Rational Maxwell knots via de Sitter space

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Rational Maxwell knots via de Sitter space

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Abstract. We set up a correspondence between solutions of the Yang–Mills equations on $\mathbb{R}\times S^3$ and in Minkowski spacetime via de Sitter space. Some known Abelian and non-Abelian exact solutions are rederived. For the Maxwell case we present a straightforward algorithm to generate an infinite number of explicit solutions, with fields and potentials in Minkowski coordinates given by rational functions of increasing complexity. We illustrate our method with some nontrivial examples.

1. Conformal equivalence of dS_4 to $\mathcal{I}\times S^3$ and two copies of $\mathbb{R}^{1,3}_+$

Four-dimensional de Sitter space is a one-sheeted hyperboloid (of radius ℓ) in $\mathbb{R}^{1,4}\ni\{Z_0,Z_1,\dots,Z_4\}$ given by

$$-Z_0^2+Z_1^2+Z_2^2+Z_3^2+Z_4^2 \;=\; \ell^2 \tag{1}$$

Constant Z_0 slices are 3-spheres of varying radius, yielding a parametrization of $dS_4\ni\{\tau,\omega_A\}$ as

$$\begin{aligned} Z_0 \;&=\; -\ell\,\cot\tau \qquad\quad \text{and} \qquad\quad Z_A \;=\; \frac{\ell}{\sin\tau}\,\omega_A \quad \text{for} \quad A=1,\dots,4 \\ \text{with} \qquad \tau\in\mathcal{I}:=(0,\pi) \qquad &\text{and} \qquad \omega_A\omega_A=1\,. \end{aligned} \tag{2}$$

The Minkowski metric

$$\mathrm{d}s^2 \;=\; -\mathrm{d}Z_0^2+\mathrm{d}Z_1^2+\mathrm{d}Z_2^2+\mathrm{d}Z_3^2+\mathrm{d}Z_4^2 \tag{3}$$

induces on dS_4 the metric

$$\mathrm{d}s^2 \;=\; \frac{\ell^2}{\sin^2\tau}\big(-\mathrm{d}\tau^2+\mathrm{d}\Omega_3^2\big) \qquad \text{with} \qquad \mathrm{d}\Omega_3^2 \quad \text{for} \quad S^3\,, \tag{4}$$

showing that dS_4 is conformally equivalent to a finite cylinder $\mathcal{I}\times S^3$.

The $Z_0{+}Z_4{<}0$ half of dS_4 is also conformally related to future Minkowski space $\mathbb{R}^{1,3}_+\ni\{t,x,y,z\}$,

$$\begin{aligned} &Z_0=\frac{t^2{-}r^2{-}\ell^2}{2\,t}\,, \quad Z_1=\ell\,\frac{x}{t}\,, \quad Z_2=\ell\,\frac{y}{t}\,, \quad Z_3=\ell\,\frac{z}{t}\,, \quad Z_4=\frac{r^2{-}t^2{-}\ell^2}{2\,t} \\ &\text{with} \qquad x,y,z\in\mathbb{R} \qquad \text{and} \qquad r^2=x^2+y^2+z^2 \qquad \text{but} \qquad t\in\mathbb{R}_+\,, \end{aligned} \tag{5}$$

since $t\in[0,\infty]$ corresponds to $Z_0\in[-\infty,\infty]$ but $Z_0{+}Z_4<0$. In these Minkowski coordinates,

$$\mathrm{d}s^2 \;=\; \frac{\ell^2}{t^2}\left(-\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right). \tag{6}$$

One may cover the entire $\mathbb{R}^{1,3}$ by gluing a second dS_4 copy and using the patch $Z_0+Z_4>0$.

We shall employ the direct relation between the cylinder and Minkowski coordinates:

$$\cot\tau = \frac{r^2-t^2+\ell^2}{2\,\ell\,t}\,, \quad \omega_1 = \gamma\,\frac{x}{\ell}\,, \quad \omega_2 = \gamma\,\frac{y}{\ell}\,, \quad \omega_3 = \gamma\,\frac{z}{\ell}\,, \quad \omega_4 = \gamma\frac{r^2-t^2-\ell^2}{2\,\ell^2} \tag{7}$$

with the convenient abbreviation

$$\gamma \;=\; \frac{2\,\ell^2}{\sqrt{4\,\ell^2t^2+(r^2-t^2+\ell^2)^2}} \tag{8}$$

Since $t=-\infty,0,\infty$ corresponds to $\tau=-\pi,0,\pi$, the cylinder gets doubled to $2\mathcal{I}\times S^3$, and full Minkowski space is covered by the cylinder patch $\omega_4\le\cos\tau$. The cylinder time τ is a regular smooth

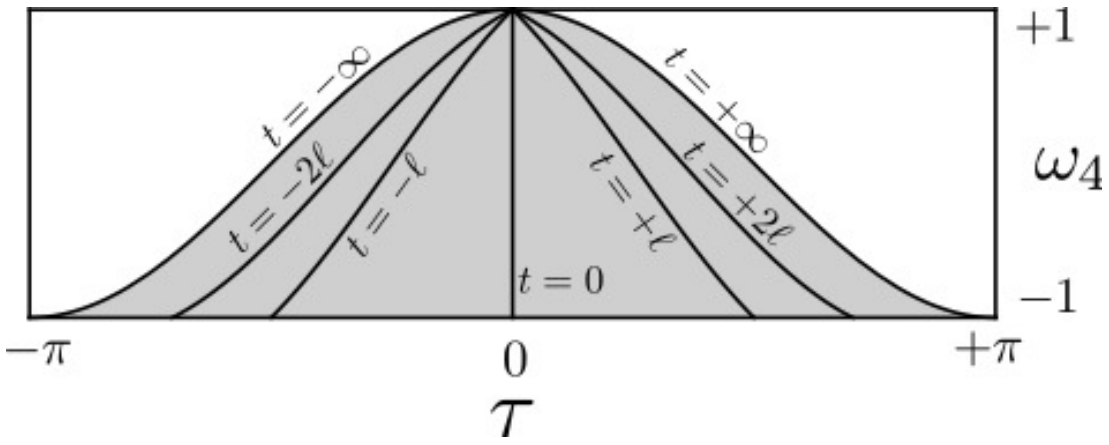

Figure 1. An illustration of the map between a cylinder $2\mathcal{I}\times S^3$ and Minkowski space $\mathbb{R}^{1,3}$. The Minkowski coordinates cover the shaded area. Its boundary is given by the curve $\omega_4=\cos\tau$. Each point is a two-sphere spanned by $\omega_{1,2,3}$, which is mapped to a sphere of constant r and t.

function of (t,x,y,z), but more useful will be

$$\exp(\mathrm{i}\,\tau) \;=\; \frac{(\ell+\mathrm{i}t)^2+r^2}{\sqrt{4\,\ell^2t^2+(r^2-t^2+\ell^2)^2}}\,. \tag{9}$$

The following is a rendition of the our publication [1].

2. The correspondence

Yang–Mills and Maxwell theory are conformally invariant in four spacetime dimensions. Therefore, we may solve their equations of motion on the cylinder $2\mathcal{I}\times S^3$ rather than directly on Minkowski space $\mathbb{R}^{1,3}$. The cylinder parametrization has the advantage that it makes manifest a hidden SO(4) covariance.

The gauge potential taking values in a Lie algebra $\mathfrak{g}$ can always be chosen as

$$A \;=\; \sum_{a=1}^{3} X_a(\tau,\omega)\,e^a \qquad \text{on} \quad 2\mathcal{I}\times S^3 \tag{10}$$

where $X_a\in\mathfrak{g}$, and $\{e^a\}$ is a basis of left-invariant one-forms on S^3. There is no $\mathrm{d}\tau$ component because we picked the temporal gauge $A_\tau=0$.

Yang–Mills or Maxwell solutions are translated from $2\mathcal{I}\times S^3$ to $\mathbb{R}^{1,3}$ simply by the coordinate change (7). The behavior at the boundary $\cos\tau = \omega_4$ yields the fall-off properties at $t \to \pm\infty$. To become explicit, we need the Minkowski-parametrization of the one-forms $e^0 \equiv \mathrm{d}\tau$ and e^a, which are subject to

$$\mathrm{d}e^a + \varepsilon^a_{\ bc}\, e^b \wedge e^c \;=\; 0 \qquad\qquad \text{and} \qquad\qquad e^a e^a \;=\; \mathrm{d}\Omega_3^2 \,. \tag{11}$$

In terms of the S^3 coordinates $(a, i, j, k = 1, 2, 3)$ they are

$$e^a \;=\; -\eta^a_{BC}\, \omega_B\, \mathrm{d}\omega_C \qquad \text{where} \quad \eta^i_{jk} = \varepsilon^i{}_{jk} \quad \text{and} \quad \eta^i_{j4} = -\eta^i_{4j} = \delta^i_j \,. \tag{12}$$

A slightly lengthy computation yields the Minkowski-coordinate expressions,

$$\begin{aligned} e^0 \;&=\; \frac{\gamma^2}{\ell^3}\Big(\tfrac{1}{2}(t^2 + r^2 + \ell^2)\,\mathrm{d}t - t\,x^k \mathrm{d}x^k\Big) \\ e^a \;&=\; \frac{\gamma^2}{\ell^3}\Big(t\,x^a \mathrm{d}t - \big(\tfrac{1}{2}(t^2 - r^2 + \ell^2)\,\delta^a_k + x^a x^k + \ell\,\varepsilon^a_{\ jk} x^j\big)\,\mathrm{d}x^k\Big) \,, \end{aligned} \tag{13}$$

with the notation

$$(x^i) = (x, y, z) \qquad \text{and (for later)} \qquad (x^\mu) = (x^0, x^i) = (t, x, y, z) \,. \tag{14}$$

The simplest Yang–Mills solutions are most symmetric. To obtain them, let us impose SO(4) symmetry by setting $X_a(\tau, \omega) = X_a(\tau)$. The Yang–Mills equations then become ordinary matrix differential equations [2, 3],

$$\tfrac{\mathrm{d}^2}{\mathrm{d}\tau^2} X_a \;=\; -4\,X_a + 3\,\varepsilon_{abc}\,[X_b, X_c] - \big[X_b, [X_a, X_b]\big] \qquad\qquad \text{and} \qquad\qquad \big[\tfrac{\mathrm{d}}{\mathrm{d}\tau} X_a, X_a\big] \;=\; 0 \,. \tag{15}$$

For $\mathfrak{g} = su(2)$, these equations admit some analytic solutions [4, 5],

$$X_a(\tau) \;=\; \big(1 + \tfrac{1}{2} q(\tau)\big)\, T_a \qquad \text{with} \qquad \frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} \;=\; -\frac{\partial V}{\partial q} \qquad \text{for} \quad V(q) \;=\; \tfrac{1}{2} q^2 (q{+}2)^2 \,, \tag{16}$$

where $\{T_a\}$ is an $su(2)$ basis normalized to obey $[T_a, T_b] = 2\varepsilon_{abc} T_c$. Notice the identification of Lie-algebra and spatial indices. So the Yang–Mills problem has been reduced to a Newtonian particle in a double-well potential $V(q)$. Its prominent trajectories are (a) the vacua $q(\tau) \equiv -2$ or 0, (b) the sphaleron $q(\tau) \equiv -1$ and (c) the bounce $q(\tau) = \sqrt{2}\mathrm{sech}(\sqrt{2}(\tau - \tau_0)) - 1$. The corresponding gauge potential takes the form

$$A \;=\; \big(1 + \tfrac{1}{2} q(\tau)\big)\, g^{-1} \mathrm{d}g \qquad \text{for} \quad g : S^3 \to \mathrm{SU}(2) \,. \tag{17}$$

The sphaleron gives the only nontrivial static homogeneous solution (on the cylinder), i.e. $A = \frac{1}{2} T_a e^a = \frac{1}{2} g^{-1}\mathrm{d}g$, which translates to a finite-action homogeneous color-magnetic Yang–Mills solution on dS_4 [6] (see also [7]).

In addition, there exist analytic Abelian symmetric solutions,

$$X_a(\tau) \;=\; \bar{X}_a(\tau)\, T_3 \qquad \text{with} \qquad \frac{\mathrm{d}^2 \bar{X}_a}{\mathrm{d}\tau^2} \;=\; -4\,\bar{X}_a \,. \tag{18}$$

Obviously, these are solutions to Maxwell's equations, taking $\mathfrak{g} = \mathbb{R}$, so we can drop the matrix T_3 and consider just real-valued functions $\bar{X}_a(\tau)$. Let us drop the bar and consider $X_a \in \mathbb{R}$ from now on. The general solution to (18) is an oscillation with frequency two,

$$X_a(\tau) \;=\; c_a\, \cos\big(2(\tau - \tau_a)\big) \,. \tag{19}$$

The task is to transfer the oscillatory cylinder solutions to Minkowski space ($x \equiv \{x^\mu\}$),

$$A \;=\; X_a(\tau(x))\, e^a(x) \;=\; A_\mu(x)\, \mathrm{d}x^\mu \qquad \text{yielding} \quad A_\mu(x) \quad \text{with} \quad A_t \neq 0\,, \tag{20}$$

$$\mathrm{d}A \;=\; \frac{\mathrm{d}}{\mathrm{d}\tau} X_a\, e^0 \wedge e^a - \varepsilon^a_{\ bc} X_a\, e^b \wedge e^c \;=\; \tfrac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \qquad \text{yielding} \quad F_{\mu\nu}(x)\,. \tag{21}$$

From this, we obtain electric and magnetic fields $E_i = F_{i0}$ and $B_i = \frac{1}{2}\varepsilon_{ijk} F_{jk}$. For the computation it is helpful to recognize that $\exp(2\mathrm{i}\tau)$ is a rational function of t and r.

We may always choose a frame where $X_3 = 0$ and $\tau_2 = 0$. The overall amplitude is irrelevant as all equations are linear, and solutions can be superposed at will. Specializing to $c_1 = c_2 = -\frac{1}{8}$ and $\tau_1 = \frac{\pi}{2}$,

$$X_1(\tau) = -\tfrac{1}{8} \sin 2\tau\,, \quad X_2(\tau) = -\tfrac{1}{8} \cos 2\tau\,, \quad X_3(\tau) = 0\,, \tag{22}$$

the result of short computation (putting $\ell = 1$) yields

$$\vec{E} + \mathrm{i}\vec{B} \;=\; \frac{1}{\big((t-\mathrm{i})^2 - r^2\big)^3} \begin{pmatrix} (x-\mathrm{i}y)^2 - (t-\mathrm{i}-z)^2 \\ \mathrm{i}(x-\mathrm{i}y)^2 + \mathrm{i}(t-\mathrm{i}-z)^2 \\ -2\,(x-\mathrm{i}y)\,(t-\mathrm{i}-z) \end{pmatrix}\,. \tag{23}$$

This is the celebrated Hopf–Rañada electromagnetic knot [8, 9]. Our approach also yields its gauge potential.

3. Construction of electromagnetic solutions

In the following, we are interested only in Maxwell solutions. The linearity of the equations then will allow us to solve for a general (not SO(4)-symmetric) potential. Therefore, let us admit arbitrary non-symmetric configurations $X_a = X_a(\tau, \omega)$ but capture the ω-dependence in an SO(4)-covariant fashion. The main ingredients are the left-invariant vector fields generating right multiplication,

$$R_a \;=\; -\eta^a_{BC}\, \omega_B \frac{\partial}{\partial \omega_C} \qquad \Rightarrow \qquad [R_a, R_b] \;=\; 2\, \varepsilon_{abc}\, R_c\,, \tag{24}$$

and the right-invariant ones generating left multiplication (by the inverse),

$$L_a \;=\; -\tilde{\eta}^a_{BC}\, \omega_B \frac{\partial}{\partial \omega_C} \qquad \Rightarrow \qquad [L_a, L_b] \;=\; 2\, \varepsilon_{abc}\, L_c\,. \tag{25}$$

They mutually commute, $[R_a, L_b] = 0$, and the right translations are dual to our left-invariant one-forms, e.g. $e^a(R_b) = \delta^a_b$. Hence, an arbitrary function Φ on S^3 obeys

$$\mathrm{d}\Phi(\omega) \;=\; e^a\, R_a \Phi(\omega)\,. \tag{26}$$

The space of functions on S^3 decomposes into irreps of $su(2)_L \oplus su(2)_R$ labelled by a common spin $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. To make contact with standard physics notation, we define hermitian "angular momenta"

$$I_a := \tfrac{\mathrm{i}}{2}\, L_a \quad \text{and} \quad J_a := \tfrac{\mathrm{i}}{2}\, R_a \qquad \Rightarrow \qquad [I_a, I_b] = \mathrm{i}\, \varepsilon_{abc}\, I_c \quad \text{and} \quad [J_a, J_b] = \mathrm{i}\, \varepsilon_{abc}\, J_c\,. \tag{27}$$

A basis of hyperspherical harmonics

$$Y_{j;m,n}(\omega) \qquad \text{with} \qquad m, n = -j, -j{+}1, \ldots, +j \qquad \text{and} \qquad 2j = 0, 1, 2, \ldots \tag{28}$$

is specified by the relations

$$\begin{aligned} I^2\, Y_{j;m,n} &= J^2\, Y_{j;m,n} = j(j+1)\, Y_{j;m,n}\,, \\ I_3\, Y_{j;m,n} &= m\, Y_{j;m,n} \qquad \text{and} \qquad J_3\, Y_{j;m,n} = n\, Y_{j;m,n}\,. \end{aligned} \tag{29}$$

For an explicit construction, one introduces two complex coordinates

$$\alpha = \omega_1 + \mathrm{i}\omega_2 \qquad \text{and} \qquad \beta = \omega_3 + \mathrm{i}\omega_4 \qquad \text{subject to} \quad \bar\alpha\alpha + \bar\beta\beta = 1\,. \tag{30}$$

The angular momenta generators in those terms read

$$I_+ = (\bar\beta\partial_{\bar\alpha} - \alpha\partial_\beta)/\sqrt{2}\,, \qquad J_+ = (\beta\partial_{\bar\alpha} - \alpha\partial_{\bar\beta})/\sqrt{2}\,, \tag{31}$$

$$I_3 = (\alpha\partial_\alpha + \bar\beta\partial_{\bar\beta} - \bar\alpha\partial_{\bar\alpha} - \beta\partial_\beta)/2\,, \qquad J_3 = (\alpha\partial_\alpha + \beta\partial_\beta - \bar\alpha\partial_{\bar\alpha} - \bar\beta\partial_{\bar\beta})/2\,, \tag{32}$$

$$I_- = (\bar\alpha\partial_{\bar\beta} - \beta\partial_\alpha)/\sqrt{2}\,, \qquad J_- = (\bar\alpha\partial_\beta - \bar\beta\partial_\alpha)/\sqrt{2}\,. \tag{33}$$

The normalized hyperspherical harmonics are represented as

$$Y_{j;m,n} = \sqrt{\frac{2j+1}{2\pi^2}}\sqrt{\frac{2^{j-m}(j+m)!}{(2j)!\,(j-m)!}\frac{2^{j-n}(j+n)!}{(2j)!\,(j-n)!}}\,(I_-)^{j-m}(J_-)^{j-n}\,\alpha^{2j} \tag{34}$$

and are homogenous polynomials of degree $2j$ in $\{\alpha, \bar\alpha, \beta, \bar\beta\}$.

To set up a left-invariant and right-covariant formulation, we parametrize the general Maxwellian gauge potential on $2\mathcal{I}\times S^3$ as

$$A = X_0(\tau,\omega)\,\mathrm{d}\tau + X_a(\tau,\omega)\,e^a \tag{35}$$

The temporal and Coulomb gauge allows us to impose

$$X_0(\tau,\omega) = 0 \qquad \text{and} \qquad J_a\, X_a(\tau,\omega) = 0\,. \tag{36}$$

Maxwell's equations then are nothing but coupled wave equations:

$$-\tfrac{1}{4}\,\partial_\tau^2 X_a = (J^2{+}1)\,X_a + \mathrm{i}\,\varepsilon_{abc}J_b\,X_c \tag{37}$$

A more transparent rewriting employs the famiiar complex linear combinations

$$X_\pm = (X_1 \pm \mathrm{i}X_2)/\sqrt{2}\,, \tag{38}$$

which provides a partial decoupling of the components,

$$\begin{aligned} -\tfrac{1}{4}\,\partial_\tau^2 X_+ &= (J^2 - J_3 + 1)\,X_+ + J_+X_3\,, \\ -\tfrac{1}{4}\,\partial_\tau^2 X_3 &= (J^2 + 1)\,X_3 - J_+X_- + J_-X_+\,, \\ -\tfrac{1}{4}\,\partial_\tau^2 X_- &= (J^2 + J_3 + 1)\,X_- - J_-X_3\,, \end{aligned} \tag{39}$$

to be supplemented by the gauge condition

$$0 = J_3X_3 + J_+X_- + J_-X_+\,. \tag{40}$$

Since the X_a live on S^3, we naturally expand in our basis of hyperspherical harmonics,

$$X_a(\tau,\omega) = \sum_{jmn} X_a^{j;m,n}(\tau)\,Y_{j;m,n}(\alpha,\beta) \tag{41}$$

From the form of the equations it is obvious that

- the equations are diagonal in j and m, so these may be kept fixed
- they only couple triplets $(X_3^{j;m,n}, X_+^{j;m,n+1}, X_-^{j;m,n-1})$, so $X_\pm \propto J_\pm X_3$ for $X_3 \propto Y_{j;m,n}$
- the ansatz $X_a^{j;m,n}(\tau) = \mathrm{e}^{\mathrm{i}\Omega_a^{j;n}\tau} c_a^{j;n}$ gives a linear system for $\Omega_a^{j;n}$ and $c_a^{j;n}$

The frequencies turn out to be integral,

$$\Omega_a^{j;n} \;=\; \pm 2(j{+}1) \quad \text{or} \quad \pm\, 2j \;, \tag{42}$$

which produces two types of basis solutions:

- type I : $\quad j{\geq}0 \;, \quad m = -j, \ldots, +j \;, \quad n = -j{-}1, \ldots, j{+}1 \;, \quad \Omega^j = \pm 2(j{+}1) \;,$

$$\begin{aligned} X_+ \;&=\; \sqrt{(j{-}n)(j{-}n{+}1)/2}\; \mathrm{e}^{\pm 2(j+1)\mathrm{i}\tau}\; Y_{j;m,n+1} \;, \\ X_3 \;&=\; \sqrt{(j{+}1)^2 - n^2}\; \mathrm{e}^{\pm 2(j+1)\mathrm{i}\tau}\; Y_{j;m,n} \;, \\ X_- \;&=\; -\sqrt{(j{+}n)(j{+}n{+}1)/2}\; \mathrm{e}^{\pm 2(j+1)\mathrm{i}\tau}\; Y_{j;m,n-1} \;, \end{aligned} \tag{43}$$

- type II : $\quad j{\geq}1 \;, \quad m = -j, \ldots, +j \;, \quad n = -j{+}1, \ldots, j{-}1 \;, \quad \Omega^j = \pm 2j \;,$

$$\begin{aligned} X_+ \;&=\; -\sqrt{(j{+}n)(j{+}n{+}1)/2}\; \mathrm{e}^{\pm 2j\,\mathrm{i}\tau}\; Y_{j;m,n+1} \;, \\ X_3 \;&=\; \sqrt{j^2 - n^2}\; \mathrm{e}^{\pm 2j\,\mathrm{i}\tau}\; Y_{j;m,n} \;, \\ X_- \;&=\; \sqrt{(j{-}n)(j{-}n{+}1)/2}\; \mathrm{e}^{\pm 2j\,\mathrm{i}\tau}\; Y_{j;m,n-1} \;. \end{aligned} \tag{44}$$

Of course, a generic solution is some linear combination of the above. Due to the linearity of the equations, the overall scale of a solution is arbitrary.

4. Some properties of the solutions

Each complex solution yields two real ones, real part and imaginary part. For fixed spin j we get $2(2j{+}1)(2j{+}3)$ type-I solutions ($j{\geq}0$) and $2(2j{+}1)(2j{-}1)$ type-II solutions ($j{>}0$). They add up to $4(2j{+}1)^2$ solutions for $j{>}0$ and 6 solutions for $j{=}0$, which is the correct number for the dimension of a spin-j representation of SO(4). Constant solutions ($\Omega{=}0$) are not allowed; the simplest ones ($\Omega{=}2$) are the three complex $j{=}0$ type I and three complex $j{=}1$ type II basis configurations $(j;m,n) = (0,\star,0)$ and $(1,\star,0)$ with $\star = -1, 0, +1$, respectively. The Hopf–Rañada solution is a real combination of $(0,+1,0)$ and $(0,-1,0)$. The classification (43) and (44) shows a general parity relation map between $(j;m,n)$ type I and $(j{+}1;n,m)$ type II. Electromagnetic duality is realized via shifting $|\Omega^j|\tau$ by $\pm\frac{\pi}{2}$; this maps $A \mapsto A_D$.

The main technical task is to transform a chosen solution on $2\mathcal{I} \times S^3$ to Minkowski coordinates (t,x,y,z), which is straightforward due to the explicit formulæ for all ingredients and will produce only rational functions. Conserved (in time) quantities are helicity and energy,

$$h \;=\; \tfrac{1}{2} \int\limits_{\mathbb{R}^3} \big(A \wedge F + A_D \wedge F_D\big) \qquad \text{and} \qquad E \;=\; \tfrac{1}{2} \int\limits_{\mathbb{R}^3} \mathrm{d}^3x\, \big(\vec{E}^2 + \vec{B}^2\big) \;. \tag{45}$$

Their common scale is determined by the amplitude of the solution, but their ratio is fixed for the basis configurations. Both quantities are best computed in "sphere frame" at $t = \tau = 0$,

$$F \;=\; \mathcal{E}_a\, e^a \wedge e^0 + \tfrac{1}{2} \mathcal{B}_a\, \varepsilon^a{}_{bc}\, e^b \wedge e^c \;, \tag{46}$$

giving, for example,

$$\int\limits_{\mathbb{R}^3} \mathrm{d}^3x\; \vec{E}^2 \;=\; \frac{1}{\ell} \int\limits_{S^3} \mathrm{d}^3\Omega_3\; (1{-}\omega_4)\, \mathcal{E}_a \mathcal{E}_a \qquad \text{and} \qquad \int\limits_{\mathbb{R}^3} \mathrm{d}^3x\; \vec{B}^2 \;=\; \frac{1}{\ell} \int\limits_{S^3} \mathrm{d}^3\Omega_3\; (1{-}\omega_4)\, \mathcal{B}_a \mathcal{B}_a \;, \tag{47}$$

by exploiting the orthogonality properties of the hyperspherical harmonics.

5. Examples

Finally we shall present two cases for illustrative purposes. For the first example, let us take the real part of the $(j;m,n)=(1;0,0)$ type-I basis solution. Combining $\mathrm{e}^{4\mathrm{i}\tau}+\mathrm{e}^{-4\mathrm{i}\tau}=2\cos 4\tau$ and reading off $Y_{1;0,\star}$ from (43), we have

$$X_+ \;=\; -\tfrac{\sqrt{3}}{\pi}\,\alpha\beta\,\cos 4\tau\,, \quad X_3 \;=\; \tfrac{\sqrt{6}}{\pi}\,(\beta\bar\beta-\alpha\bar\alpha)\,\cos 4\tau\,, \quad X_- \;=\; -\tfrac{\sqrt{3}}{\pi}\,\bar\alpha\bar\beta\,\cos 4\tau\,. \tag{48}$$

This solution has $h=12$ and $E=48/\ell$ and takes the explicit form

$$\begin{aligned}
(E{+}\mathrm{i}B)_x \;&=\; \frac{-2\mathrm{i}}{\left((t-\mathrm{i})^2-x^2-y^2-z^2\right)^5}\,\times\\
&\times\Big\{2y+3\mathrm{i}ty-xz+2t^2y+2\mathrm{i}txz-8x^2y-8y^3+4yz^2\\
&\quad+\;4\mathrm{i}t^3y-6t^2xz-8\mathrm{i}tx^2y-8\mathrm{i}ty^3+4\mathrm{i}tyz^2+10x^3z+10xy^2z-2xz^3\\
&\quad+\;2(\mathrm{i}txz+x^2y+y^3+yz^2)(-t^2+x^2+y^2+z^2)+(\mathrm{i}ty-xz)(-t^2+x^2+y^2+z^2)^2\Big\}\,,\\
(E{+}\mathrm{i}B)_y \;&=\; \frac{2\mathrm{i}}{\left((t-\mathrm{i})^2-x^2-y^2-z^2\right)^5}\,\times\\
&\times\Big\{2x+3\mathrm{i}tx+yz+2t^2x-2\mathrm{i}tyz-8x^3-8xy^2+4xz^2\\
&\quad+\;4\mathrm{i}t^3x+6t^2yz-8\mathrm{i}tx^3-8\mathrm{i}txy^2+4\mathrm{i}txz^2-10x^2yz-10y^3z+2yz^3\\
&\quad+\;2(-\mathrm{i}tyz+x^3+xy^2+xz^2)(-t^2+x^2+y^2+z^2)+(\mathrm{i}tx+yz)(-t^2+x^2+y^2+z^2)^2\Big\}\,,\\
(E{+}\mathrm{i}B)_z \;&=\; \frac{\mathrm{i}}{\left((t-\mathrm{i})^2-x^2-y^2-z^2\right)^5}\,\times\\
&\times\Big\{1+2\mathrm{i}t+t^2-11x^2-11y^2+3z^2+4\mathrm{i}t^3-16\mathrm{i}tx^2-16\mathrm{i}ty^2+4\mathrm{i}tz^2\\
&\quad-\;t^4-2t^2x^2-2t^2y^2-2t^2z^2+11x^4+22x^2y^2+10x^2z^2+11y^4-10y^2z^2+3z^4\\
&\quad+\;2\mathrm{i}t(t^2-3x^2-3y^2-z^2)(t^2-x^2-y^2-z^2)-(t^2+x^2+y^2-z^2)(-t^2+x^2+y^2+z^2)^2\Big\}\,.
\end{aligned} \tag{49}$$

Figures 2 and 3 below show $t{=}0$ energy density level surfaces and a particular closed magnetic field line for this example. For the second example, a concrete $\left(\frac{3}{2};\frac{1}{2},\frac{3}{2}\right)$ type-I solution, $t{=}0$ energy density level surfaces are displayed in Figure 4.

6. Summary and discussion

- Rational electromagnetic fields with nontrivial topology have been investigated since 1989
- We introduced a new construction method based on two insights:
  - the simplicity of solving Maxwell's equations on a temporal cylinder over a three-sphere
  - the conformal equivalence of a cylinder patch to four-dimensional Minkowski space
- $A=X_\nu(\tau,\omega)\,e^\nu=X_\nu(\tau(x),\omega(x))\,e^\nu{}_\mu(x)\,\mathrm{d}x^\mu$
- Only finite-time $\tau\in(-\pi,+\pi)$ dynamics is required on the cylinder
- Our solutions have finite energy and action, by construction
- A complete basis was discovered for sufficiently fast spatially and temporally decaying fields
- The non-Abelian extension couples different j components of X_a and is expected to be much harder
- The method may be useful for a numerical study of Yang–Mills dynamics in Minkowski space

References

[1] Lechtenfeld O and Zhilin G 2018 *Phys. Lett. A* **382** 1528
[2] Ivanova T A, Lechtenfeld O and Popov A D 2017 *Phys. Rev. Lett.* **119** 061601
[3] Ivanova T A, Lechtenfeld O and Popov A D 2017 *J. High Energy Phys.* JHEP11(2017)017
[4] Lüscher M 1977 *Phys. Lett.* **70B** 321
[5] Actor A 1979 *Rev. Mod. Phys.* **51** 461
[6] De Alfaro V, Fubini S and Furlan G 1976 *Phys. Lett.* **65B** 163
[7] Gibbons G W and Steif A R 1995 *Phys. Lett.* **346B** 255
[8] Rañada A R 1989 *Lett. Math. Phys.* **18** 97
[9] Arrayás M, Bouwmeester D and Trueba J L 2017 *Phys. Rept.* **667** 1

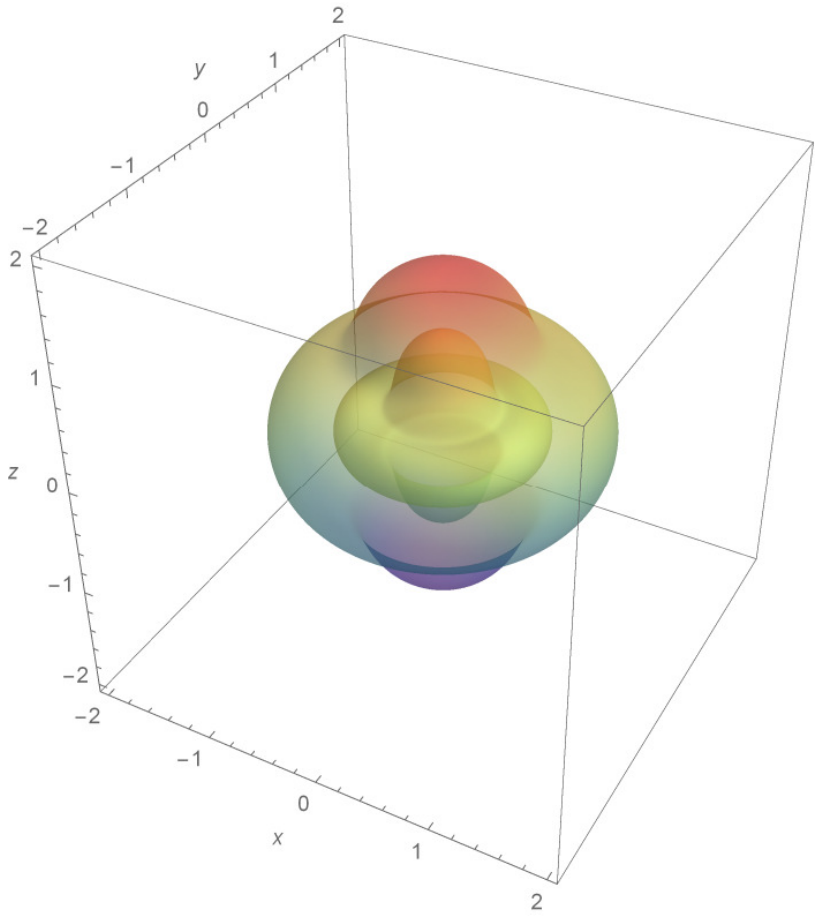

Figure 2. Energy density level surfaces at $t{=}0$ for the $(1;0,0)$ solution (49).

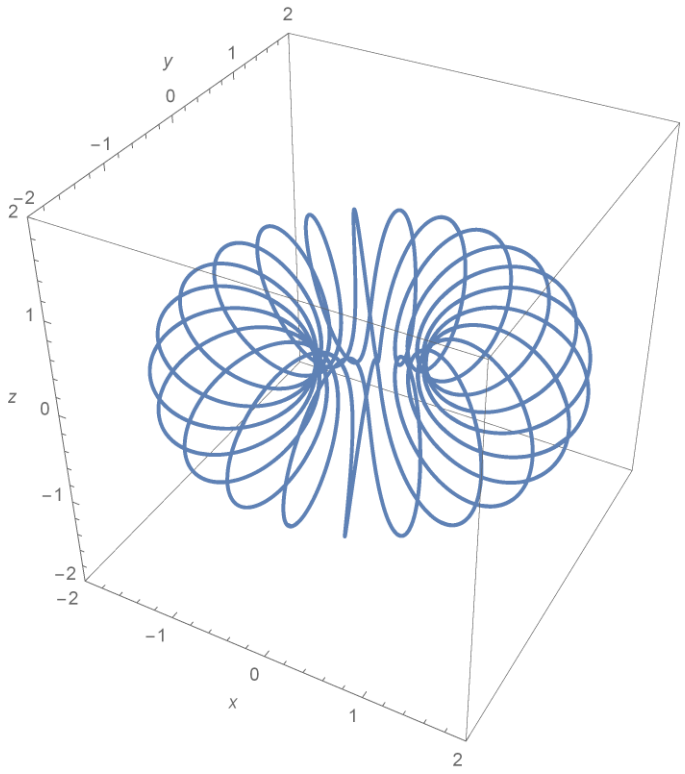

Figure 3. A particular magnetic field line for the $(1; 0, 0)$ solution (49).

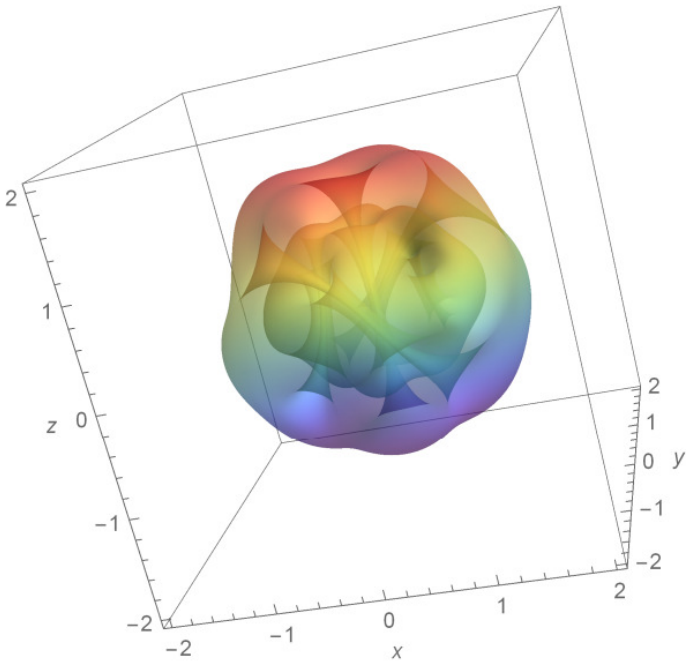

Figure 4. Energy density level surfaces at t=0 for a particular $\left(\frac{3}{2}; \frac{1}{2}, \frac{3}{2}\right)$ solution.