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Application of geometrical methods to study the systems of differential equations for quantum-mechanical problems

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Abstract. A geometrical method based on the structural stability theory is used to study systems of differential equations which arise in quantum-mechanical problems. We consider a 1/2-spin particle in external Coulomb field or in the presence of magnetic charge on the background of the de-Sitter space, and a free 3/2-spin particle in spherical coordinates of the flat space. It turns out that the first and the second Kosambi-Cartan-Chern invariants are nontrivial for the corresponding systems, while the 3-d, 4-th and 5-th invariants identically vanish. From physical point of view, the second invariant determines how rapidly the different branches of the solution diverge from or converge to the intersection points, while the most interesting are the singular points. The convergence (divergence) near the singular points $r = 0, \infty$ are shown to correlate with the behavior of solutions for quantum mechanical states (discrete and continuous spectra). The vanishing of the 3-d, 4-th and 5-th invariants geometrically implies the existence of a nonlinear connection on the tangent bundle, having zero torsion and curvature.

1. Introduction

In many situations of physical interest, the equations of motion of a dynamical system follow from a Lagrangian L via the Euler-Lagrange equations which may be written as:

$$\dot{y}^i(r) + 2Q^i(r, x, y) = 0, \quad (1)$$

where x^i are r -dependent coordinates along the paths, $y^i = dx^i/dr = \dot{x}^i$ and

$$Q^i = \frac{1}{4}g^{il} \left(\frac{\partial^2 L}{\partial x^k \partial y^l} y^k - \frac{\partial L}{\partial x^l} + \frac{\partial^2 L}{\partial y^l \partial r} \right), \quad g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}. \quad (2)$$

The geometrical study of these systems of equations is based on the use of the KCC (Kosambi–Cartan–Chern) invariants. The general KCC-theory is developed in detail in numerous works, e.g., [1, 2, 3].

The first KCC-invariant ε^i , which can be interpreted as an external force, is given by

$$\varepsilon^i(r, x, y) = \frac{\partial Q^i}{\partial y^j} y^j - 2Q^i. \quad (3)$$



The second KCC-invariant P_j^i is defined by [1]:

$$P_j^i = 2 \frac{\partial Q^i}{\partial x^j} + 2Q^s \frac{\partial^2 Q^i}{\partial y^j \partial y^s} - \frac{\partial^2 Q^i}{\partial y^j \partial x^s} y^s - \frac{\partial Q^i}{\partial y^s} \frac{\partial Q^s}{\partial y^j} - \frac{\partial^2 Q^i}{\partial y^j \partial r}, \quad (4)$$

and relates to the Jacobi stability of the dynamical system. There is an analogy between the equations of geodesic deviation expressed in terms of the Riemann curvature:

$$\frac{D^2 \xi^i}{Ds^2} = R_{kjl}^i \frac{dx^k}{ds} \frac{dx^l}{ds} \xi^j = -K_j^i \xi^j$$

and in terms of the second KCC-invariant

$$\frac{D^2 \xi^i}{Dr^2} = P_j^i \xi^j.$$

It is known that a pencil of geodesic curves emanating from some point r_0 converges (or diverges) if the real parts of all the eigenvalues of the 2-nd KCC-invariant P_j^i are negative (or positive ones).

The third KCC-invariant

$$R_{jk}^i = \frac{1}{3} \left(\frac{\partial P_j^i}{\partial y^k} - \frac{\partial P_k^i}{\partial y^j} \right) \quad (5)$$

determines the torsion of the Berwald connection.

The fourth KCC-invariant is an extension of the Riemann–Christoffel tensor,

$$B_{jkl}^i = \frac{\partial R_{jk}^i}{\partial y^l}. \quad (6)$$

Finally, the fifth KCC-invariant, which extends the Douglas tensor, is provided by

$$D_{jkl}^i = \frac{\partial^3 Q^i}{\partial y^j \partial y^k \partial y^l}. \quad (7)$$

It is important to note that if the vector field Q^i is linear in the coordinates x^i and y^i , then the first and the second invariants are functions of the radial coordinate r , do not depend on x^i and y^i , and the remaining invariants identically vanish.

2. Spin 1/2 particle in the Coulomb field in Minkowski space

The Dirac equation for the particle in external Coulomb field gives the system [4]:

$$\begin{aligned} \left(\frac{d}{dr} + \frac{\nu}{r} \right) f + \left(E + \frac{e}{r} + m \right) g &= 0, \\ \left(\frac{d}{dr} - \frac{\nu}{r} \right) g - \left(E + \frac{e}{r} - m \right) f &= 0, \end{aligned} \quad (8)$$

where $\nu = j + 1/2$ and $j = 1/2, 3/2, \dots$. In terms of the variable $z = 2\lambda r$, $\lambda = \sqrt{m^2 - E^2}$, and of the new functions f_1, g_1 given by the relations:

$$f = \sqrt{m + E}(f_1 + g_1)\phi(z), \quad g = \sqrt{m - E}(f_1 - g_1)\phi(z), \quad (9)$$

where $\mu = em/\lambda$, $\epsilon = eE/\lambda$, one gets the equations with separated variables:

$$\begin{aligned}\frac{d^2 f_1}{dz^2} + \left(\frac{2\epsilon + 1}{2z} - \frac{1}{4} + \frac{1 + 4(\mu^2 - \nu^2 - \epsilon^2)}{4z^2} \right) f_1 &= 0, \\ \frac{d^2 g_1}{dz^2} + \left(\frac{2\epsilon - 1}{2z} - \frac{1}{4} + \frac{1 + 4(\mu^2 - \nu^2 - \epsilon^2)}{4z^2} \right) g_1 &= 0.\end{aligned}\quad (10)$$

Geometrically, the first and the second invariants (3–4) for the last system (10) are given by the formulas

$$\begin{aligned}\varepsilon^1 &= \left(\frac{1}{4} - \frac{2\epsilon+1}{2z} - \frac{1+4(\mu^2-\nu^2-\epsilon^2)}{4z^2} \right) f_1, \\ \varepsilon^2 &= \left(\frac{1}{4} - \frac{2\epsilon-1}{2z} - \frac{1+4(\mu^2-\nu^2-\epsilon^2)}{4z^2} \right) g_1, \\ (P^i_j) &= \begin{pmatrix} -\frac{1}{4} + \frac{2\epsilon+1}{2z} + \frac{1+4(\mu^2-\nu^2-\epsilon^2)}{4z^2} & 0 \\ 0 & -\frac{1}{4} + \frac{2\epsilon-1}{2z} + \frac{1+4(\mu^2-\nu^2-\epsilon^2)}{4z^2} \end{pmatrix}.\end{aligned}$$

We further study the behavior of the eigenvalues of P^i_j near the singular points $z = 0$, $z = \infty$:

$$\begin{aligned}z \rightarrow 0 &\Rightarrow \lambda_{1,2} \rightarrow \frac{1+4(\epsilon^2-\nu^2)}{2z^2} \approx \frac{1-4\nu^2}{2z^2} < 0; \\ z \rightarrow \infty &\Rightarrow \lambda_{1,2} \rightarrow -\frac{1}{4} < 0.\end{aligned}$$

This correlates with the behavior of solutions near the points $r = 0, \infty$ for quantum mechanical states (discrete and continuous spectra).

We can construct the Lagrangian in the form

$$L = g_{ij} y^i y^j + b_j(x, r) y^j. \quad (11)$$

Taking into account the linearity of the differential equations (10) in the coordinates, we will search the 1-form $b_j(x, r)$ as a linear combination of the coordinates, namely $b_j(x, r) = h_{ij} x^i$. Then, using the above definition (2), and substituting L from (11) ($L \rightsquigarrow Q^i$), one gets

$$Q^i = \frac{1}{4} g^{il} \left(h_{jl} y^j - h_{lj} y^l + \frac{\partial g_{jl}}{\partial r} y^j + \frac{\partial h_{jl}}{\partial r} x^j \right). \quad (12)$$

Assuming the symmetry $h_{jl} = h_{lj}$ and $g_{ij} = \delta_{ij}$, we find the semispray coefficients

$$\begin{aligned}Q^1 &= \frac{1}{4} \frac{dh_{11}}{dz} = \frac{1}{2} \left(\frac{2\epsilon+1}{2z} - \frac{1}{4} + \frac{1+4(\mu^2-\nu^2-\epsilon^2)}{4z^2} \right), \\ Q^2 &= \frac{1}{4} \frac{dh_{22}}{dz} = \frac{1}{2} \left(\frac{2\epsilon-1}{2z} - \frac{1}{4} + \frac{1+4(\mu^2-\nu^2-\epsilon^2)}{4z^2} \right).\end{aligned}$$

By integrating the last relations, one gets the Lagrangian:

$$L = \dot{f}_1^2 + \dot{g}_1^2 - \left(\frac{1}{2} z - 2\epsilon \ln z + \frac{1 + 4(\mu^2 - \nu^2 - \epsilon^2)}{2z} \right) (\dot{f}_1 f_1 + \dot{g}_1 g_1) + \ln z (\dot{f}_1 f_1 - \dot{g}_1 g_1).$$

A similar study can be done for the original system (8) in terms of $f(r)$ and $g(r)$. By differentiating the equations with respect to r , the system becomes

$$\begin{aligned}\frac{d^2 f}{dr^2} + \frac{\nu}{r} \frac{df}{dr} + \left(E + \frac{e}{r} + m \right) \frac{dg}{dr} - \frac{\nu f + eg}{r^2} &= 0, \\ \frac{d^2 g}{dr^2} - \frac{\nu}{r} \frac{dg}{dr} - \left(E + \frac{e}{r} - m \right) \frac{df}{dr} + \frac{\nu g + ef}{r^2} &= 0.\end{aligned}\quad (13)$$

Now, the first and second invariants are

$$\begin{aligned}\varepsilon^1 &= -\frac{1}{2} \left(E + \frac{e}{r} + m \right) \frac{dg}{dr} - \frac{\nu}{2r} \frac{df}{dr} + \frac{e}{r^2} g + \frac{\nu}{r^2} f, \\ \varepsilon^2 &= \frac{1}{2} \left(E + \frac{e}{r} - m \right) \frac{df}{dr} + \frac{\nu}{2r} \frac{dg}{dr} - \frac{e}{r^2} f - \frac{\nu}{r^2} g, \\ (P^i_j) &= \begin{pmatrix} \frac{1}{4} \left(E + \frac{e}{r} \right)^2 - \frac{m^2}{4} - \frac{\nu(\nu+2)}{4r^2} & -\frac{e}{2r^2} \\ \frac{e}{2r^2} & \frac{1}{4} \left(E + \frac{e}{r} \right)^2 - \frac{m^2}{4} - \frac{\nu(\nu-2)}{4r^2} \end{pmatrix}.\end{aligned}$$

The eigenvalues of P^i_j are given by the expressions:

$$\begin{aligned}\lambda_1 &= \frac{-2\sqrt{\nu^2-e^2}-\nu^2}{4r^2} + \frac{1}{4} \left(E + \frac{e}{r} \right)^2 - \frac{m^2}{4}, \\ \lambda_2 &= \frac{2\sqrt{\nu^2-e^2}-\nu^2}{4r^2} + \frac{1}{4} \left(E + \frac{e}{r} \right)^2 - \frac{m^2}{4}.\end{aligned}$$

Near the singularities of the system, the eigenvalues turn to

$$\begin{aligned}r \rightarrow 0 \Rightarrow \begin{cases} \lambda_1 \rightarrow \frac{1-(1+\sqrt{\nu^2-e^2})^2}{4r^2} < 0, \\ \lambda_2 \rightarrow \frac{1-(1-\sqrt{\nu^2-e^2})^2}{4r^2}, \text{ and } \lambda_2 < 0 \text{ for } \nu^2 - e^2 > 4, \end{cases} \\ r \rightarrow \infty \Rightarrow \lambda_1 = \lambda_2 \rightarrow (E^2 - m^2)/4.\end{aligned}$$

The behavior of eigenvalues near the point $r = 0$ is slightly different, comparing with the previous case of separated variables f_1, g_1 . This may be explained as follows. Strictly speaking, direct differentiation of the first order system leads to the non-equivalent mathematical problem

$$\begin{aligned}\hat{A}F &= 0 \rightsquigarrow \frac{d}{dr}\hat{A}F = 0, \\ \hat{A}\tilde{F} &= const \rightsquigarrow \frac{d}{dr}\hat{A}\tilde{F} = 0, \quad \tilde{F} \neq F.\end{aligned}$$

The system for the radial functions $f(r), g(r)$ (8) may be written in the equivalent form as

$$\begin{aligned}\frac{d^2 f}{dr^2} - \left(\frac{\nu(\nu+1)}{r^2} - \left(E + \frac{e}{r} \right)^2 + m^2 \right) f - \frac{e}{r^2} g &= 0, \\ \frac{d^2 g}{dr^2} - \left(\frac{\nu(\nu-1)}{r^2} - \left(E + \frac{e}{r} \right)^2 + m^2 \right) g + \frac{e}{r^2} f &= 0.\end{aligned}\tag{14}$$

Now, the first and the second invariants are

$$\begin{aligned}\varepsilon^1 &= \left(\frac{\nu(\nu+1)}{r^2} - \left(E + \frac{e}{r} \right)^2 + m^2 \right) f + \frac{e}{r^2} g, \\ \varepsilon^2 &= \left(\frac{\nu(\nu-1)}{r^2} - \left(E + \frac{e}{r} \right)^2 + m^2 \right) g - \frac{e}{r^2} f, \\ (P^i_j) &= \begin{pmatrix} \left(E + \frac{e}{r} \right)^2 - m^2 - \frac{\nu(\nu+1)}{r^2} & -\frac{e}{r^2} \\ \frac{e}{r^2} & \left(E + \frac{e}{r} \right)^2 - m^2 - \frac{\nu(\nu-1)}{r^2} \end{pmatrix}.\end{aligned}$$

The eigenvalues of P^i_j are

$$\begin{aligned}\lambda_1 &= \frac{-\sqrt{\nu^2-e^2}-\nu^2}{r^2} + \left(E + \frac{e}{r} \right)^2 - m^2, \\ \lambda_2 &= \frac{\sqrt{\nu^2-e^2}-\nu^2}{r^2} + \left(E + \frac{e}{r} \right)^2 - m^2.\end{aligned}$$

In this case, near the singular points $r = 0$ and ∞ the eigenvalues behave as

$$r \rightarrow 0 \Rightarrow \begin{cases} \lambda_1 \rightarrow \frac{1-(1+2\sqrt{\nu^2-e^2})^2}{4r^2} < 0, \\ \lambda_2 \rightarrow \frac{1-(1-2\sqrt{\nu^2-e^2})^2}{4r^2}, \text{ with } \lambda_2 < 0 \text{ for } \nu^2 - e^2 > 1, \end{cases}$$

$$r \rightarrow \infty, \Rightarrow \lambda_1 = \lambda_2 \rightarrow (E^2 - m^2).$$

So, for bound states, $(E^2 - m^2) < 0$. We conclude that the choice of the concrete representation for the system of equations slightly influences the behavior of the eigenvalues of the second invariant.

The corresponding Lagrangian is given by the expression:

$$L = \dot{f}^2 - \dot{g}^2 + 2 \left(\frac{\nu^2 - e^2}{r} + (E^2 - m^2) r + 2eE \ln r \right) (\dot{f}f - \dot{g}g) + \frac{2\nu}{r} (\dot{f}f + \dot{g}g) + \frac{2e}{r} (gf + fg). \quad (15)$$

3. Spin 1/2 particle in the field of Abelian monopole on the background of de Sitter space-time

The study of Dirac equation in the field of Dirac monopole on the background of de Sitter space-time (in static coordinates) may be reduced to two separated equations [5]:

$$\begin{aligned} \frac{d^2 f}{dz^2} + \frac{3 + 4z(1-z)[4U_1(z) - 1]}{16z^2(1-z)^2} f(z) &= 0, \\ \frac{d^2 g}{dz^2} + \frac{3 + 4z(1-z)[4U_2(z) - 1]}{16z^2(1-z)^2} g(z) &= 0, \end{aligned} \quad (16)$$

where the following notations are used:

$$\begin{aligned} U_1(z) &= -\frac{1}{4} \left(M - \frac{i}{2} \right)^2 + \frac{\epsilon(\epsilon - i)}{4(1-z)} - \frac{\alpha(\alpha + 1t)}{4z}, \\ U_2(z) &= -\frac{1}{4} \left(M - \frac{i}{2} \right)^2 + \frac{\epsilon(\epsilon + i)}{4(1-z)} - \frac{\alpha(\alpha - 1)}{4z}. \end{aligned}$$

We note the symmetry between the two equations:

$$f \rightsquigarrow g, \quad \epsilon \rightsquigarrow -\epsilon, \quad \alpha \rightsquigarrow -\alpha. \quad (17)$$

Now the first and second invariants are given by expressions:

$$\begin{aligned} \varepsilon^1 &= \frac{4\epsilon(3-\epsilon)z - ((3-2M)^2 + 4)z(z-1) - 4\alpha(\alpha+1)(z-1) - 3}{16(z-1)^2 z^2} f, \\ \varepsilon^2 &= \frac{-4\epsilon(3+\epsilon)z - ((3-2M)^2 + 4)z(z-1) - 4\alpha(\alpha-1)(z-1) - 3}{16(z-1)^2 z^2} g, \\ (P^i_j) &= \begin{pmatrix} \frac{((3-2M)^2 + 4)z(z-1) + 4\epsilon(\epsilon-3)z + 4\alpha(\alpha+1)(z-1) + 3}{16(z-1)^2 z^2} & 0 \\ 0 & \frac{((3-2M)^2 + 4)z(z-1) + 4\epsilon(\epsilon+3)z + 4\alpha(\alpha-1)(z-1) + 3}{16(z-1)^2 z^2} \end{pmatrix}. \end{aligned}$$

The behavior of the eigenvalues of the second invariant P^i_j in the neighborhood of the singular points is as follows

$$z \rightarrow 0 \Rightarrow \lambda_{1,2} \rightarrow \frac{3 - 4\alpha(\alpha \pm 1)}{16z^2} < 0, \quad \alpha = \sqrt{(j + 1/2)^2 - k^2} > 1;$$

$$z \rightarrow 1 \Rightarrow \lambda_{1,2} \rightarrow \frac{3 + 4\epsilon(\epsilon \mp i)}{16(z - 1)^2}, \quad \text{Re}[\lambda_{1,2}] > 0.$$

The behavior at $z \rightarrow 1$ correlates with the structure of the solutions near the de Sitter horizon. For this model, the Lagrangian is determined by the expression

$$\begin{aligned} L = & \dot{f}^2 + \dot{g}^2 \\ & + \frac{1}{8} \left[\frac{-3+4\epsilon(i-\epsilon)}{z-1} + \frac{-3+4\alpha(1+\alpha)}{z} + 2(-2 + 4\epsilon(i-\epsilon) + 4\alpha(1+\alpha) + (i-2M)^2) \arctan[1-2z] \right] f\dot{f} \\ & + \frac{1}{8} \left[\frac{-3-4\epsilon(i+\epsilon)}{z-1} + \frac{-3+4\alpha(\alpha-1)}{z} + 2(-2 - 4\epsilon(i+\epsilon) + 4\alpha(\alpha-1) + (i-2M)^2) \arctan[1-2z] \right] g\dot{g}. \end{aligned}$$

4. Spherical solutions for the spin 3/2 particle

As has been shown in [6], the problem of finding of the spherically symmetric solutions of the wave equation for the spin 3/2 particle reduces to two systems, each of them consisting of three equations ($i, j \in \{1, 2, 3\}$)

$$\begin{aligned} \frac{d^2 F^i}{dr^2} + (\epsilon^2 - m^2)F^i - \frac{1}{r^2}T^i_j F^j &= 0, \\ \frac{d^2 G^i}{dr^2} + (\epsilon^2 - m^2)G^i - \frac{1}{r^2}T'^i_j G^j &= 0, \end{aligned} \quad (18)$$

where the mixing matrices T and T' have the form

$$T^i_j = \begin{pmatrix} b^2 & \sqrt{2}b & b \\ \sqrt{2}b & a^2 + a + 2 & \sqrt{2}(a+1) \\ b & \sqrt{2}(a+1) & b^2 + 2 \end{pmatrix}, \quad (19)$$

$$T'^i_j = \begin{pmatrix} b^2 & \sqrt{2}b & -b \\ \sqrt{2}b & a^2 - a + 2 & \sqrt{2}(a-1) \\ -b & \sqrt{2}(a-1) & b^2 + 2 \end{pmatrix}; \quad (20)$$

the parameters a and b depend on the total angular momentum

$$a = j + 1/2, \quad b = \sqrt{(j - 1/2)(j + 3/2)}.$$

The second invariants in explicit form are

$$(P_F)^i_j = \begin{pmatrix} -\frac{b^2}{r^2} + \epsilon^2 - m^2 & -\frac{\sqrt{2}b}{r^2} & -\frac{b}{r^2} \\ -\frac{\sqrt{2}b}{r^2} & -\frac{a^2+a+2}{r^2} + \epsilon^2 - m^2 & -\frac{\sqrt{2}(a+1)}{r^2} \\ -\frac{b}{r^2} & -\frac{\sqrt{2}(a+1)}{r^2} & -\frac{b^2+2}{r^2} + \epsilon^2 - m^2 \end{pmatrix}, \quad (21)$$

$$(P_G)^i_j = \begin{pmatrix} -\frac{b^2}{r^2} + \epsilon^2 - m^2 & -\frac{\sqrt{2}b}{r^2} & \frac{b}{r^2} \\ -\frac{\sqrt{2}b}{r^2} & -\frac{a^2-a+2}{r^2} + \epsilon^2 - m^2 & -\frac{\sqrt{2}(a-1)}{r^2} \\ \frac{b}{r^2} & -\frac{\sqrt{2}(a-1)}{r^2} & -\frac{b^2+2}{r^2} + \epsilon^2 - m^2 \end{pmatrix}. \quad (22)$$

The eigenvalues λ_F^i and λ_G^i for the invariants P_F and P_G are given by the formulas:

$$\begin{aligned}\lambda_F^1 &= \lambda_F^2 = \epsilon^2 - m^2 - \frac{(2j-1)(2j+1)}{4r^2}, & \lambda_F^3 &= \epsilon^2 - m^2 - \frac{(2j+3)(2j+5)}{4r^2}; \\ \lambda_G^1 &= \lambda_G^2 = \epsilon^2 - m^2 - \frac{(2j+1)(2j+3)}{4r^2}, & \lambda_G^3 &= \epsilon^2 - m^2 - \frac{(2j-1)(2j-3)}{4r^2}.\end{aligned}\quad (23)$$

Near the singular point, they become simpler

$$r \rightarrow 0 \Rightarrow \lambda_i \rightarrow -\infty < 0, \quad r \rightarrow \infty \Rightarrow \lambda_i \rightarrow \epsilon^2 - m^2 > 0,$$

which agree with the structure of the solution of continuous spectrum.

5. Conclusions

We have considered a 1/2-spin particle in Coulomb field and in the presence of magnetic charge on the background of the de-Sitter space, and a free 3/2-spin particle in spherical coordinates of the flat space. It turns out that the first and the second Kosambi-Cartan-Chern invariants are nontrivial for these systems, while the 3-d, 4-th and 5-th invariants identically vanish. The first invariant determines the vector field on the configuration space of the differential system, and is interpreted as an external field potential. From physical point of view, the second invariant determines how rapidly the different branches of the solutions diverge from or converge to the intersection point, while the singular points are most relevant. The convergence (divergence) near the singular points $r = 0, \infty$ correlates with the behavior of solutions for quantum mechanical states (discrete and continuous spectra).

It is shown that the choice of the concrete representation for the system of equations slightly influences on behavior of the eigenvalues of the second invariant.

The vanishing of the 3-d, 4-th and 5-th invariants means that, in geometrical terms, there exists a nonlinear connection on the tangent bundle, with zero torsion and curvature.

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