\mathbf{P}}^{(L)}\rangle = \frac{1}{N_{\mathbf{P}} \sqrt{\Omega}} \int d\mathbf{R} \phi_{\mathbf{P}}(\mathbf{r} - \mathbf{R}) \exp\left(i\mathbf{P} \cdot \mathbf{R} + \sum_{\mathbf{k}} \left(u_{\mathbf{k}} \mathbf{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{R}} - \frac{1}{2} u_{\mathbf{k}}^2\right)\right) |0\rangle \quad (99)$$

and Hamiltonian

$$\begin{aligned} \mathbf{H} = & \frac{1}{2} \left(\mathbf{P}^2 - 2 \sum_{\mathbf{k}} \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{a}_{\mathbf{k}} \mathbf{k} \cdot \mathbf{P} + \left(\sum_{\mathbf{k}} \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{a}_{\mathbf{k}} \mathbf{k} \right)^2 \right) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{a}_{\mathbf{k}} \\ & + \frac{g}{\sqrt{\Omega}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(e^{i\mathbf{k} \cdot \mathbf{r}} \mathbf{a}_{\mathbf{k}} + e^{-i\mathbf{k} \cdot \mathbf{r}} \mathbf{a}_{\mathbf{k}}^\dagger \right). \end{aligned} \quad (100)$$

The normalization constant $N_{\mathbf{P}}$ is found from the condition

$$\langle \Psi_{\mathbf{P}}^{(L)} | \Psi_{\mathbf{P}}^{(L)} \rangle = 1. \quad (101)$$

In order to evaluate equation (101), we use equation (97), in which $n_{\mathbf{k}} = n_{\mathbf{j}} = 0$ and $f(\mathbf{a}_{\mathbf{l}}, \mathbf{a}_{\mathbf{l}}^\dagger) = \delta_{\mathbf{l},0}$. This gives immediately the result

$$|N_{\mathbf{P}}|^2 = \int d\mathbf{R} \int d\mathbf{r} \phi_{\mathbf{P}}^*(\mathbf{r}) \phi_{\mathbf{P}}(\mathbf{r} - \mathbf{R}) \exp\left(\sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 (e^{-i\mathbf{k} \cdot \mathbf{R}} - 1) + i\mathbf{P} \cdot \mathbf{R}\right). \quad (102)$$

The expectation value of the energy is performed in exactly the same way. First of all, the matrix elements of the field states are calculated with the help of equation (97). For example, if the function f is chosen as $f = l \mathbf{a}_{\mathbf{l}}^\dagger \mathbf{a}_{\mathbf{l}}$, $n_{\mathbf{k}} = n_{\mathbf{j}} = 0$, we immediately find

$$\begin{aligned} Q = & \langle \sum_{\mathbf{l}} l \mathbf{a}_{\mathbf{l}}^\dagger \mathbf{a}_{\mathbf{l}} \rangle \\ = & \int d\mathbf{R} d\mathbf{R}' d\mathbf{r} \phi_{\mathbf{P}}^*(\mathbf{r} - \mathbf{R}') \phi_{\mathbf{P}}(\mathbf{r} - \mathbf{R}) \sum_{\mathbf{l}} l |u_{\mathbf{l}}|^2 \\ & \times \exp\left(\sum_{\mathbf{m}} |u_{\mathbf{m}}|^2 (e^{-i\mathbf{m} \cdot (\mathbf{R} - \mathbf{R}')}) - 1) + i(\mathbf{P} - \mathbf{l}) \cdot (\mathbf{R} - \mathbf{R}')\right). \end{aligned} \quad (103)$$

Then by carrying out the change of variables $\mathbf{R} - \mathbf{R}' = \mathbf{R}_1$ and $\mathbf{r} - \mathbf{R}' = \boldsymbol{\rho}$, we obtain

$$\begin{aligned} Q &= \frac{1}{|N_P|^2} \sum_{\mathbf{k}} \mathbf{k} |u_{\mathbf{k}}|^2 \int d\mathbf{R} d\mathbf{r} \phi_{\mathbf{P}}^*(\mathbf{r}) \phi_{\mathbf{P}}(\mathbf{r} - \mathbf{R}) e^{\Phi(\mathbf{R}) + i(\mathbf{P} - \mathbf{k}) \cdot \mathbf{R}}, \\ \Phi(\mathbf{R}) &= \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 (e^{-i\mathbf{k} \cdot \mathbf{R}} - 1). \end{aligned} \quad (104)$$

All other matrix elements are evaluated in exactly the same fashion. Consequently, we obtain expression (44) of the manuscript:

$$\begin{aligned} E_L^{(0)}(\mathbf{P}, g) &= \frac{P^2}{2} - \mathbf{P} \cdot \mathbf{Q} + G + E_f(\mathbf{P}) + E_{\text{int}}(\mathbf{P}); \\ Q &= \frac{1}{|N_P|^2} \sum_{\mathbf{k}} \mathbf{k} |u_{\mathbf{k}}|^2 \int d\mathbf{R} d\mathbf{r} \phi_{\mathbf{P}}^*(\mathbf{r}) \phi_{\mathbf{P}}(\mathbf{r} - \mathbf{R}) e^{\Phi(\mathbf{R}) + i(\mathbf{P} - \mathbf{k}) \cdot \mathbf{R}}, \\ G &= \frac{1}{2} \frac{1}{|N_P|^2} \sum_{\mathbf{m}, \mathbf{l}} \mathbf{m} \cdot \mathbf{l} |u_{\mathbf{m}}|^2 |u_{\mathbf{l}}|^2 \int d\mathbf{r} d\mathbf{R} \phi^*(\mathbf{r}) \phi(\mathbf{r} - \mathbf{R}) e^{i\mathbf{P} \cdot \mathbf{R} + \Phi(\mathbf{R}) - i(\mathbf{m} + \mathbf{l}) \cdot \mathbf{R}}, \\ E_f(\mathbf{P}) &= \frac{1}{|N_P|^2} \sum_{\mathbf{k}} \left(k + \frac{k^2}{2} \right) |u_{\mathbf{k}}|^2 \int d\mathbf{R} d\mathbf{r} \phi_{\mathbf{P}}^*(\mathbf{r}) \phi_{\mathbf{P}}(\mathbf{r} - \mathbf{R}) e^{\Phi(\mathbf{R}) + i(\mathbf{P} - \mathbf{k}) \cdot \mathbf{R}}, \\ E_{\text{int}}(\mathbf{P}) &= \frac{g}{|N_P|^2} \sum_{\mathbf{k}} \frac{u_{\mathbf{k}}}{\sqrt{2k\Omega}} \int d\mathbf{R} d\mathbf{r} \left(\phi_{\mathbf{P}}^*(\mathbf{r} + \mathbf{R}) \phi_{\mathbf{P}}(\mathbf{r}) + \phi_{\mathbf{P}}^*(\mathbf{r}) \phi_{\mathbf{P}}(\mathbf{r} - \mathbf{R}) \right) e^{\Phi(\mathbf{R}) + i(\mathbf{P} \cdot \mathbf{R} + \mathbf{k} \cdot \mathbf{r})}, \\ \Phi(\mathbf{R}) &= \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 (e^{-i\mathbf{k} \cdot \mathbf{R}} - 1); \\ |N_P|^2 &= \int d\mathbf{R} d\mathbf{r} \phi_{\mathbf{P}}^*(\mathbf{r}) \phi_{\mathbf{P}}(\mathbf{r} - \mathbf{R}) e^{\Phi(\mathbf{R}) + i\mathbf{P} \cdot \mathbf{R}}. \end{aligned} \quad (105)$$

We notice here one more time that the Fourier component of the function reads

$$\phi(\mathbf{r}) = \frac{\lambda^{\frac{3}{4}}}{\pi^{\frac{3}{4}}} e^{-\frac{\lambda^2 r^2}{2}}. \quad (106)$$

and the classical component of the field look like

$$u_{\mathbf{k}} = -\frac{g}{\sqrt{2\Omega}} \frac{1}{\sqrt{k^3}} \int d\mathbf{r} |\phi(\mathbf{r})|^2 e^{-i\mathbf{k} \cdot \mathbf{r}} = -\frac{g}{\sqrt{2\Omega}} \frac{e^{-\frac{k^2}{4\lambda^2}}}{\sqrt{k^3}}; \quad (107)$$

$$\phi_{\mathbf{k}} = \int d\mathbf{r} \phi(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} = 2\sqrt{2} \frac{\pi^{\frac{3}{4}}}{\lambda^{\frac{3}{2}}} e^{-\frac{k^2}{2\lambda^2}} = \phi_0 e^{-\frac{k^2}{2\lambda^2}}. \quad (108)$$

In order to calculate the energy, we firstly neglect the function

$$\Phi(\mathbf{R}) = \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 (e^{-i\mathbf{k} \cdot \mathbf{R}} - 1) = g^2 \frac{1}{4\pi^2} \int_0^\infty dt \frac{e^{-\frac{t^2}{2}}}{t} \left(\frac{\sin \lambda R t}{\lambda R t} - 1 \right) \sim g^2 \quad (109)$$

in equation (105). The remaining quantities can be rewritten employing the Fourier transform

of the function $\phi(\mathbf{r})$. As for example

$$\int d\mathbf{R}_1 d\rho \phi^*(\rho) \phi(\rho - \mathbf{R}_1) e^{-i\mathbf{k} \cdot \mathbf{R}_1} = \int d\rho \phi^*(\rho) e^{-i\mathbf{k} \cdot \rho} \int d\mathbf{R} \phi(\mathbf{R}) e^{i\mathbf{k} \cdot \mathbf{R}} = \phi_{\mathbf{k}}^* \phi_{-\mathbf{k}} = \phi_{\mathbf{k}}^2, \quad (110)$$

$$\int d\mathbf{R}_1 d\rho \phi^*(\rho) \phi(\rho - \mathbf{R}_1) = |\phi_0|^2 = \phi_0^2, \quad (111)$$

$$\int d\mathbf{R}_1 d\rho \phi^*(\rho) \phi(\rho - \mathbf{R}_1) e^{-i\mathbf{k} \cdot \rho} = \phi_{\mathbf{k}}^* \phi_0 = \phi_{\mathbf{k}} \phi_0, \quad (112)$$

and by plugging equations (110-112) into equation (105), one finds

$$E_L^{(0)}(0, g) = \frac{1}{2} \sum_{\mathbf{m}, l} \mathbf{m} \cdot l |u_{\mathbf{m}}|^2 |u_l|^2 \frac{\phi_{l+\mathbf{m}}^2}{\phi_0^2} + \sum_{\mathbf{k}} \left(k + \frac{k^2}{2} \right) |u_{\mathbf{k}}|^2 \frac{\phi_{\mathbf{k}}^2}{\phi_0^2} + \frac{2g}{\sqrt{2\Omega}} \sum_{\mathbf{k}} \frac{u_{\mathbf{k}}}{\sqrt{k}} \frac{\phi_{\mathbf{k}}}{\phi_0}. \quad (113)$$

By insertion of the definitions of the classical component of the field $u_{\mathbf{k}}$ and the Fourier transform of the function $\phi_{\mathbf{k}}$, defined in equations (107) and (108), we find

$$\begin{aligned} E_L^{(0)}(0, g) &= \frac{g^4}{8(2\pi)^6} \int dldm \frac{\mathbf{m} \cdot l}{m^3 l^3} e^{-\frac{3}{2} \frac{m^2}{\lambda^2} - \frac{3}{2} \frac{l^2}{\lambda^2} - \frac{2\mathbf{m} \cdot l}{\lambda^2}} + \frac{g^2}{2(2\pi)^3} \int \frac{d\mathbf{k}}{k^2} \left(1 + \frac{k}{2} \right) e^{-\frac{3}{2} \frac{k^2}{\lambda^2}} \\ &\quad - \frac{g^2}{(2\pi^3)} \int \frac{d\mathbf{k}}{k^2} e^{-\frac{3}{2} \frac{k^2}{\lambda^2}} = \frac{g^4}{8(2\pi)^5} \int \frac{dl}{l^2} e^{-\frac{3}{2} \frac{l^2}{\lambda^2}} \int dme^{-\frac{3}{2} \frac{m^2}{\lambda^2}} \frac{-2lm\lambda^2 \cosh \frac{2lm}{\lambda^2} + \lambda^4 \sinh \frac{2lm}{\lambda^2}}{2l^2 m^2} \\ &\quad + \frac{g^2}{24\pi^2} (\lambda(-4 + \sqrt{2}) \sqrt{3\pi} + \lambda^2) = \frac{g^4 \lambda^2}{16(2\pi)^4} \int_0^\infty \frac{du}{u^2} \left(4u - e^{\frac{2}{3} u^2} \sqrt{6\pi} \text{Erf} \left(\sqrt{\frac{2}{3}} u \right) \right) e^{-\frac{3}{2} u^2} \\ &\quad + \frac{g^2}{24\pi^2} (\lambda(-4 + \sqrt{2}) \sqrt{3\pi} + \lambda^2) \\ &= -\frac{g^4 \lambda^2}{2^8 \pi^4} \alpha + \frac{g^2}{24\pi^2} (\lambda(-4 + \sqrt{2}) \sqrt{3\pi} + \lambda^2), \end{aligned} \quad (114)$$

where $\alpha = 0.736559$.

To find λ , we minimize the energy, which results in the equation

$$\frac{\partial E_L^{(0)}(0, g)}{\partial \lambda} = -\frac{2g^4 \lambda}{2^8 \pi^4} \alpha + \frac{g^2}{24\pi^2} ((-4 + \sqrt{2}) \sqrt{3\pi} + 2\lambda), \quad (115)$$

from here we find

$$\lambda = -\frac{\sqrt{3\pi}}{2} (-4 + \sqrt{2}) \frac{1}{1 - \frac{3\alpha g^2}{32\pi^2}} \approx -\frac{\sqrt{3\pi}}{2} (-4 + \sqrt{2}) \left(1 + \frac{3\alpha g^2}{32\pi^2} \right) \quad (116)$$

and by plugging λ in equation (114)

$$E_L^{(0)}(0, g) = -g^2 \frac{(-4 + \sqrt{2})^2}{32\pi} - \frac{3\alpha g^4 (-4 + \sqrt{2})^2}{2^{10} \pi^3} + O(g^6). \quad (117)$$

APPENDIX D: MASS RENORMALIZATION IN ZERO-ORDER APPROXIMATION

In the weak coupling limit we find the renormalized mass in zeroth-order approximation. For this purpose, we rewrite the energy through Fourier components for the case $\mathbf{P} \neq 0$. This yields

$$E_L^{(0)}(\mathbf{P}, g) = \frac{P^2}{2} - \mathbf{P} \cdot \mathbf{Q} + G + E_f(\mathbf{P}) + E_{\text{int}}(\mathbf{P}); \quad (118)$$

$$\mathbf{Q} = \frac{1}{|N_{\mathbf{P}}|^2} \sum_{\mathbf{k}} \mathbf{k} |u_{\mathbf{k}}|^2 \phi_{\mathbf{P}-\mathbf{k}}^2; \quad (119)$$

$$G = \frac{1}{2} \frac{1}{|N_{\mathbf{P}}|^2} \sum_{\mathbf{m}, \mathbf{l}} \mathbf{m} \cdot \mathbf{l} |u_{\mathbf{m}}|^2 |u_{\mathbf{l}}|^2 \phi_{\mathbf{P}-\mathbf{l}-\mathbf{m}}^2; \quad (120)$$

$$E_f(\mathbf{P}) = \frac{1}{|N_{\mathbf{P}}|^2} \sum_{\mathbf{k}} \left(k + \frac{k^2}{2} \right) |u_{\mathbf{k}}|^2 \phi_{\mathbf{P}-\mathbf{k}}^2; \quad (121)$$

$$E_{\text{int}}(\mathbf{P}) = \frac{g}{|N_{\mathbf{P}}|^2} \sum_{\mathbf{k}} \frac{u_{\mathbf{k}}}{\sqrt{2k\Omega}} (\phi_{\mathbf{P}} \phi_{\mathbf{P}-\mathbf{k}} + \phi_{\mathbf{P}} \phi_{\mathbf{P}+\mathbf{k}}); \quad (122)$$

$$|N_{\mathbf{P}}|^2 = \phi_{\mathbf{P}}^2. \quad (123)$$

From here we can immediately conclude that the quantity G yields only a correction of the order of g^4 and can be neglected. Let us expand the Fourier transform of the function $\phi(\mathbf{r})$ into Taylor series over \mathbf{P} up to second-order

$$\phi_{\mathbf{P}}^2 = \phi_0^2 e^{-\frac{P^2}{\lambda^2}} \approx \phi_0^2 \left(1 - \frac{P^2}{\lambda^2} \right), \quad (124)$$

$$\phi_{\mathbf{P}-\mathbf{k}}^2 = \phi_0^2 e^{-\frac{(\mathbf{P}-\mathbf{k})^2}{\lambda^2}} = \phi_0^2 \left(e^{-\frac{k^2}{\lambda^2}} + e^{-\frac{k^2}{\lambda^2}} \frac{2\mathbf{P} \cdot \mathbf{k}}{\lambda^2} + e^{-\frac{k^2}{\lambda^2}} \frac{2(\mathbf{P} \cdot \mathbf{k})^2}{\lambda^4} - e^{-\frac{k^2}{\lambda^2}} \frac{P^2}{\lambda^2} \right), \quad (125)$$

$$\phi_{\mathbf{P}-\mathbf{k}} = \phi_0 e^{-\frac{(\mathbf{P}-\mathbf{k})^2}{2\lambda^2}} = \phi_0 \left(e^{-\frac{k^2}{2\lambda^2}} + e^{-\frac{k^2}{2\lambda^2}} \frac{\mathbf{P} \cdot \mathbf{k}}{\lambda^2} + e^{-\frac{k^2}{2\lambda^2}} \frac{(\mathbf{P} \cdot \mathbf{k})^2}{2\lambda^4} - e^{-\frac{k^2}{2\lambda^2}} \frac{P^2}{2\lambda^2} \right). \quad (126)$$

By plugging equations (124-126) into equations (119-123) and taking into account only the terms of the order of P^2 , we find for the vector \mathbf{Q}

$$\begin{aligned} \mathbf{Q} &= \frac{1}{1 - P^2/\lambda^2} \frac{2}{\lambda^2} \sum_{\mathbf{k}} \mathbf{k} |u_{\mathbf{k}}|^2 e^{-\frac{k^2}{\lambda^2}} (\mathbf{P} \cdot \mathbf{k}) = \frac{1}{1 - P^2/\lambda^2} \frac{g^2}{\lambda^2} \frac{\mathbf{P}}{4\pi^2} \int_0^\infty dk k e^{-\frac{3}{2} \frac{k^2}{\lambda^2}} \int_{-1}^1 t^2 dt \\ &= \frac{g^2}{9\pi^2} \frac{\mathbf{P}}{2} \end{aligned} \quad (127)$$

and for the field energy

$$E_f = \frac{1}{1 - P^2/\lambda^2} \sum_{\mathbf{k}} \left(k + \frac{k^2}{2} \right) |u_{\mathbf{k}}|^2 \left(e^{-\frac{k^2}{\lambda^2}} + e^{-\frac{k^2}{\lambda^2}} \frac{2\mathbf{P} \cdot \mathbf{k}}{\lambda^2} + e^{-\frac{k^2}{\lambda^2}} \frac{2(\mathbf{P} \cdot \mathbf{k})^2}{\lambda^4} - e^{-\frac{k^2}{\lambda^2}} \frac{P^2}{\lambda^2} \right). \quad (128)$$

In this expression, in round brackets the first and the last terms cancel each other after decom-

position of the normalization constant in the Taylor series in \mathbf{P} . The result reads as

$$\begin{aligned}
 E_f &= E_f(0) + \frac{2}{\lambda^2} \sum_{\mathbf{k}} \left(k + \frac{k^2}{2} \right) |u_{\mathbf{k}}|^2 e^{-\frac{k^2}{\lambda^2}} (\mathbf{P} \cdot \mathbf{k})^2 \\
 &= E_f(0) + \frac{g^2}{\lambda^4} \frac{P^2}{4\pi^2} \int_0^\infty k \left(k + \frac{k^2}{2} \right) e^{-\frac{3}{2} \frac{k^2}{\lambda^2}} dk \int_{-1}^1 t^2 dt \\
 &= E_f(0) + \frac{P^2}{2} \frac{g^2}{9\pi^2} \frac{1}{6\lambda} (\sqrt{6\pi} + 2\lambda).
 \end{aligned} \tag{129}$$

The remaining energy is calculated in exactly the same way

$$\begin{aligned}
 E_{\text{int}} &= \frac{g}{\sqrt{2\Omega}} \frac{1}{1 - P^2/(2\lambda^2)} \sum_{\mathbf{k}} \frac{u_{\mathbf{k}}}{\sqrt{k}} \left[e^{-\frac{k^2}{2\lambda^2}} + e^{-\frac{k^2}{2\lambda^2}} \frac{\mathbf{P} \cdot \mathbf{k}}{\lambda^2} + e^{-\frac{k^2}{2\lambda^2}} \frac{(\mathbf{P} \cdot \mathbf{k})^2}{2\lambda^4} - e^{-\frac{k^2}{2\lambda^2}} \frac{P^2}{2\lambda^2} \right. \\
 &\quad \left. + e^{-\frac{k^2}{2\lambda^2}} - e^{-\frac{k^2}{2\lambda^2}} \frac{\mathbf{P} \cdot \mathbf{k}}{\lambda^2} + e^{-\frac{k^2}{2\lambda^2}} \frac{(\mathbf{P} \cdot \mathbf{k})^2}{2\lambda^4} - e^{-\frac{k^2}{2\lambda^2}} \frac{P^2}{2\lambda^2} \right].
 \end{aligned} \tag{130}$$

In a full analogy to the field energy E_f , the first and the last terms are cancelled. The remaining terms are

$$\begin{aligned}
 E_{\text{int}} &= E_{\text{int}}(0) + \frac{g}{\sqrt{2\Omega}} \frac{1}{\lambda^4} \sum_{\mathbf{k}} \frac{u_{\mathbf{k}}}{\sqrt{k}} e^{-\frac{k^2}{2\lambda^2}} (\mathbf{P} \cdot \mathbf{k})^2 \\
 &= E_{\text{int}}(0) - \frac{g^2}{\lambda^4} \frac{P^2}{8\pi^2} \int_0^\infty k^2 e^{-\frac{3}{4} \frac{k^2}{\lambda^2}} dk \int_{-1}^1 t^2 dt \\
 &= E_{\text{int}}(0) - \frac{g^2}{9\pi^2} \frac{1}{\lambda} \sqrt{\frac{\pi}{3}} \frac{P^2}{2}.
 \end{aligned} \tag{131}$$

By combining all results together, we find the equation for the total energy of the system with a renormalized mass, in the zeroth-order approximation:

$$\begin{aligned}
 E_L^{(0)}(\mathbf{P}, g) &= E_L^{(0)}(0, g) + \frac{P^2}{2} \left[1 - \frac{g^2}{9\pi^2} \left(1 + \frac{1}{\lambda} \sqrt{\frac{\pi}{3}} - \frac{(\sqrt{6\pi} + 2\lambda)}{6\lambda} \right) \right] \\
 &= E_L^{(0)}(0, g) + \frac{P^2}{2} \left[1 - \frac{g^2}{9\pi^2} \left(\frac{2}{3} + \frac{1}{\lambda} \frac{\sqrt{6\pi}(\sqrt{2} - 1)}{6} \right) \right],
 \end{aligned} \tag{132}$$

or by plugging in for λ according to equation (116) we finally obtain

$$E_L^{(0)}(\mathbf{P}, g) = E_L^{(0)}(0, g) + \frac{P^2}{2} \left[1 - \frac{g^2}{9\pi^2} \frac{17 - \sqrt{2}}{21} \right]. \tag{133}$$

Concluding, the renormalized mass is equal to

$$m^{*(0)} = 1 + \frac{g^2}{9\pi^2} \frac{17 - \sqrt{2}}{21}. \tag{134}$$

APPENDIX E: CALCULATION OF MATRIX ELEMENTS IN THE SECOND ITERATION FOR THE ENERGY OF THE SYSTEM

The calculation of the second iteration of the energy of the system requires the evaluation of the transition matrix elements $\langle \Psi_{\mathbf{P}}^{(L)} | \mathbf{H} | \Psi_{P_1, \{n_{\mathbf{k}}\}} \rangle$, $\langle \Psi_{\mathbf{P}}^{(L)} | \Psi_{P_1, \{n_{\mathbf{k}}\}} \rangle$ and $\langle \Psi_{P_1, \{n_{\mathbf{k}}\}} | \mathbf{H} | \Psi_{P_1, \{n_{\mathbf{k}}\}} \rangle$ from

the full Hamiltonian of the system, equation (100), with the function

$$\begin{aligned}
|\Psi_{P_1, n_k}^{(0)}\rangle &= \frac{1}{N_{P_1, 1_k} \sqrt{\Omega}} \int d\mathbf{R} \phi_{P_1}(\mathbf{r} - \mathbf{R}) \exp \{i(\mathbf{P}_1 - \mathbf{k} n_k) \cdot \mathbf{R}\} \\
&\times \exp \left[\sum_{\mathbf{k}} (u_{\mathbf{k}} e^{-i\mathbf{k} \cdot \mathbf{R}} \mathbf{a}_{\mathbf{k}}^\dagger - u_{\mathbf{k}}^* e^{i\mathbf{k} \cdot \mathbf{R}} \mathbf{a}_{\mathbf{k}}) \right] |n_k\rangle \\
&= \frac{1}{N_{P_1, 1_k} \sqrt{\Omega}} \int d\mathbf{R} \phi_{P_1}(\mathbf{r} - \mathbf{R}) \exp \left\{ i(\mathbf{P}_1 - \mathbf{k} n_k) \cdot \mathbf{R} - \frac{1}{2} \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 + \sum_{\mathbf{k}} u_{\mathbf{k}} e^{-i\mathbf{k} \cdot \mathbf{R}} \mathbf{a}_{\mathbf{k}}^\dagger \right\} \\
&\times \frac{(\mathbf{a}_{\mathbf{k}}^\dagger - u_{\mathbf{k}}^* e^{i\mathbf{k} \cdot \mathbf{R}})^{n_k}}{\sqrt{n_k!}} |0\rangle. \tag{135}
\end{aligned}$$

The normalization constant in equation (135) is calculated with the help of equation (97) and has the form

$$\begin{aligned}
|N_{P_1, 1_k}|^2 &= \int d\mathbf{R}_1 d\rho \phi_{P_1}^*(\rho) \phi_{P_1}(\rho - \mathbf{R}_1) e^{i(\mathbf{P}_1 - \mathbf{k}) \cdot \mathbf{R}_1 + \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1)} \\
&\times (2|u_{\mathbf{k}}|^2 (\cos \mathbf{k} \cdot \mathbf{R}_1 - 1) + 1). \tag{136}
\end{aligned}$$

The calculation of the transition matrix element is performed with the help of equation (97):

$$\begin{aligned}
\langle \Psi_{P_1, n_k} | H | \Psi_P^{(L)} \rangle &= \frac{(2\pi)^3 \delta(\mathbf{P} - \mathbf{P}_1)}{N_{P_1, 1_k}^* N_P \Omega} \int d\mathbf{R}_1 d\rho \phi_{P_1}^*(\rho) \phi_P(\rho - \mathbf{R}_1) \\
&\times e^{i\mathbf{P} \cdot \mathbf{R}_1 + \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1)} \frac{u_{\mathbf{k}}^{n_k} (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1)^{n_k}}{\sqrt{n_k!}} \\
&\times \left[\frac{P_1^2}{2} + \left(\frac{1}{2} k^2 + k - \mathbf{P}_1 \cdot \mathbf{k} \right) n_k e^{-i\mathbf{k} \cdot \mathbf{R}_1} (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1)^{-1} \right. \\
&\quad + \frac{g}{\sqrt{2\Omega}} \frac{e^{-i\mathbf{k} \cdot \rho}}{\sqrt{\omega_k}} n_k u_{\mathbf{k}}^{-1} (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1)^{-1} \\
&\quad + \sum_{\mathbf{m}} \mathbf{m} |u_{\mathbf{m}}|^2 e^{-i\mathbf{m} \cdot \mathbf{R}_1} \cdot (-\mathbf{P}_1 + \mathbf{k} n_k e^{-i\mathbf{k} \cdot \mathbf{R}_1} (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1)^{-1}) \\
&\quad + \sum_{\mathbf{m}} \left(\frac{1}{2} m^2 + m \right) |u_{\mathbf{m}}|^2 e^{-i\mathbf{m} \cdot \mathbf{R}_1} + \frac{1}{2} k^2 n_k (n_k - 1) e^{-2i\mathbf{k} \cdot \mathbf{R}_1} (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1)^{-2} \\
&\quad \left. + \frac{1}{2} \left(\sum_{\mathbf{m}} \mathbf{m} |u_{\mathbf{m}}|^2 e^{-i\mathbf{m} \cdot \mathbf{R}_1} \right)^2 + \frac{g}{\sqrt{2\Omega}} \sum_{\mathbf{m}} \frac{u_{\mathbf{m}} e^{i\mathbf{m} \cdot \rho}}{\sqrt{\omega_m}} (e^{-i\mathbf{m} \cdot \mathbf{R}_1} + 1) \right]. \tag{137}
\end{aligned}$$

We further obtain the cover integral

$$\begin{aligned}
\langle \Psi_{P_1, n_k} | \Psi_P^{(L)} \rangle &= \frac{(2\pi)^3 \delta(\mathbf{P} - \mathbf{P}_1)}{N_{P_1, 1_k}^* N_P \Omega} \int d\mathbf{R}_1 d\rho \phi_{P_1}^*(\rho) \phi_P(\rho - \mathbf{R}_1) e^{i\mathbf{P} \cdot \mathbf{R}_1 + \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1)} \\
&\times \frac{u_{\mathbf{k}}^{n_k} (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1)^{n_k}}{\sqrt{n_k!}} \tag{138}
\end{aligned}$$

and the expectation value of the Hamiltonian

$$\begin{aligned}
\langle \Psi_{P_1, 1_k} | H | \Psi_{P_1, 1_k} \rangle &= \frac{P_1^2}{2} + \frac{1}{|N_{P_1, 1_k}|^2} \int d\mathbf{R}_1 d\rho \phi_{P_1}^*(\rho) \phi_{P_1}(\rho - \mathbf{R}_1) e^{i(P_1 - \mathbf{k}) \cdot \mathbf{R}_1 + \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1)} \\
&\times \left\{ k^2 |u_{\mathbf{k}}|^2 e^{-i\mathbf{k} \cdot \mathbf{R}_1} \right. \\
&+ \left(2|u_{\mathbf{k}}|^2 (\cos \mathbf{k} \cdot \mathbf{R}_1 - 1) + 1 \right) \left[- \sum_{\mathbf{m}} \mathbf{P}_1 \cdot \mathbf{m} |u_{\mathbf{m}}|^2 e^{-i\mathbf{m} \cdot \mathbf{R}_1} + \sum_{\mathbf{m}} \left(\frac{m^2}{2} + m \right) |u_{\mathbf{m}}|^2 e^{-i\mathbf{m} \cdot \mathbf{R}_1} \right. \\
&\quad \left. + \frac{1}{2} \left(\sum_{\mathbf{m}} \mathbf{m} |u_{\mathbf{m}}|^2 e^{-i\mathbf{m} \cdot \mathbf{R}_1} \right)^2 + \frac{g}{\sqrt{2\Omega}} \sum_{\mathbf{m}} \frac{u_{\mathbf{m}} e^{i\mathbf{m} \cdot \rho}}{\sqrt{\omega_{\mathbf{m}}}} (e^{-i\mathbf{m} \cdot \mathbf{R}_1} + 1) \right] \\
&+ \left(2|u_{\mathbf{k}}|^2 (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1) + 1 \right) \left[- \mathbf{k} \cdot \mathbf{P}_1 + \sum_{\mathbf{m}} \mathbf{k} \cdot \mathbf{m} |u_{\mathbf{m}}|^2 e^{-i\mathbf{m} \cdot \mathbf{R}_1} + \frac{k^2}{2} + k \right] \\
&\left. + \frac{g}{\sqrt{2\Omega}} \left[\frac{u_{\mathbf{k}} e^{i\mathbf{k} \cdot \rho}}{\sqrt{\omega_{\mathbf{k}}}} (e^{-i\mathbf{k} \cdot \mathbf{R}_1} - 1) + \frac{u_{\mathbf{k}}^* e^{-i\mathbf{k} \cdot \rho}}{\sqrt{\omega_{\mathbf{k}}}} (1 - e^{i\mathbf{k} \cdot \mathbf{R}_1}) \right] \right\}. \tag{139}
\end{aligned}$$

Equations (137-139) are valid for arbitrary coupling constants. However, in the weak coupling limit they are significantly simplified as they can be expressed via Fourier transforms. Another significant simplification comes from the fact that the action of one field mode on the system is inversely proportional to the square root of the normalization volume Ω , that is $u_{\mathbf{k}} \sim 1/\sqrt{\Omega}$. Consequently, such terms can be kept only within the sum. Within this approximation, equations (137-139) can be rewritten as

$$\langle \Psi_{P_1, n_k} | \Psi_{\mathbf{P}}^{(L)} \rangle = \frac{(2\pi)^3 \delta(\mathbf{P} - \mathbf{P}_1)}{\Omega} \frac{u_{\mathbf{k}} (\phi_{\mathbf{P}-\mathbf{k}}^2 - \phi_{\mathbf{P}}^2)}{\phi_{\mathbf{P}} \phi_{\mathbf{P}-\mathbf{k}}}, \tag{140}$$

and

$$\begin{aligned}
\langle \Psi_{P_1, n_k} | H | \Psi_{\mathbf{P}}^{(L)} \rangle &= \frac{(2\pi)^3 \delta(\mathbf{P} - \mathbf{P}_1)}{\Omega} \frac{1}{\phi_{\mathbf{P}} \phi_{\mathbf{P}-\mathbf{k}}} \\
&\times \left[\frac{P_1^2}{2} u_{\mathbf{k}} (\phi_{\mathbf{P}-\mathbf{k}}^2 - \phi_{\mathbf{P}}^2) + \left(\frac{k^2}{2} + k - \mathbf{P}_1 \cdot \mathbf{k} \right) u_{\mathbf{k}} \phi_{\mathbf{P}-\mathbf{k}}^2 + \frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{P}-\mathbf{k}} \phi_{\mathbf{P}}}{\sqrt{k}} \right. \\
&- u_{\mathbf{k}} (\mathbf{P}_1 - \mathbf{k}) \sum_{\mathbf{m}} \mathbf{m} |u_{\mathbf{m}}|^2 \phi_{\mathbf{P}-\mathbf{m}-\mathbf{k}}^2 \\
&+ u_{\mathbf{k}} \sum_{\mathbf{m}} \left(\frac{m^2}{2} + m \right) |u_{\mathbf{m}}|^2 \phi_{\mathbf{P}-\mathbf{m}-\mathbf{k}}^2 + \frac{u_{\mathbf{k}}}{2} \sum_{\mathbf{l}, \mathbf{m}} \mathbf{l} \cdot \mathbf{m} |u_{\mathbf{l}}|^2 |u_{\mathbf{m}}|^2 \phi_{\mathbf{P}-\mathbf{l}-\mathbf{m}-\mathbf{k}}^2 \\
&\left. + \frac{g}{\sqrt{2\Omega}} u_{\mathbf{k}} \sum_{\mathbf{m}} \frac{u_{\mathbf{m}}}{\sqrt{m}} \phi_{\mathbf{P}-\mathbf{k}} (\phi_{\mathbf{P}-\mathbf{k}-\mathbf{m}} + \phi_{\mathbf{P}-\mathbf{k}+\mathbf{m}}) - u_{\mathbf{k}} \phi_{\mathbf{P}}^2 \tilde{E}_L^{(0)} \right], \tag{141}
\end{aligned}$$

as well as

$$\begin{aligned}
\langle \Psi_{P_1, n_k} | H | \Psi_{P_1, n_k} \rangle &= \frac{P_1^2}{2} + \frac{1}{\phi_{\mathbf{P}-\mathbf{k}}^2} \left[\phi_{\mathbf{P}-\mathbf{k}}^2 \left(-\mathbf{k} \cdot \mathbf{P}_1 + \frac{k^2}{2} + k \right) - (\mathbf{P}_1 - \mathbf{k}) \cdot \sum_{\mathbf{m}} \mathbf{m} |u_{\mathbf{m}}|^2 \phi_{\mathbf{P}-\mathbf{m}-\mathbf{k}}^2 \right. \\
&+ \sum_{\mathbf{m}} \left(\frac{m^2}{2} + m \right) |u_{\mathbf{m}}|^2 \phi_{\mathbf{P}-\mathbf{m}-\mathbf{k}}^2 + \frac{g}{\sqrt{2\Omega}} \sum_{\mathbf{m}} \frac{u_{\mathbf{m}}}{\sqrt{m}} \phi_{\mathbf{P}-\mathbf{k}} (\phi_{\mathbf{P}-\mathbf{m}-\mathbf{k}} + \phi_{\mathbf{P}+\mathbf{m}-\mathbf{k}}) \\
&\left. + \frac{1}{2} \sum_{\mathbf{l}, \mathbf{m}} \mathbf{l} \cdot \mathbf{m} |u_{\mathbf{l}}|^2 |u_{\mathbf{m}}|^2 \phi_{\mathbf{P}-\mathbf{l}-\mathbf{m}-\mathbf{k}}^2 \right]. \tag{142}
\end{aligned}$$

In equations (140-142) we have used expressions for the normalization constants

$$N_P = \phi_P, \quad N_{P_1, l_k} = \phi_{P_1 - k} \quad (143)$$

and sorted out the energy of the zeroth-order approximation

$$\begin{aligned} \tilde{E}_L^{(0)} = & -P \sum_m m |u_m|^2 \frac{\phi_{P-m}^2}{\phi_P^2} + \frac{1}{2} \sum_{l,m} m \cdot l |u_m|^2 |u_l|^2 \frac{\phi_{P-m-l}^2}{\phi_P^2} \\ & + \sum_m \left(\frac{m^2}{2} + m \right) |u_m|^2 \frac{\phi_{P-m}^2}{\phi_P^2} + \frac{g}{\sqrt{2}\Omega} \sum_m \frac{u_m}{\sqrt{m}} \phi_P \frac{(\phi_{P-m} + \phi_{P+m})}{\phi_P}. \end{aligned} \quad (144)$$

APPENDIX F: SECOND ITERATION FOR THE ENERGY OF THE SYSTEM WITH PARTICLE AT REST

For the evaluation of the second iteration for the particle energy the knowledge of the behavior of different terms in expressions (140-142) is required. In order to determine those, we will carry out the summations over m . For the first sum we can write

$$\begin{aligned} I_k^{(1)} &= \frac{1}{g^2 \phi_k^2} \sum_m m |u_m|^2 \phi_{m+k}^2 = \frac{\phi_0^2}{g^2 \phi_k^2} \frac{g^2}{2(2\pi)^3} \int d\mathbf{m} \begin{pmatrix} m \sin \theta \cos \phi \\ m \sin \theta \sin \phi \\ m \cos \theta \end{pmatrix} \frac{e^{-\frac{m^2}{2\lambda^2}}}{m^3} e^{-\frac{m^2+k^2+2\mathbf{m} \cdot \mathbf{k}}{\lambda^2}} \\ &= \frac{1}{8\pi^2} \frac{k}{k} \int d\mathbf{m} e^{-\frac{3}{2} \frac{m^2}{\lambda^2}} \frac{-2km\lambda^2 \cosh \frac{2km}{\lambda^2} + \lambda^4 \sinh \frac{2km}{\lambda^2}}{2k^2 m^2} \\ &= \frac{\lambda^2}{32\pi^2} \frac{k}{k} \frac{4k - e^{\frac{2}{3} \frac{k^2}{\lambda^2}} \sqrt{6\pi} \lambda \text{Erf} \frac{\sqrt{\frac{2}{3}} k}{\lambda}}{k^2} \xrightarrow{k \rightarrow \infty} -\frac{\lambda^2}{32\pi^2} \frac{k}{k^3} \left(-4k + e^{\frac{2}{3} \frac{k^2}{\lambda^2}} \sqrt{6\pi} \lambda \right) \\ &\sim -\frac{\phi_0^2}{\phi_k^2} \frac{\lambda^3}{32\pi^2} \frac{\sqrt{6\pi}}{k^3} e^{-\frac{1}{3} \frac{k^2}{\lambda^2}} k \end{aligned} \quad (145)$$

as for the second

$$\begin{aligned} I_k^{(2)} &= \frac{1}{g^2 \phi_k^2} \sum_m \left(\frac{m^2}{2} + m \right) |u_m|^2 \phi_{m+k}^2 = \frac{1}{2(2\pi)^3} \int \frac{d\mathbf{m}}{m^2} \left(1 + \frac{m}{2} \right) e^{-\frac{3}{2} \frac{m^2}{\lambda^2}} e^{-\frac{2\mathbf{m} \cdot \mathbf{k}}{\lambda^2}} \\ &= \frac{\lambda^2}{8\pi^2} \int d\mathbf{m} \left(1 + \frac{m}{2} \right) e^{-\frac{3}{2} \frac{m^2}{\lambda^2}} \frac{\sinh \frac{2km}{\lambda^2}}{km} \\ &= \frac{\lambda^2}{96k} \left(\frac{\sqrt{6\pi} \lambda e^{\frac{2}{3} \frac{k^2}{\lambda^2}} \text{Erf} \frac{\sqrt{\frac{2}{3}} k}{\lambda}}{\pi^2} + \frac{6 \text{Erfi} \frac{\sqrt{\frac{2}{3}} k}{\lambda}}{\pi} \right) \\ &\xrightarrow{k \rightarrow \infty} \frac{\lambda^3}{96k^2 \pi^{\frac{3}{2}}} \left(\sqrt{6} e^{\frac{2}{3} \frac{k^2}{\lambda^2}} (3+k) - \frac{6i k \sqrt{\pi}}{\lambda} \right) \sim \frac{\phi_0^2}{\phi_k^2} \frac{\lambda^3}{96\pi^2} \frac{\sqrt{6\pi}}{k} e^{-\frac{1}{3} \frac{k^2}{\lambda^2}} \end{aligned} \quad (146)$$

and for the third

$$\begin{aligned}
I_{\mathbf{k}}^{(3)} &= \frac{1}{g^2 \phi_{\mathbf{k}}^2} \phi_{\mathbf{k}} \frac{g}{\sqrt{2\Omega}} \sum_{\mathbf{m}} \frac{u_{\mathbf{m}}}{\sqrt{m}} (\phi_{\mathbf{m}+\mathbf{k}} + \phi_{\mathbf{m}-\mathbf{k}}) = -\frac{1}{2(2\pi)^3} \int \frac{d\mathbf{m}}{m^2} e^{-\frac{3}{4} \frac{m^2}{\lambda^2}} \left(e^{-\frac{\mathbf{m} \cdot \mathbf{k}}{\lambda^2}} + e^{\frac{\mathbf{m} \cdot \mathbf{k}}{\lambda^2}} \right) \\
&= -\frac{\lambda^2}{2\pi^2} \int d\mathbf{m} e^{-\frac{3}{4} \frac{m^2}{\lambda^2}} \frac{\sinh \frac{\mathbf{k} \cdot \mathbf{m}}{\lambda^2}}{km} = -\frac{\lambda^2}{4\pi} \frac{\text{Erfi} \frac{k}{\sqrt{3}\lambda}}{k} \\
&\xrightarrow{k \rightarrow \infty} -\frac{\lambda^2}{4k^2 \pi^{\frac{3}{2}}} (-ik \sqrt{\pi} + \sqrt{3} \lambda e^{\frac{k^2}{3\lambda^2}}) \sim -\frac{\phi_0^2}{\phi_{\mathbf{k}}^2} \frac{\lambda^3 \sqrt{3\pi}}{4\pi^2} \frac{e^{-\frac{2}{3} \frac{k^2}{\lambda^2}}}{k^2}.
\end{aligned} \tag{147}$$

In the above expressions $\text{Erf}(x) = 2/\sqrt{\pi} \int_0^x e^{-z^2} dz$ and $\text{Erfi}(x) = -i\text{Erf}(ix)$ are the error function and the imaginary error functions, respectively.

Consequently, we can introduce the abbreviations, which were used in equations (55-59) of the manuscript, namely

$$I_{\mathbf{k}} = \mathbf{k} \cdot \mathbf{I}_{\mathbf{k}}^{(1)} + I_{\mathbf{k}}^{(2)} + I_{\mathbf{k}}^{(3)} \underset{k \rightarrow \infty}{\sim} -\frac{\lambda^3 \sqrt{6\pi}}{48\pi^2} \frac{e^{\frac{2k^2}{\lambda^2}}}{k}. \tag{148}$$

After the determination of the asymptotic behavior of the different terms, we can find the second iteration for the energy of the system

$$E^{(2)} = \frac{A}{B}. \tag{149}$$

For the numerator we have

$$\begin{aligned}
A &= E_L^{(0)} + \sum_{P_1, \{n_{\mathbf{k}}\}} C_{P_1, \{n_{\mathbf{k}}\}}^{(1)} \langle \Psi_P^{(L)} | H | \Psi_{P_1, \{n_{\mathbf{k}}\}} \rangle \\
&= E_L^{(0)} + \sum_{\mathbf{k} < \mathbf{k}_0} \frac{-\left(u_{\mathbf{k}} \frac{\phi_{\mathbf{k}}}{\phi_0} \left(\frac{k^2}{2} + k\right) + \frac{g}{\sqrt{2\Omega}} \frac{1}{\sqrt{k}}\right)^2}{\left(\frac{k^2}{2} + k\right)} + \sum_{\mathbf{k} > \mathbf{k}_0} \frac{\left(-\frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{k}} \phi_0}{\sqrt{k}}\right) (-E_0 u_{\mathbf{k}})}{\phi_{\mathbf{k}}^2 I_{\mathbf{k}} g^2}
\end{aligned} \tag{150}$$

and for the denominator we obtain

$$\begin{aligned}
B &= 1 + \sum_{P_1, \{n_{\mathbf{k}}\}} C_{P_1, \{n_{\mathbf{k}}\}}^{(1)} \langle \Psi_P^{(L)} | \Psi_{P_1, \{n_{\mathbf{k}}\}} \rangle \\
&= 1 + \sum_{\mathbf{k} < \mathbf{k}_0} \frac{-\left(u_{\mathbf{k}} \frac{\phi_{\mathbf{k}}}{\phi_0} \left(\frac{k^2}{2} + k\right) + \frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{k}}}{\phi_0 \sqrt{k}}\right) u_{\mathbf{k}} \left(1 - \frac{\phi_0^2}{\phi_{\mathbf{k}}^2}\right)}{\left(\frac{k^2}{2} + k\right)} + \sum_{\mathbf{k} > \mathbf{k}_0} \frac{\left(-\frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{k}} \phi_0}{\sqrt{k}}\right) u_{\mathbf{k}} \left(\frac{\phi_{\mathbf{k}}^2}{\phi_0^2} - 1\right)}{\phi_{\mathbf{k}}^2 I_{\mathbf{k}} g^2}.
\end{aligned} \tag{151}$$

Further explicit calculations yield

$$\begin{aligned}
A &= E_L^{(0)} + \sum_{\mathbf{k} < \mathbf{k}_0} \frac{-\left(u_{\mathbf{k}} \frac{\phi_{\mathbf{k}}}{\phi_0} \left(\frac{k^2}{2} + k\right) + \frac{g}{\sqrt{2\Omega}} \frac{1}{\sqrt{k}}\right)^2}{\left(\frac{k^2}{2} + k\right)} + \sum_{\mathbf{k} > \mathbf{k}_0} \frac{\left(-\frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{k}} \phi_0}{\sqrt{k}}\right) (-E_0 u_{\mathbf{k}})}{\phi_{\mathbf{k}}^2 I_{\mathbf{k}} g^2} \\
&= E_L^{(0)} - \sum_{\mathbf{k} < \mathbf{k}_0} \left(u_{\mathbf{k}}^2 \left(\frac{\phi_{\mathbf{k}}}{\phi_0} \right)^2 \left(\frac{k^2}{2} + k \right) + \frac{2g}{\sqrt{2\Omega}} \frac{u_{\mathbf{k}} \phi_{\mathbf{k}}}{\sqrt{k} \phi_0} + \frac{g^2}{2\Omega} \frac{1}{k^2 (k/2 + 1)} \right) \\
&\quad + \frac{g}{\sqrt{2\Omega}} \frac{E_L^{(0)}}{g^2} \sum_{\mathbf{k} > \mathbf{k}_0} \frac{\phi_{\mathbf{k}} \phi_0 u_{\mathbf{k}}}{\sqrt{k} \phi_{\mathbf{k}}^2 I_{\mathbf{k}}}
\end{aligned}$$

$$\begin{aligned}
&= E_L^{(0)} - \sum_{\mathbf{k} < \mathbf{k}_0} \left(u_{\mathbf{k}}^2 \left(\frac{\phi_{\mathbf{k}}}{\phi_0} \right)^2 \left(\frac{k^2}{2} + k \right) + \frac{2g}{\sqrt{2\Omega}} \frac{u_{\mathbf{k}}}{\sqrt{k}} \frac{\phi_{\mathbf{k}}}{\phi_0} \right) - \frac{g^2}{2\pi^2} \ln \left(\frac{k_0}{2} + 1 \right) \\
&+ \frac{E_L^{(0)}}{2(2\pi)^3} \int_{\mathbf{k}_0}^{\infty} d\mathbf{k} \frac{e^{-\frac{k^2}{4\lambda^2}}}{k^2} \frac{e^{-\frac{k^2}{2\lambda^2}}}{\frac{\lambda^3 \sqrt{6\pi}}{48\pi^2} e^{-\frac{1}{2} \frac{k^2}{\lambda^2}} k} \\
&= E_L^{(0)} - \sum_{\mathbf{k} < \mathbf{k}_0} \left(u_{\mathbf{k}}^2 \left(\frac{\phi_{\mathbf{k}}}{\phi_0} \right)^2 \left(\frac{k^2}{2} + k \right) + \frac{2g}{\sqrt{2\Omega}} \frac{u_{\mathbf{k}}}{\sqrt{k}} \frac{\phi_{\mathbf{k}}}{\phi_0} \right) - \frac{g^2}{2\pi^2} \ln \left(\frac{k_0}{2} + 1 \right) \\
&+ E_L^{(0)} \frac{12}{\lambda^3 \sqrt{6\pi}} \int_{k_0}^{\infty} dk k e^{-\frac{5}{12} \frac{k^2}{\lambda^2}} \\
&= E_L^{(0)} - \left[\frac{g^2 \lambda}{24\pi^2} \left(\sqrt{6\pi} \text{Erf} \left(\frac{\sqrt{\frac{3}{2}} k_0}{\lambda} \right) + \lambda - \lambda e^{-\frac{3k_0^2}{2\lambda^2}} \right) - \frac{g^2 \lambda}{2 \sqrt{3} \pi^{3/2}} \text{Erf} \left(\frac{\sqrt{3} k_0}{2\lambda} \right) \right] \\
&- \frac{g^2}{2\pi^2} \ln \left(\frac{k_0}{2} + 1 \right) + E_L^{(0)} \frac{12 \sqrt{6\pi}}{5\lambda\pi} e^{-\frac{5k_0^2}{12\lambda^2}} \tag{152}
\end{aligned}$$

and

$$\begin{aligned}
B &= 1 + \sum_{\mathbf{k} < \mathbf{k}_0} \frac{-\left(u_{\mathbf{k}} \frac{\phi_{\mathbf{k}}^2}{\phi_0^2} \left(\frac{k^2}{2} + k \right) + \frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{k}}}{\phi_0} \sqrt{k} \right) u_{\mathbf{k}} \left(1 - \frac{\phi_0^2}{\phi_{\mathbf{k}}^2} \right)}{\left(\frac{k^2}{2} + k \right)} + \sum_{\mathbf{k} > \mathbf{k}_0} \frac{\left(-\frac{g}{\sqrt{2\Omega}} \frac{\phi_{\mathbf{k}} \phi_0}{\sqrt{k}} \right) u_{\mathbf{k}} \left(\frac{\phi_{\mathbf{k}}^2}{\phi_0^2} - 1 \right)}{\phi_{\mathbf{k}}^2 I_{\mathbf{k}} g^2} \\
&= 1 + \frac{g^2}{2(2\pi)^3} \int_0^{k_0} dk \frac{k^2}{\lambda^2} \left[\frac{e^{-\frac{3k^2}{2\lambda^2}}}{k^3} - \frac{e^{-\frac{3k^2}{4\lambda^2}}}{k^2(k^2/2 + k)} \right] - \frac{12}{\lambda^5 \sqrt{6\pi}} \int_{k_0}^{\infty} dk k^3 e^{-\frac{5}{12} \frac{k^2}{\lambda^2}} \\
&= 1 + \frac{g^2}{12\pi^2} \left(1 - e^{-\frac{3}{2} \frac{k_0^2}{\lambda^2}} \right) - g^2 f \left(\frac{k_0}{\lambda} \right) - \frac{144 \sqrt{6\pi}}{25\lambda\pi} \left(1 + \frac{5}{12} \frac{k_0^2}{\lambda^2} \right) e^{-\frac{5k_0^2}{12\lambda^2}} \tag{153}
\end{aligned}$$

with

$$f(x) = \frac{1}{4\pi^2} \int_0^x \frac{tdt}{1+t/2} e^{-\frac{3}{4} t^2}.$$

-
- [1] F. J. Dyson, *Phys. Rev.* **75**, 1736 (1949); *Phys. Rev.* **85**, 631 (1952); M. Gell-Mann and F. E. Low, *Phys. Rev.* **95**, 1300 (1954); E. Stueckelberg and A. Petermann, *Helvetica Physica Acta* **26**, 499 (1953).
- [2] G. 't Hooft, *Nuclear Physics B* **33**, 173 (1971); *Nuclear Physics B* **35**, 167 (1971); G. 't Hooft and M. Veltman, *Nuclear Physics B* **44**, 189 (1972).
- [3] R. P. Feynman, *Science* **153**, 699 (1966), <http://www.sciencemag.org/content/153/3737/699.full.pdf>.
- [4] P. A. M. Dirac, *AIP Conference Proceedings* **74**, 129 (1981).
- [5] P. Dirac, *The Principles of Quantum Mechanics*, International series of monographs on physics (Clarendon Press, 1981).
- [6] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.* **108**, 1175 (1957).
- [7] K. Cahill, *Phys. Rev. D* **88**, 125014 (2013); *Phys. Rev. D* **87**, 065024 (2013).
- [8] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*, Course of theoretical physics (Butterworth-Heinemann, 1982).
- [9] V. Chung, *Phys. Rev.* **140**, B1110 (1965).
- [10] P. Kulish and L. Faddeev, *Theoretical and Mathematical Physics* **4**, 745 (1970).

- [11] D. Zwanziger, *Phys. Rev. D* **11**, 3481 (1975).
- [12] T. W. B. Kibble, *Phys. Rev.* **173**, 1527 (1968); *Phys. Rev.* **174**, 1882 (1968); *Phys. Rev.* **175**, 1624 (1968).
- [13] I. D. Feranchuk, A. Ivanov, V. H. Le, and A. P. Ulyanenko, *Non-perturbative Description of Quantum Systems*, Lecture Notes in Physics (Springer International Publishing, 2015).
- [14] I. Feranchuk, L. Komarov, I. Nichipor, and A. Ulyanenko, *Annals of Physics* **238**, 370 (1995).
- [15] I. D. Feranchuk and O. D. Skoromnik, *Phys. Rev. A* **82**, 052703 (2010); V. G. Baryshevskii, I. D. Feranchuk, and P. B. Kats, *Phys. Rev. A* **70**, 052701 (2004).
- [16] W. Healy, *Non-relativistic quantum electrodynamics* (Academic Press, 1982).
- [17] Y. Toyozawa, *Progress of Theoretical Physics* **26**, 29 (1961).
- [18] J. Smit, *Introduction to Quantum Fields on a Lattice*, Cambridge Lecture Notes in Physics (Cambridge University Press, 2002).
- [19] H. Spohn, *Letters in Mathematical Physics* **44**, 9 (1998); *Communications in Mathematical Physics* **123**, 277 (1989).
- [20] A. Amann, *Annals of Physics* **208**, 414 (1991); A. Arai and M. Hirokawa, *Journal of Functional Analysis* **151**, 455 (1997).
- [21] E. Nelson, *Journal of Mathematical Physics* **5**, 1190 (1964).
- [22] V. Bach, J. Fröhlich, and I. M. Sigal, *Advances in Mathematics* **137**, 299 (1998); V. Bach, T. Chen, J. Fröhlich, and I. M. Sigal, *Journal of Functional Analysis* **243**, 426 (2007); T. Chen, *Journal of Functional Analysis* **254**, 2555 (2008).
- [23] H. Fröhlich, *Advances in Physics* **3**, 325 (1954), <http://dx.doi.org/10.1080/00018735400101213>.
- [24] B. Gerlach and H. Löwen, *Rev. Mod. Phys.* **63**, 63 (1991); T. Mitra, A. Chatterjee, and S. Mukhopadhyay, *Physics Reports* **153**, 91 (1987).
- [25] R. P. Feynman, *Phys. Rev.* **97**, 660 (1955).
- [26] H. Spohn, *Annals of Physics* **175**, 278 (1987).
- [27] A. Messiah, *Quantum Mechanics*, Quantum Mechanics No. v. 2 (North-Holland, 1981).
- [28] I. Feranchuk and L. Komarov, *Physics Letters A* **88**, 211 (1982).
- [29] M. O. Scully and S. Zubairy, *Quantum Optics* (Cambridge University Press, 1997).
- [30] I. D. Feranchuk, S. I. Fisher, and L. I. Komarov, *Journal of Physics C: Solid State Physics* **17**, 4309 (1984).
- [31] J. C. Collins, *Renormalization* (Cambridge University Press, 1984).
- [32] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-relativistic Theory*, A-W series in advanced physics (Pergamon Press, 1965).