

**INITIAL BOUNDARY VALUE PROBLEM FOR A SEMILINEAR
PARABOLIC EQUATION WITH ABSORPTION AND
NONLINEAR NONLOCAL BOUNDARY CONDITION**

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ABSTRACT. In this paper we consider an initial boundary value problem for a semilinear parabolic equation with absorption and nonlinear nonlocal Neumann boundary condition. We prove comparison principle, the existence theorem of a local solution and study the problem of uniqueness and nonuniqueness.

1. INTRODUCTION

In this paper we consider the initial boundary value problem for the following semilinear parabolic equation

$$u_t = \Delta u - c(x, t)u^p, \quad x \in \Omega, \quad t > 0, \quad (1.1)$$

with nonlinear nonlocal boundary condition

$$\frac{\partial u(x, t)}{\partial \nu} = \int_{\Omega} k(x, y, t)u^l(y, t) dy, \quad x \in \partial\Omega, \quad t \geq 0, \quad (1.2)$$

and initial datum

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.3)$$

where $p > 0$, $l > 0$, Ω is a bounded domain in \mathbb{R}^n for $n \geq 1$ with smooth boundary $\partial\Omega$, ν is unit outward normal on $\partial\Omega$.

Throughout this paper we suppose that the functions $c(x, t)$, $k(x, y, t)$ and $u_0(x)$ satisfy the following conditions:

$$\begin{aligned} c(x, t) &\in C_{loc}^{\alpha}(\overline{\Omega} \times [0, +\infty)), \quad 0 < \alpha < 1, \quad c(x, t) \geq 0; \\ k(x, y, t) &\in C(\partial\Omega \times \overline{\Omega} \times [0, +\infty)), \quad k(x, y, t) \geq 0; \\ u_0(x) &\in C^1(\overline{\Omega}), \quad u_0(x) \geq 0 \text{ in } \Omega, \quad \frac{\partial u_0(x)}{\partial \nu} = \int_{\Omega} k(x, y, 0)u_0^l(y) dy \text{ on } \partial\Omega. \end{aligned}$$

A lot of articles have been devoted to the investigation of initial boundary value problems for parabolic equations and systems with nonlinear nonlocal Dirichlet boundary condition (see, for example, [6, 8, 12, 13, 14, 17, 18, 19, 23, 24, 26, 27] and the references therein). In particular, the initial boundary value problem for equation (1.1) with nonlocal boundary condition

$$u(x, t) = \int_{\Omega} k(x, y, t)u^l(y, t) dy, \quad x \in \partial\Omega, \quad t > 0,$$

was considered for $c(x, t) \leq 0$ and $c(x, t) \geq 0$ in [13, 14] and [17, 18] respectively. The problem (1.1)–(1.3) with $c(x, t) \leq 0$ were investigated in [15, 16].

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We note that for $p < 1$ and $l < 1$ the nonlinearities in equation (1.1) and boundary condition (1.2) are non-Lipschitzian. The problem of uniqueness and nonuniqueness for different parabolic nonlinear equations with non-Lipschitzian data in bounded domain has been addressed by several authors (see, for example, [2, 4, 5, 7, 9, 14, 18, 21] and the references therein).

The aim of this paper is to study problem (1.1)–(1.3) for any $p > 0$ and $l > 0$. We prove existence of a local solution and establish some uniqueness and nonuniqueness results.

This paper is organized as follows. In the next section we prove the existence of a local solution. Comparison principle and the problem of uniqueness and nonuniqueness for (1.1)–(1.3) are investigated in Section 3.

2. LOCAL EXISTENCE

In this section a local existence theorem for (1.1)–(1.3) will be proved. We begin with definitions of a supersolution, a subsolution and a maximal solution of (1.1)–(1.3). Let $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$, $\Gamma_T = S_T \cup \bar{\Omega} \times \{0\}$, $T > 0$.

Definition 2.1. We say that a nonnegative function $u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a supersolution of (1.1)–(1.3) in Q_T if

$$u_t \geq \Delta u - c(x, t)u^p, \quad (x, t) \in Q_T, \quad (2.1)$$

$$\frac{\partial u(x, t)}{\partial \nu} \geq \int_{\Omega} k(x, y, t)u^l(y, t) dy, \quad x \in \partial\Omega, \quad 0 \leq t < T, \quad (2.2)$$

$$u(x, 0) \geq u_0(x), \quad x \in \Omega, \quad (2.3)$$

and $u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a subsolution of (1.1)–(1.3) in Q_T if $u \geq 0$ and it satisfies (2.1)–(2.3) in the reverse order. We say that $u(x, t)$ is a solution of problem (1.1)–(1.3) in Q_T if $u(x, t)$ is both a subsolution and a supersolution of (1.1)–(1.3) in Q_T .

Definition 2.2. We say that a solution $u(x, t)$ of (1.1)–(1.3) in Q_T is a maximal solution if for any other solution $v(x, t)$ of (1.1)–(1.3) in Q_T the inequality $v(x, t) \leq u(x, t)$ is satisfied for $(x, t) \in Q_T \cup \Gamma_T$.

Definition 2.3. We say that u is a strict supersolution of problem (1.1)–(1.3) in Q_T if it is a supersolution in Q_T and the inequality in (2.2) is strict. Analogously we say that u is a strict subsolution of problem (1.1)–(1.3) in Q_T if it is a subsolution in Q_T and the inequality in (2.2) is reversed and strict.

Let $\{\varepsilon_m\}$ be decreasing to 0 sequence such that $0 < \varepsilon_m < 1$. For $\varepsilon = \varepsilon_m$ let $u_{0\varepsilon}(x)$ be the functions with the following properties: $u_{0\varepsilon}(x) \in C^1(\bar{\Omega})$, $u_{0\varepsilon}(x) \geq \varepsilon$, $u_{0\varepsilon_i}(x) \geq u_{0\varepsilon_j}(x)$ for $\varepsilon_i > \varepsilon_j$, $u_{0\varepsilon}(x) \rightarrow u_0(x)$ as $\varepsilon \rightarrow 0$ and

$$\frac{\partial u_{0\varepsilon}(x)}{\partial \nu} = \int_{\Omega} k(x, y, 0)u_{0\varepsilon}^l(y) dy,$$

for $x \in \partial\Omega$. Since the nonlinearities in (1.1), (1.2), the Lipschitz condition can be not satisfied, and thus we need to consider the following auxiliary problem:

$$\begin{cases} u_t = \Delta u - c(x, t)u^p + c(x, t)\varepsilon^p & \text{for } x \in \Omega, \quad t > 0, \\ \frac{\partial u(x, t)}{\partial \nu} = \int_{\Omega} k(x, y, t)u^l(y, t) dy & \text{for } x \in \partial\Omega, \quad t \geq 0, \\ u(x, 0) = u_{0\varepsilon}(x) & \text{for } x \in \Omega, \end{cases} \quad (2.4)$$

where $\varepsilon = \varepsilon_m$. The notion of a solution u_ε for problem (2.4) can be defined in a similar way as in the Definition 2.1.

Theorem 2.4. *Problem (2.4) has a unique solution in Q_T for small values of T .*

Proof. We start the proof with the construction of a supersolution of (2.4). Let $\sup_{\Omega} u_{0\varepsilon}(x) \leq M$. Denote $K = \sup_{\partial\Omega \times Q_1} k(x, y, t)$ and introduce an auxiliary function $\psi(x)$ with the following properties:

$$\psi(x) \in C^2(\overline{\Omega}), \inf_{\Omega} \psi(x) \geq 1, \inf_{\partial\Omega} \frac{\partial\psi(x)}{\partial\nu} \geq KM^{l-1} \max\{1, \exp(l-1)\} \int_{\Omega} \psi^l(y) dy.$$

Set $\alpha = \sup_{\Omega} \Delta\psi(x)$. Then it is not difficult to check that

$$w(x, t) = M \exp(\alpha t) \psi(x)$$

is a supersolution of (2.4) in Q_T if $T \leq \min\{1/\alpha, 1\}$.

To prove the existence of a solution for (2.4) we introduce the set

$$B = \{h(x, t) \in C(\overline{Q_T}) : \varepsilon \leq h(x, t) \leq w(x, t), h(x, 0) = u_{0\varepsilon}(x)\}$$

and consider the following problem

$$\begin{cases} u_t = \Delta u - c(x, t)u^p + c(x, t)\varepsilon^p & \text{for } x \in \Omega, \quad 0 < t < T, \\ \frac{\partial u(x, t)}{\partial\nu} = \int_{\Omega} k(x, y, t)v^l(y, t) dy & \text{for } x \in \partial\Omega, \quad 0 \leq t < T, \\ u(x, 0) = u_{0\varepsilon}(x) & \text{for } x \in \Omega, \end{cases} \quad (2.5)$$

where $v \in B$. It is obvious, B is a nonempty convex subset of $C(\overline{Q_T})$. By classical theory [22] problem (2.5) has a solution $u \in C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q_T})$. Let us call $Av = u$. In order to show that A has a fixed point in B we verify that A is a continuous mapping from B into itself such that AB is relatively compact. Thanks to a comparison principle for (2.5) we have that A maps B into itself.

Let $G(x, y; t)$ denote the Green's function for a heat equation given by

$$u_t - \Delta u = 0 \text{ for } x \in \Omega, t > 0$$

with homogeneous Neumann boundary condition. Then $u(x, t)$ is a solution of (2.5) in Q_T if and only if

$$\begin{aligned} u(x, t) &= \int_{\Omega} G(x, y; t) u_{0\varepsilon}(y) dy + \int_0^t \int_{\Omega} G(x, y; t - \tau) c(y, \tau) (\varepsilon^p - u^p(y, \tau)) dy d\tau \\ &+ \int_0^t \int_{\partial\Omega} G(x, \xi; t - \tau) \int_{\Omega} k(\xi, y, \tau) v^l(y, \tau) dy dS_{\xi} d\tau \end{aligned} \quad (2.6)$$

for $(x, t) \in Q_T$.

We claim that A is continuous. In fact let v_k be a sequence in B converging to $v \in B$ in $C(\overline{Q_T})$. Denote $u_k = Av_k$. Then by (2.6) we see that

$$\begin{aligned} |u - u_k| &\leq \theta \sup_{Q_T} |u - u_k| \int_0^t \int_{\Omega} G(x, y; t - \tau) c(y, \tau) dy d\tau \\ &+ \sup_{Q_T} |v^l - v_k^l| \int_0^t \int_{\partial\Omega} G(x, \xi; t - \tau) \int_{\Omega} k(\xi, y, \tau) dy dS_{\xi} d\tau, \end{aligned}$$

where $\theta = p \max\{\varepsilon^{p-1}, \sup_{Q_T} w^{p-1}\}$. We note that (see [11])

$$\theta \sup_{Q_T} \int_0^t \int_{\Omega} G(x, y; t - \tau) c(y, \tau) dy d\tau < 1$$

for small values of T . Now we can conclude that u_k converges to u_ε in $C(\overline{Q_T})$.

The equicontinuity of AB follows from (2.6) and the properties of the Green's function (see, for example, [25]). The Ascoli-Arzelá theorem guarantees the relative compactness of AB . Thus we are able to apply the Schauder-Tychonoff fixed point theorem and conclude that A has a fixed point in B if T is small. Now if u_ε is a fixed point of A , $u_\varepsilon \in C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q_T})$ and it is a solution of (2.4) in Q_T . Uniqueness of the solution follows from a comparison principle for (2.4) which can be proved in a similar way as in the next section. \square

Now, let $\varepsilon_2 > \varepsilon_1$. Then it is easy to see that $u_{\varepsilon_2}(x, t)$ is a supersolution of problem (2.4) with $\varepsilon = \varepsilon_1$. Applying to problem (2.4) a comparison principle we have $u_{\varepsilon_1}(x, t) \leq u_{\varepsilon_2}(x, t)$. Using the last inequality and the continuation principle of solutions we deduce that the existence time of u_ε does not decrease as $\varepsilon \searrow 0$. Taking $\varepsilon \rightarrow 0$, we get $u_m(x, t) = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t) \geq 0$ and $u_m(x, t)$ exists in Q_T for some $T > 0$. By dominated convergence theorem, $u_m(x, t)$ satisfies the following equation

$$\begin{aligned} u_m(x, t) &= \int_{\Omega} G(x, y; t) u_0(y) dy - \int_0^t \int_{\Omega} G(x, y; t - \tau) c(y, \tau) u_m^p(y, \tau) dy d\tau \\ &+ \int_0^t \int_{\partial\Omega} G(x, \xi; t - \tau) \int_{\Omega} k(\xi, y, \tau) u_m^l(y, \tau) dy d\xi d\tau \end{aligned}$$

for $(x, t) \in Q_T$. Further, the interior regularity of $u_m(x, t)$ follows from the continuity of $u_m(x, t)$ in Q_T and the properties of the Green's function. Obviously, $u_m(x, t)$ satisfies (1.1)–(1.3).

Theorem 2.5. *Problem (1.1)–(1.3) has a solution in Q_T for small values of T .*

3. UNIQUENESS AND NONUNIQUENESS

We start this section with a comparison principle for problem (1.1)–(1.3) which is used below.

Theorem 3.1. *Let \overline{u} and \underline{u} be a supersolution and a subsolution of problem (1.1)–(1.3) in Q_T , respectively. Suppose that $\underline{u}(x, t) > 0$ or $\overline{u}(x, t) > 0$ in $Q_T \cup \Gamma_T$ if $l < 1$. Then $\overline{u}(x, t) \geq \underline{u}(x, t)$ in $Q_T \cup \Gamma_T$.*

Proof. Suppose that $l \geq 1$. Let $T_0 \in (0, T)$ and $u_{0\varepsilon}(x)$ have the same properties as in previous section but $u_{0\varepsilon}(x) \rightarrow \underline{u}(x, 0)$ as $\varepsilon \rightarrow 0$. We can construct a solution $u_m(x, t)$ of (1.1)–(1.3) with $u_0(x) = \underline{u}(x, 0)$ in the following way $u_m(x, t) = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t)$ where $u_\varepsilon(x, t)$ is a solution of (2.4). To establish theorem we shall show that

$$\underline{u}(x, t) \leq u_m(x, t) \leq \overline{u}(x, t) \text{ in } \overline{Q_{T_0}} \text{ for any } T_0. \quad (3.1)$$

We prove the second inequality in (3.1) only since the proof of the first one is similar. Let $\varphi(x, t) \in C^{2,1}(\overline{Q_{T_0}})$ be a nonnegative function such that

$$\frac{\partial \varphi(x, t)}{\partial \nu} = 0, \quad (x, t) \in S_{T_0}.$$

If we multiply the first equation in (2.4) by $\varphi(x, t)$ and then integrate over Q_t for $0 < t < T_0$, we get

$$\int_{\Omega} u_\varepsilon(x, t) \varphi(x, t) dx \leq \int_{\Omega} u_\varepsilon(x, 0) \varphi(x, 0) dx + \varepsilon^p \int_0^t \int_{\Omega} c(x, \tau) \varphi dx d\tau$$

$$\begin{aligned}
 & + \int_0^t \int_{\Omega} (u_{\varepsilon} \varphi_{\tau} + u_{\varepsilon} \Delta \varphi - c(x, \tau) u_{\varepsilon}^p \varphi) dx d\tau \\
 & + \int_0^t \int_{\partial\Omega} \varphi(x, \tau) \int_{\Omega} k(x, y, \tau) u_{\varepsilon}^l(y, \tau) dy dS_x d\tau, \quad (3.2)
 \end{aligned}$$

On the other hand, \bar{u} satisfies (3.2) with reversed inequality and with $\varepsilon = 0$. Set $w(x, t) = u_{\varepsilon}(x, t) - \bar{u}(x, t)$. Then $w(x, t)$ satisfies

$$\begin{aligned}
 \int_{\Omega} w(x, t) \varphi(x, t) dx & \leq \int_{\Omega} w(x, 0) \varphi(x, 0) dx + \varepsilon^p \int_0^t \int_{\Omega} c(x, \tau) \varphi dx d\tau \\
 & + \int_0^t \int_{\Omega} (\varphi_{\tau} + \Delta \varphi - c(x, \tau) p \theta_1^{p-1} \varphi) w dx d\tau \\
 & + \int_0^t \int_{\partial\Omega} \varphi(x, \tau) \int_{\Omega} k(x, y, \tau) l \theta_2^{l-1} w(y, \tau) dy dS_x d\tau, \quad (3.3)
 \end{aligned}$$

where θ_1 and θ_2 are some continuous functions between u_{ε} and \bar{u} . Note here that by hypotheses for $c(x, t)$, $k(x, y, t)$, $u_{\varepsilon}(x, t)$ and $\bar{u}(x, t)$, we have

$$\begin{aligned}
 0 \leq c(x, t) \leq M, \quad 0 \leq \bar{u}(x, t) \leq M \quad \varepsilon \leq u_{\varepsilon}(x, t) \leq M \quad \text{in } \bar{Q}_{T_0} \\
 \text{and } 0 \leq k(x, y, t) \leq M \quad \text{in } \partial\Omega \times \bar{Q}_{T_0}, \quad (3.4)
 \end{aligned}$$

where M is some positive constant. Then, it is easy to see from (3.4) that θ_1^{p-1} and θ_2^{l-1} are positive and bounded functions in \bar{Q}_{T_0} and moreover, $\theta_2^{l-1} \leq M^{l-1}$. Define a sequence $\{a_n\}$ in the following way: $a_n \in C^{\infty}(\bar{Q}_{T_0})$, $a_n \geq 0$ and $a_n \rightarrow c(x, t) p \theta_1^{p-1}$ as $n \rightarrow \infty$ in $L^1(Q_{T_0})$. Now, consider a backward problem given by

$$\begin{cases} \varphi_{\tau} + \Delta \varphi - a_n \varphi = 0 & \text{for } x \in \Omega, \quad 0 < \tau < t, \\ \frac{\partial \varphi(x, \tau)}{\partial \nu} = 0 & \text{for } x \in \partial\Omega, \quad 0 \leq \tau < t, \\ \varphi(x, t) = \psi(x) & \text{for } x \in \Omega, \end{cases} \quad (3.5)$$

where $\psi(x) \in C_0^{\infty}(\Omega)$ and $0 \leq \psi(x) \leq 1$. Denote a solution of (3.5) as $\varphi_n(x, \tau)$. Then by the standard theory for linear parabolic equations (see [22], for example), we find that $\varphi_n \in C^{2,1}(\bar{Q}_t)$, $0 \leq \varphi_n(x, \tau) \leq 1$ in \bar{Q}_t . Putting $\varphi = \varphi_n$ in (3.3) and passing then to the limit as $n \rightarrow \infty$ we infer

$$\int_{\Omega} w(x, t) \psi(x) dx \leq \int_{\Omega} w(x, 0)_+ dx + \varepsilon^p M T_0 |\Omega| + l M^l |\partial\Omega| \int_0^t \int_{\Omega} w(y, \tau)_+ dy d\tau, \quad (3.6)$$

where $w_+ = \max\{w, 0\}$, $|\partial\Omega|$ and $|\Omega|$ are the Lebesgue measures of $\partial\Omega$ in \mathbb{R}^{n-1} and Ω in \mathbb{R}^n , respectively. Since (3.6) holds for every $\psi(x)$, we can choose a sequence $\{\psi_n\}$ converging on Ω to $\psi(x) = 1$ if $w(x, t) > 0$ and $\psi(x) = 0$ otherwise. Hence, from (3.6) we get

$$\int_{\Omega} w(x, t)_+ dx \leq \int_{\Omega} w(x, 0)_+ dx + \varepsilon^p M T_0 |\Omega| + l M^l |\partial\Omega| \int_0^t \int_{\Omega} w(y, \tau)_+ dy d\tau.$$

Applying now Gronwall's inequality and passing to the limit $\varepsilon \rightarrow 0$, the conclusion of this theorem follows for $l \geq 1$. For the case $l < 1$ we can consider $w(x, t) = \underline{u}(x, t) - \bar{u}(x, t)$ and prove the theorem in a similar way using the positiveness of a subsolution or a supersolution. \square

Corollary 3.2. *Problem (1.1)–(1.3) has a maximal solution in Q_T for small values of T .*

Proof. In the previous section we prove the existence of a local solution $u_m(x, t) = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t)$. Let $v(x, t)$ be any other solution of (1.1)–(1.3) in Q_T . By Theorem 3.1 we have $u_\varepsilon(x, t) \geq v(x, t)$. Taking $\varepsilon \rightarrow 0$, we conclude $u_m(x, t) \geq v(x, t)$. Obviously, $u_m(x, t)$ is a maximal solution of (1.1)–(1.3) in Q_T . \square

Next lemma shows the positiveness of all nontrivial solutions for $t > 0$ if $p \geq 1$.

Lemma 3.3. *Let u_0 is a nontrivial function in Ω , $p \geq 1$ or $c(x, t) \equiv 0$. Suppose u is a solution of (1.1)–(1.3) in Q_T . Then $u > 0$ in $Q_T \cup S_T$.*

Proof. As $c(x, t)$ and $u(x, t)$ are continuous in $\overline{Q_T}$ functions then we have

$$\max\{c(x, t), u(x, t)\} \leq M, \quad (x, t) \in \overline{Q_T} \quad (3.7)$$

with some positive constant M . Now we put $v = u \exp(\lambda t)$ where $\lambda \geq M^p$. It is easy to verify that $v_t - \Delta v \geq 0$. Since $v(x, 0) = u_0(x) \not\equiv 0$ in Ω and $v(x, t) \geq 0$ in Q_T , by the strong maximum principle $v(x, t) > 0$ in Q_T . Let $v(x_0, t_0) = 0$ in some point $(x_0, t_0) \in S_T$. Then according to Theorem 3.6 of [10] it yields $\partial v(x_0, t_0)/\partial \nu < 0$, which contradicts (1.2). \square

As a simple consequence of Theorem 3.1 and Lemma 3.3, we get the following uniqueness result for problem (1.1)–(1.3).

Theorem 3.4. *Let problem (1.1)–(1.3) has a solution in Q_T with nonnegative initial data for $l \geq 1$ and with positive initial data under the conditions $l < 1$, $p \geq 1$ or a positive in $Q_T \cup \Gamma_T$ solution if $\max(p, l) < 1$. Then a solution of (1.1)–(1.3) is unique in Q_T .*

Now we shall prove the nonuniqueness of a solution of our problem with trivial initial datum for $l < p$. We note that problem (1.1)–(1.3) with trivial initial datum has trivial solution.

Theorem 3.5. *Let $l < \min\{1, p\}$ and $u_0(x) \equiv 0$. Suppose that*

$$k(x, y_0, t_0) > 0 \text{ for any } x \in \partial\Omega \text{ and some } y_0 \in \partial\Omega \text{ and } t_0 \in [0, T]. \quad (3.8)$$

Then maximal solution $u_m(x, t)$ of problem (1.1)–(1.3) is nontrivial function in Q_T .

Proof. As we showed in Theorem 2.5 and Corollary 3.2 a maximal solution $u_m(x, t) = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t)$, where $u_\varepsilon(x, t)$ is some positive in $\overline{Q_T}$ supersolution of (1.1)–(1.3). To prove theorem we construct a nontrivial nonnegative subsolution $\underline{u}(x, t)$ of (1.1)–(1.3) with trivial initial datum. By Theorem 3.1 then we have $u_\varepsilon(x, t) \geq \underline{u}(x, t)$ and therefore maximal solution $u_m(x, t)$ is nontrivial function.

To construct a subsolution we use the change of variables in a neighborhood of $\partial\Omega$ as in [3]. Let \overline{x} be a point in $\partial\Omega$. We denote by $\hat{n}(\overline{x})$ the inner unit normal to $\partial\Omega$ at the point \overline{x} . Since $\partial\Omega$ is smooth it is well known that there exists $\delta > 0$ such that the mapping $\psi : \partial\Omega \times [0, \delta] \rightarrow \mathbb{R}^n$ given by $\psi(\overline{x}, s) = \overline{x} + s\hat{n}(\overline{x})$ defines new coordinates (\overline{x}, s) in a neighborhood of $\partial\Omega$ in $\overline{\Omega}$.

A straightforward computation shows that, in these coordinates, Δ applied to a function $g(\overline{x}, s) = g(s)$, which is independent of the variable \overline{x} , evaluated at a point (\overline{x}, s) is given by

$$\Delta g(\overline{x}, s) = \frac{\partial^2 g}{\partial s^2}(\overline{x}, s) - \sum_{j=1}^{n-1} \frac{H_j(\overline{x})}{1 - sH_j(\overline{x})} \frac{\partial g}{\partial s}(\overline{x}, s), \quad (3.9)$$

where $H_j(\bar{x})$ for $j = 1, \dots, n-1$, denotes the principal curvatures of $\partial\Omega$ at \bar{x} .

Under the assumptions of the theorem there exists $\bar{t} > 0$ such that $k(x, y, t) > 0$ for $t_0 \leq t \leq t_0 + \bar{t}$, $x \in \partial\Omega$ and $y \in V(y_0)$, where $V(y_0)$ is some neighborhood of y_0 in $\bar{\Omega}$.

Let $1/(1-l) < \alpha \leq 1/(1-p)$ for $p < 1$ and $\alpha > 1/(1-l)$ for $p \geq 1$, $2 < \beta < 2/(1-p)$ for $p < 1$ and $\beta > 2$ for $p \geq 1$ and assume that $A > 0$, $0 < \xi_0 \leq 1$ and $0 < T_0 < \min(T - t_0, \bar{t}, \delta^2)$. For points in $\partial\Omega \times [0, \delta] \times (t_0, t_0 + T_0]$ of coordinates (\bar{x}, s, t) define

$$\underline{u}(\bar{x}, s, t) = A(t - t_0)^\alpha \left(\xi_0 - \frac{s}{\sqrt{t - t_0}} \right)_+^\beta \quad (3.10)$$

and extend \underline{u} as zero to the whole of $\overline{Q_\tau}$ with $\tau = t_0 + T_0$. Using (3.9), we get that

$$\begin{aligned} & \underline{u}_t(\bar{x}, s, t) - \Delta \underline{u}(\bar{x}, s, t) + c(x, t) \underline{u}^p(\bar{x}, s, t) = \alpha A(t - t_0)^{\alpha-1} \left(\xi_0 - \frac{s}{\sqrt{t - t_0}} \right)_+^\beta \\ & + \frac{\beta}{2} A s (t - t_0)^{\alpha-3/2} \left(\xi_0 - \frac{s}{\sqrt{t - t_0}} \right)_+^{\beta-1} - \beta(\beta-1) A (t - t_0)^{\alpha-1} \left(\xi_0 - \frac{s}{\sqrt{t - t_0}} \right)_+^{\beta-2} \\ & - \beta A (t - t_0)^{\alpha-1/2} \left(\xi_0 - \frac{s}{\sqrt{t - t_0}} \right)_+^{\beta-1} \sum_{j=1}^{n-1} \frac{H_j(\bar{x})}{1 - s H_j(\bar{x})} \\ & + A^p c(x, t) (t - t_0)^{\alpha p} \left(\xi_0 - \frac{s}{\sqrt{t - t_0}} \right)_+^{\beta p} \leq 0 \end{aligned}$$

for $(\bar{x}, s, t) \in \partial\Omega \times (0, \delta] \times (t_0, \tau)$ and small values of ξ_0 .

It is obvious,

$$\frac{\partial \underline{u}}{\partial \nu}(\bar{x}, 0, t) = -\frac{\partial \underline{u}}{\partial s}(\bar{x}, 0, t) = \beta A (t - t_0)^{\alpha-1/2} \xi_0^{\beta-1}, \quad \bar{x} \in \partial\Omega, \quad t_0 < t < \tau.$$

To prove that \underline{u} is the subsolution of (1.1)–(1.3) in Q_τ it is enough to check the validity of the following inequality

$$\beta A (t - t_0)^{\alpha-1/2} \xi_0^{\beta-1} \leq A^l (t - t_0)^{\alpha l} \int_{\partial\Omega \times [0, \delta]} k(x, (\bar{y}, s), t) |J(\bar{y}, s)| \left(\xi_0 - \frac{s}{\sqrt{t - t_0}} \right)_+^{\beta l} d\bar{y} ds \quad (3.11)$$

for $x \in \partial\Omega$ and $t_0 < t < \tau$. Here $J(\bar{y}, s)$ is Jacobian of the change of variables. Estimating the integral I in the right-hand side of (3.11)

$$I = (t - t_0)^{\frac{1}{2}} \int_{\partial\Omega} d\bar{y} \int_0^{\xi_0} k(x, (\bar{y}, z\sqrt{t - t_0}), t) |J(\bar{y}, z\sqrt{t - t_0})| (\xi_0 - z)_+^{\beta l} dz \geq C (t - t_0)^{\frac{1}{2}},$$

where positive constant C does not depend on t , we obtain that (3.11) is true if we take T_0 sufficiently small. \square

Remark 3.6. Let $c(x, t_0) \leq c_0$ and $k(x, y, t_0) \geq k_0$ for $x \in \Omega$, $y \in \partial\Omega$, some $t_0 \in [0, T)$ and positive constants c_0 and k_0 . Then nonuniqueness of trivial solution for our problem holds for $l = p < 1$ and large values of k_0/c_0 . To prove this we can take in (3.10) $\alpha = 1/(1-l)$, $\beta = 2/(1-l)$ and A in a suitable way.

To prove the uniqueness of trivial solution of (1.1)–(1.3) with $u_0(x) \equiv 0$ we need the following comparison principle.

Lemma 3.7. *Let \underline{u} and \bar{u} be a subsolution and a strict supersolution of (1.1)–(1.3) in Q_T , respectively. Then $\bar{u} \geq \underline{u}$ in \bar{Q}_T .*

Proof. Let $v = \bar{u} + \varepsilon$ for $\varepsilon > 0$. Note that for any $T_0 < T$ under suitable choice of ε we have

$$\frac{\partial v(x, t)}{\partial \nu} \geq \int_{\Omega} k(x, y, t) v^l(y, t) dy \text{ for } x \in \partial\Omega, \ 0 \leq t \leq T_0.$$

Now we can apply Theorem 3.1 to get the inequality $v \geq \underline{u}$ in \bar{Q}_{T_0} . Passing here to the limit as $\varepsilon \rightarrow 0$ we prove the lemma. \square

Theorem 3.8. *Let $p < l < 1$ and $u_0(x) \equiv 0$. Suppose that $c(x, t) \geq c_1 > 0$ for $(x, t) \in \bar{Q}_T$. Then the solution $u \equiv 0$ of problem (1.1)–(1.3) is unique in Q_T .*

Proof. Since there is the comparison principle for a solution and a strict supersolution of (1.1)–(1.3), it is sufficiently to construct arbitrarily small strict supersolutions which have a positive values on $\partial\Omega$ (see [5] for another problem).

We shall use the change of variables in a neighborhood of $\partial\Omega$ which we introduced in Theorem 3.5. Let $(1-l)/2 < \gamma < (1-p)/2$, $0 < \varepsilon < \delta^{1/\gamma}$, $A > 0$ and $0 < \xi_0 \leq 1$. For points in $\partial\Omega \times [0, \delta] \times [0, T]$ of coordinates (\bar{x}, s, t) define

$$\bar{u}_\varepsilon(\bar{x}, s, t) = \varepsilon A (\xi_0 - \varepsilon^{-\gamma} s)_+^{1/\gamma} \quad (3.12)$$

and extend $\underline{u}_\varepsilon$ as zero to the whole of \bar{Q}_T . Using (3.9), we get that

$$\bar{u}_{\varepsilon t}(\bar{x}, s, t) - \Delta \bar{u}_\varepsilon(\bar{x}, s, t) + c(x, t) \bar{u}_\varepsilon^p(\bar{x}, s, t) \geq 0$$

in Q_T for small values of ξ_0 .

To show that \bar{u}_ε is a strict supersolution we need to prove the following inequality

$$\frac{A}{\gamma} \varepsilon^{1-\gamma} \xi_0^{(1-\gamma)/\gamma} > (\varepsilon A)^l \int_{\partial\Omega \times [0, \delta]} k(x, (\bar{y}, s), t) |J(\bar{y}, s)| (\xi_0 - \varepsilon^{-\gamma} s)_+^{l/\gamma} d\bar{y} ds \quad (3.13)$$

for $x \in \partial\Omega$ and $0 \leq t < T$. Let $k(x, y, t) \leq k_1$ in Q_T . Then estimating the right-hand side of (3.13) J , we get

$$J \leq k_1 C A^l \varepsilon^{l+\gamma} \xi_0^{(l+\gamma)/\gamma},$$

where positive constant C depend only on p, l and $\partial\Omega$. Hence (3.13) holds if we take ξ_0 small enough.

We are in a position to complete the proof. Suppose for a contradiction that there exists a solution of (1.1)–(1.3) with trivial initial function which is not identically zero in Q_T . Then by Lemma 3.7 it follows that

$$u(x, t) \leq u_\varepsilon(x, t) \text{ in } Q_T$$

for all $0 < \varepsilon < \delta^{1/\gamma}$. This is a contradiction since $u_\varepsilon \rightarrow 0$ uniformly on \bar{Q}_T as $\varepsilon \rightarrow 0$. \square

Remark 3.9. Let the assumptions of Theorem 3.8 fulfill but only $l = p$. Then the conclusion of Theorem 3.8 holds for large values of c_1/k_1 , where k_1 was defined in the proof. To prove this we can take in (3.12) $\gamma = (1-l)/2$ and A in a suitable way.

Remark 3.10. Note that under the assumptions of Theorem 3.8 or Remark 3.9 there exists a class of nontrivial initial data such that $u(x, t) \equiv 0$ for large values of t . To prove this we can modify function \bar{u}_ε from (3.12) in the following way

$$\bar{u}_\varepsilon(\bar{x}, s, t) = \varepsilon (\xi_0 - \varepsilon^{-\gamma} s - \mu t)_+^{1/\gamma}, \quad \mu > 0.$$

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