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Abstract. The plane motion of thin layers of viscous immiscible liquids on the outer surface of a horizontal cylinder rotating at a constant angular velocity in a field of gravity and inertia is explored. In the case of sufficiently slow motion, neglecting the inertial terms of the Navier-Stokes equations, circumferential and radial velocity components were obtained, as well as an interrelated system of evolution equations for the inner and outer layers in a gravitational field. The problem of determining the type of surfaces of two layers in the case of unsteady motion of liquids on the outer surface of a rotating cylindrical shell is solved.

1. Introduction

The development of a number of promising technologies, which are based on film flows on rotating surfaces in the field of centrifugal forces, is of interest to study the movement of two immiscible layers of viscous liquids on the outer surface of a rotating cylinder. The results of the study of such a flow can be used in the technology of applying two-layer coatings on cylindrical surfaces with moderate rotation of the cylinder and the formation of composite metal fibers from the melt during its rapid rotation.

The article [1] gives a solution to the flat problem of determining the type of surfaces of two layers in the case of steady motion of liquids on the outer surface of a rotating cylindrical shell. In the work [2], the complete formulation of the problem of the motion of two immiscible flat layers of viscous liquids on the inner surface of a horizontal cylinder is given and the problem is solved in the case of a flat steady motion in a gravity field with moderate rotation of the cylinder. The flat perturbed motion of one thin layer on a rotating cylinder is considered in [3, 4]. The results of experimental studies of single-layer flow are described in [3, 5]. The perturbed motion of a non-thin layer of a viscous fluid on the outer surface of a rotating cylinder with full allowance for inertial forces was explored in [6].

2. Problem Statement

Consider in the stationary polar coordinate system O, y, θ the case of a flat perturbed motion of two thin layers (films) of viscous immiscible liquids on the outer surface of a rotating cylinder. The flow pattern is shown in Figure 1.

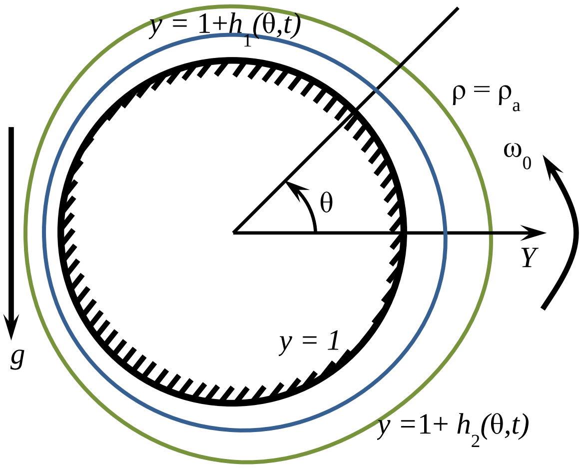

Figure 1. Motion pattern of fluid layers on the outer surface of a rotating cylinder. ω_0 – rotational speed of cylinder, g – gravity acceleration, p_a – unperturbed environmental pressure.

Fluid motion is described by the Navier-Stokes, continuity, unknown boundary line and free surface equations [2]. The boundary conditions express sticking to the moving boundary, the equality of velocities, tangential and normal forces at the boundary line of two liquids $y=1+h_1(\theta,t)$, the absence of tangential stresses at the free boundary $y=1+h_2(\theta,t)$. At the time $t=0$ initial conditions are specified – initial surface forms and initial velocity distribution, and also the condition of periodicity in θ with a period of 2π.

The equations of motion and the boundary conditions are dimensionless along the cylinder radius R_0, its rotational speed $\omega_0 R_0$ and the density of the inner fluid layer ρ_1. They contain dimensionless parameters – Reynolds $Re_i=R_0^2\omega_0 v_i^{-1}$, Froude $Fr=R_0\omega_0^2 g^{-1}$, Weber $We_i=\rho_1 R_0^3\omega_0^2\sigma_i^{-1}$ numbers, density ratio $k=\rho_1\rho_2^{-1}$. Here v_i – kinematic viscosity coefficients, g – gravity acceleration, σ_i – surface tension coefficients for liquids. The index i=1 everywhere denotes the values of the inner layer with the boundary $y=1+h_1$, index i=2 – the outer layer with a free boundary $y=1+h_2$.

We introduce a small parameter ε, which has the meaning of the ratio of the average film thickness to the radius of the cylinder. The starting point for deriving the thin film equations is the idea [4]:

$$h_i=\varepsilon H_i, \qquad y=1+\varepsilon r, \qquad v_i=\varepsilon V_i, \qquad w_i=W_i. \tag{1}$$

Substituting relations (1) into the system of equations, which is the formulation of the problem, from the Navier-Stokes equation in the projection on the radial direction, taking into account the boundary conditions, we determine the pressure $p_i(r,\theta,t)$. Substituting the obtained p_i into the Navier-Stokes equations in the circumferential direction, we obtain the equations:

$$\mathrm{Re}_1\varepsilon^2\left(\frac{\partial W_1}{\partial t}+V_1\frac{\partial W_1}{\partial r}+W_1\frac{\partial W_1}{\partial\theta}+O(\varepsilon)\right)=\frac{\partial^2 W_1}{\partial r^2}+O(\varepsilon)+We_1^{-1}\,\mathrm{Re}_1\,\varepsilon^3\left(\frac{\partial H_1}{\partial\theta}+\frac{\partial^3 H_1}{\partial\theta^3}\right)$$
$$+We_2^{-1}\,\mathrm{Re}_1\,\varepsilon^3\left(\frac{\partial H_2}{\partial\theta}+\frac{\partial^3 H_2}{\partial\theta^3}\right)-Fr^{-1}\,\mathrm{Re}_1\,\varepsilon^2(\cos\theta+O(\varepsilon)), \tag{2}$$

$$\mathrm{Re}_2\varepsilon^2\left(\frac{\partial W_2}{\partial t}+V_2\frac{\partial W_2}{\partial r}+W_2\frac{\partial W_2}{\partial\theta}+O(\varepsilon)\right)=\frac{\partial^2 W_2}{\partial r^2}+O(\varepsilon)+kWe_2^{-1}\,\mathrm{Re}_2\,\varepsilon^3\left(\frac{\partial H_2}{\partial\theta}+\frac{\partial^3 H_2}{\partial\theta^3}\right)$$
$$-Fr^{-1}\,\mathrm{Re}_2\,\varepsilon^2(\cos\theta+O(\varepsilon)). \tag{3}$$

We introduce the notation: $Re_i\varepsilon^2=\kappa_i$, $We_i^{-1}Re_i\varepsilon^3=\chi_i$, $Fr^{-1}Re_i\varepsilon^2=\mu_i$, i=1,2. Suppose that in the process of approaching transition as $\varepsilon\to0$ dimensions κ_i, χ_i, μ_i tend to finite values. Physically, this assumption means that in equations (2), (3) the forces of inertia, viscosity, capillarity and gravity with small ε have the same order. Approaching to the limit in (2) – (3) as $\varepsilon\to0$, we obtain the equations:

$$\kappa_1\left(\frac{\partial W_1}{\partial t}+V_1\frac{\partial W_1}{\partial r}+W_1\frac{\partial W_1}{\partial\theta}\right)=\frac{\partial^2 W_1}{\partial r^2}-\mu_1\cos\theta+\chi_1\left(\frac{\partial H_1}{\partial\theta}+\frac{\partial^3 H_1}{\partial\theta^3}\right)+\frac{We_2^{-1}}{We_1^{-1}}\chi_1\left(\frac{\partial H_2}{\partial\theta}+\frac{\partial^3 H_2}{\partial\theta^3}\right), \quad (4)$$

$$\kappa_2\left(\frac{\partial W_2}{\partial t}+V_2\frac{\partial W_2}{\partial r}+W_2\frac{\partial W_2}{\partial\theta}\right)=\frac{\partial^2 W_2}{\partial r^2}-\mu_2\cos\theta+k\chi_2\left(\frac{\partial H_2}{\partial\theta}+\frac{\partial^3 H_2}{\partial\theta^3}\right). \quad (5)$$

Substituting equations (1) into the equations of continuity, free surface and boundary conditions, at $\varepsilon\to0$ we get the following relations:

$$\partial V_i/\partial r+\partial W_i/\partial\theta=0, \quad (6)$$

$$\partial H_i/\partial t+W_i\partial H_i/\partial\theta=V_i, \qquad r=H_i(\theta,t), \quad (7)$$

$$V_1=0,\ \ W_1=1,\ \ r=0, \quad (8)$$

$$V_1=V_2, W_1=W_2,\quad Re_2Re_1^{-1}k\partial W_1/\partial r=\partial W_2/\partial r,\quad r=H_1(\theta,t), \quad (9)$$

$$\partial W_2/\partial r=0,\quad r=H_2(\theta,t), \quad (10)$$

To the boundary conditions (6) – (10) for the system (4) – (5), we add periodicity conditions for V_i, W_i, H_i on to θ with a period of 2π and initial conditions:

$$H_i=H_{i0}(\theta),\qquad W_i=W_{i0}(r,\theta),\quad t=0, \quad (11)$$

where H_{i0}, W_{i0} – specified functions. The initial condition for W_i at t=0 uniquely defines the initial value of the functions V_i.

Equations (4) – (11) form a boundary value problem, the solution of which is interpreted as motion in two thin layers of liquids on the outer surface of a rotating cylinder with unknown boundary lines and the outer flow region.

3. Derivation of Evolution Equations

Consider the case of slow rotation, when you can neglect the inertial terms, i.e. κ_i=0. For small changes in the surface, we also consider that the contribution of the surface tension forces is small, i.e. χ_i=0. It should be noted that the research method proposed below is applicable when taking into account surface forces, but the evolution equations will be more complicated. Equations (4) – (5) are reduced to:

$$\partial^2 W_i/\partial r^2-\mu_i\cos\theta=0,\quad i=1{,}2. \quad (12)$$

We introduce the notation: $l=Re_1Re_2^{-1}\mu_2k^{-1}$. Solving the system (12) taking into account the boundary conditions (8) – (10), we find the circumferential velocity components W_i:

$$W_1=1+2^{-1}r^2\mu_1\cos\theta-\mu_1H_1r\cos\theta+l(H_1-H_2)r\cos\theta, \quad (13)$$

$$W_2=1+2^{-1}\mu_2r^2\cos\theta-\mu_2H_2r\cos\theta+\left[l-2^{-1}(\mu_1+\mu_2)\right]H_1^2\cos\theta+(\mu_2-l)H_1H_2\cos\theta. \quad (14)$$

From the continuity equation (6) taking into account (13) – (14) we obtain the radial components:

$$V_1=\frac{1}{6}\mu_1r^3\sin\theta+\frac{1}{2}\left[l(H_1-H_2)-\mu_1H_1\right]r^2\sin\theta+\frac{1}{2}\left[\mu_1\frac{\partial H_1}{\partial\theta}+l\left(\frac{\partial H_2}{\partial\theta}-\frac{\partial H_1}{\partial\theta}\right)\right]r^2\cos\theta, \quad (15)$$

$$V_2 = \frac{1}{6}\mu_2 r^3 \sin\theta + \left[\frac{1}{6}(\mu_1 + 2\mu_2) - \frac{1}{2}l\right]H_1^3 \sin\theta + \frac{1}{2}(l-\mu_2)H_1^2 H_2 \sin\theta + (\mu_2 - l)H_1 H_2 \frac{\partial H_1}{\partial\theta}\cos\theta$$
$$-\frac{1}{2}\mu_2 H_2 r^2 \sin\theta + \frac{1}{2}\mu_2 \frac{\partial H_2}{\partial\theta} r^2 \cos\theta + \left(\frac{3}{2}l - \frac{1}{2}\mu_1 + \mu_2\right)H_1^2 \frac{\partial H_1}{\partial\theta}\cos\theta$$
$$\frac{1}{2}(\mu_2 - l)H_1^2 \frac{\partial H_2}{\partial\theta}\cos\theta + \left(l - \frac{1}{2}(\mu_1+\mu_2)\right)H_1^2 r\sin\theta + (\mu_2 - l)H_1 H_2 r \sin\theta$$
$$+(l-\mu_2)H_2 \frac{\partial H_1}{\partial\theta} r\cos\theta + (\mu_1+\mu_2-2l)H_1 \frac{\partial H_1}{\partial\theta} r\cos\theta + (l-\mu_2)H_1 \frac{\partial H_2}{\partial\theta} r\cos\theta. \quad (16)$$

Substituting the found values of the speeds V_i, W_i (13) – (16) into (7), we get the system:

$$\frac{\partial H_1}{\partial t} + \left(\frac{\mu_1}{3} - \frac{1}{2}l\right)H_1^3 \sin\theta + lH_1^2 H_2 \sin\theta + \frac{\partial H_1}{\partial\theta} + \left(\frac{3}{2}l - \mu_1\right)H_1^2 \frac{\partial H_1}{\partial\theta}\cos\theta - lH_1H_2\frac{\partial H_1}{\partial\theta}\cos\theta$$
$$-\frac{1}{2}lH_1^2 \frac{\partial H_2}{\partial\theta}\cos\theta = 0, \quad (17)$$

$$\frac{\partial H_2}{\partial t} + \left(\frac{1}{2}l - \frac{\mu_1}{6} - \frac{\mu_2}{3}\right)H_1^3 \sin\theta + \left(\frac{1}{2}\mu_1 + \mu_2 - \frac{3}{2}l\right)H_1^2\left(H_2\sin\theta + \frac{\partial H_1}{\partial\theta}\cos\theta\right)$$
$$+(l-\mu_2)H_1H_2^2 \sin\theta + \frac{1}{3}\mu_2 H_2^3 \sin\theta + \frac{\partial H_2}{\partial\theta} + (\mu_2 - l)H_2^2 \frac{\partial H_1}{\partial\theta}\cos\theta$$
$$+\left(\frac{3}{2}l - \frac{1}{2}\mu_1 - \mu_2\right)H_1^2 \frac{\partial H_2}{\partial\theta}\cos\theta - \mu_2 H_2^2 \frac{\partial H_2}{\partial\theta}\cos\theta$$
$$+(3l - \mu_1 - 2\mu_2)H_1H_2\frac{\partial H_1}{\partial\theta}\cos\theta + (2\mu_2 - 2l)H_1H_2\frac{\partial H_2}{\partial\theta}\cos\theta = 0. \quad (18)$$

The resulting system of differential equations (17) – (18) is the evolution equations for a two-layer film flow of viscous immiscible liquids on the outer surface of a horizontal cylinder that rotates with constant angular velocity in a gravity field, when the inertial terms of the motion equations can be neglected.

4. Solving Evolution Equations

The numerical method for studying system (17) – (18) with initial conditions (11) and periodicity conditions along the angular coordinate with a period of 2π is based on the method of straight lines followed by integration using fourth-order Runge-Kutta formulas. Consider the numerical solution of the system (17) – (18). At the initial time, the films have a constant thickness h_{10} and h_{20}. Now we compare the problem solutions in two cases: in the first case, the viscosity and density of the inner layer is 2 times greater than that of the outer; in the second case, on the contrary, the outer layer is more viscous and dense. Solutions obtained on the time interval $[0, 200\pi]$, correspond to 100 full turns of the cylinder. The flow parameters for both solutions are presented in the Table 1.

Table 1. Dimensionless flow parameters.

Case	ε	h_{10}	h_{20}	$k=\rho_1\rho_2^{-1}$	Re_1	Re_2	Fr	μ_1	μ_2
«1»	0.01	0.01	0.02	2	13.09	26.18	0.0987	0.01326	0.02653
«2»	0.01	0.01	0.02	0.5	26.18	13.09	0.0987	0.02653	0.01326

In the first case, when the viscosity and density of the inner layer is greater than that of the outer, changes in the surface are significantly less than in the second case. The radial velocity components have values of the order of 10^{-4}, which is much less than the velocities in the circumferential direction.

The circumferential velocity components reach their maximum values on the descent of the liquid from the cylinder surface with $\theta \in [\pi/2, 3\pi/2]$, and the layer thicknesses in this area, on the contrary, take the minimum values. The maximum thickness of the layers is achieved on the rise of the fluid through the cylinder in a symmetric region.

Figure 2 shows the differences in changes in the shapes of the surfaces of the inner layers $y=h_1(\theta,t)$ for two different sets of flow parameters from Table 1, and Figure 3 shows the speeds w_1 of the motion of the same layers in the circumferential direction.

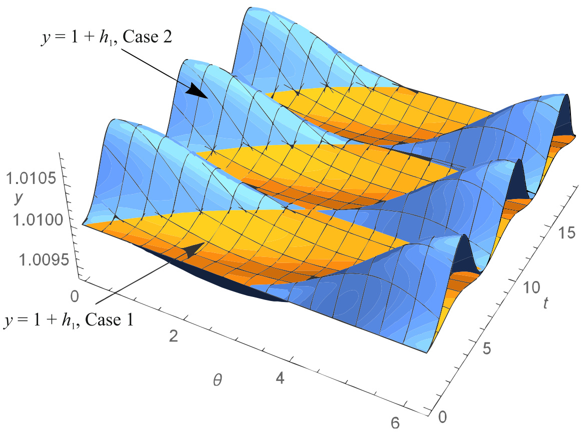

Figure 2. Three-dimensional plot of the surfaces of the inner layers of liquids $y=1+h_1(\theta,t)$.

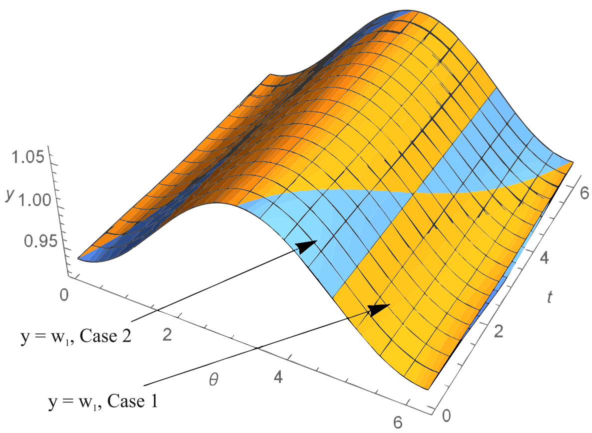

Figure 3. The circumferential velocity components $y=w_1(\theta,t)$.

Table 2 shows the maximum and minimum values of h_i, as well as the amplitudes of changes in the forms of the surfaces of the layers for the two variants of liquids from the Table 1.

Table 2. Extreme and amplitude values.

Case	Max h_1	Min h_1	Amplitude h_1	Max h_2	Min h_2	Amplitude h_2
«1»	0.01016	0.00985	0.00031	0.02063	0.01943	0.0012
«2»	0.01083	0.00937	0.00146	0.02259	0.01815	0.00444

The problem solutions are carried out for thin layers, therefore the absolute value of the layer deviations from the initial value is very small. To display the effect on the change in the form of the surface of a more dense and viscous external fluid, we give the ratios of the deviations of the maxima of the layers from the initial values h_{10} и h_{20}. In the second case, the maximum of the inner layer h_1 deviates from the initial value of the layer thickness by an amount 5.2 times larger than in the first. The outer layer in the second case deviates from the h_{20} value by an amount 4.1 times larger than in the first case.

Let us construct the two-dimensional plots at fixed points in time. In Figure 4, the forms of the surfaces $y=1+h_i$ for both variants of the calculation at a fixed time $t=\pi$ are presented in the same graphic area. Now we demonstrate how the form of the surface of the layer changes over time in the period of one turn. Figure 5 shows the surface form of the outer layer $y=1+h_2$ for the first variant of dimensionless flow criteria from Table 1 at fixed points in the first turn of the cylinder $[0, 2\pi]$.

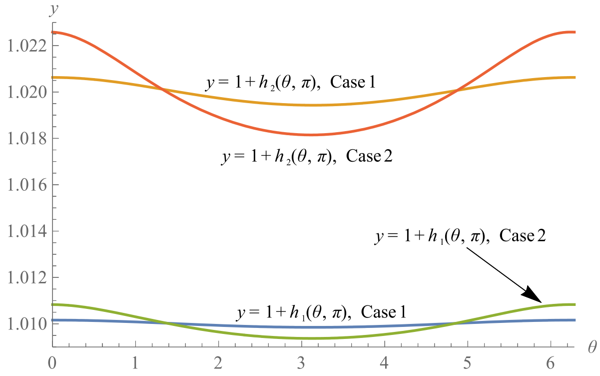

Figure 4. Forms of surfaces of the inner and outer layers y=1+h_i(θ,t) with t=π.

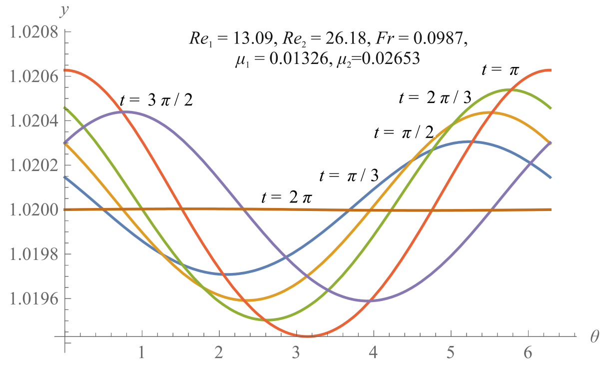

Figure 5. Forms of the surface of the outer layer y=1+h_2(θ,t) at different times t.

We will also construct plots of each of the surfaces of the layers in different plot areas. Figure 6 shows the forms of the surfaces of only the inner layer y=1+h_1 for two different variants of flow parameters from Table 1 at a fixed time t=π/2, in Figure 7 – only the outer layer y=1+h_2 at time t=3π/2.

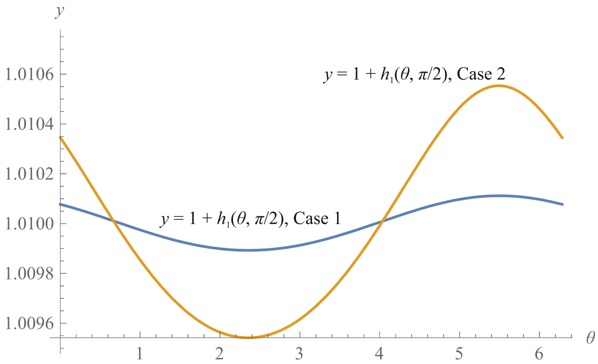

Figure 6. Forms of surfaces of the inner layers y=1+h_1(θ,π/2).

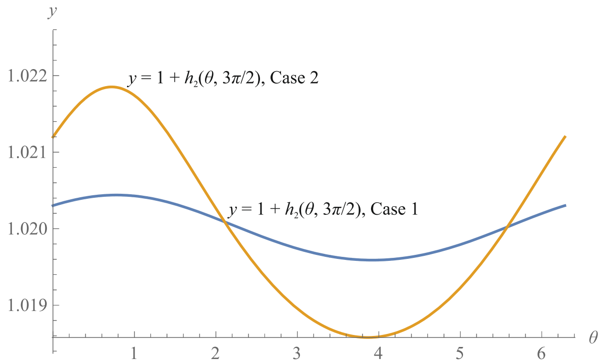

Figure 7. Forms of surfaces of the outer layers y=1+h_2(θ,3π/2).

We construct the surface forms of the layers in several complete turns. Figure 8 shows the plots of the surfaces of the outer layer y=1+h_2(θ,t) at times t={0, 2π, 4π, 6π, 8π, 10π}: (a) – for the first case of flow characteristics from table 1, (b) – for the second case.

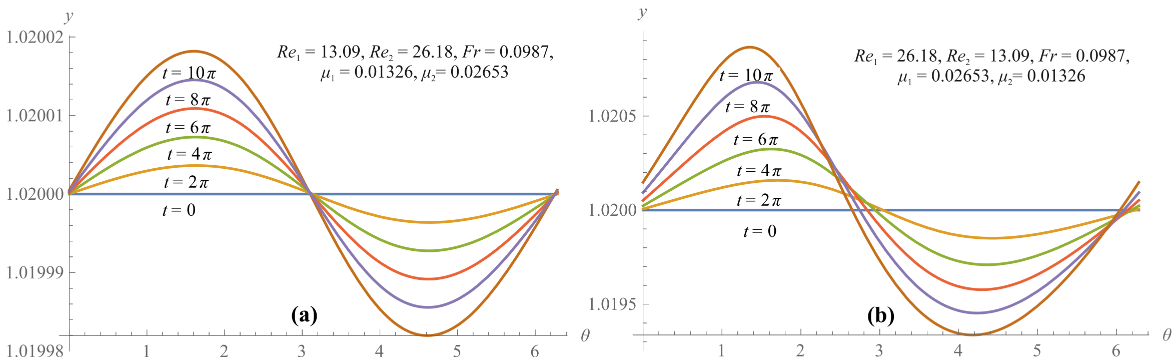

Figure 8. Surface forms y=1+h_2 with t={0, 2π, 4π, 6π, 8π, 10π}.

The surfaces of the outer layers through a full turn have a similar form, but each time they increasingly deviate from the original constant thickness h_{20}=0.02 by values of the order of 10^{-4} due to the development of nonlinear perturbations due to the action of gravity and inertia.

5. Conclusions

Using the basic postulates of the thin-layer theory, an interconnected system of evolution equations is obtained to determine the type of the inner and outer layers in the case of a two-layer film flow of viscous immiscible liquids on the outer surface of a rotating cylinder in a gravitational force field. Solutions are obtained for the problem of the flow of two thin layers (films) of viscous liquids on the outer surface of a cylinder in a field of gravity for the case of slow and moderate rotation. The velocities of motion of the layers of liquids, the boundary between the layers, and the free surface were determined and analyzed depending on the dimensionless flow criteria.

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