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A New Definition of t-Entropy for Transfer Operators

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Received: 7 September 2017; Accepted: 22 October 2017; Published: 25 October 2017

Abstract: This article presents a new definition of t-entropy that makes it more explicit and simplifies the process of its calculation.

Keywords: transfer operator; variational principle; t-entropy

MSC: 37A35; 47B37; 47C15

1. Introduction

The spectral analysis of operators associated with dynamical systems is of considerable importance. In particular, in the series of articles [1–6], a relation between t-entropy and spectral radii of the corresponding operators has been established. Here, the authors have uncovered a new dynamical invariant—t-entropy—that is related to the Legendre transform of the spectral exponent of the operators in question. The t-entropy plays a significant role in various nonlinear phenomena. In particular, it serves as a principal object in thermodynamical formalism (see [2,6,7], and the sources quoted therein). The description of t-entropy is not elementary and its calculation is rather sophisticated. In the present article, we give a new definition of t-entropy that makes it more explicit and essentially simplifies the process of its calculation.

The article consists of two sections. In Section 2, we consider t-entropy for the model example. Here, Theorem 2 gives a new definition of t-entropy, that simplifies its calculation. The general situation of arbitrary C^*-dynamical system is discussed in Section 3. To illustrate similarity and difference between the objects considered in the model and general situations, we present here a number of examples and finally introduce the general new definition of t-entropy in Theorem 3.

2. A New Definition of t-Entropy for Continuous Dynamical Systems

In this Section, we consider a model example. Here, we use definitions, notation, and results from [4,5]. We denote by X a Hausdorff compact space, and by $C(X)$ we denote the algebra of continuous functions on X taking real values and equipped with the max-norm. Consider an arbitrary continuous mapping $\alpha\colon X \to X$. The corresponding dynamical system will be denoted by (X, α).

The main object under investigation is a *transfer operator* $A\colon C(X) \to C(X)$, associated with a given dynamical system. Its definition is given in the following way:

(a) A is a positive operator (that is it maps nonnegative functions to nonnegative) and
(b) the following *homological identity* for A is valid:

$$A(g \circ \alpha \cdot f) = gAf, \qquad g, f \in C(X). \tag{1}$$

The set of linear positive normalized functionals on $C(X)$ will simply be denoted by M. The Riesz theorem states that elements of M can be identified with regular Borel probability measures on X and henceforth we assume this identification and, therefore, elements of M will be called *probability measures*.

Let us recall the classical definition of an invariant measure: $\mu \in M$ is *α-invariant* if $\mu(g) = \mu(g \circ \alpha)$ for $g \in C(X)$. The family of α-invariant probability measures on X is denoted by M_α.

A continuous *partition of unity* in $C(X)$ is a finite set $G = \{g_1, \dots, g_k\}$ consisting of nonnegative functions $g_i \in C(X)$ satisfying the identity $g_1 + \dots + g_k \equiv 1$.

According to [5], *t-entropy* is the functional $\tau(\mu)$ on M which is defined in three steps.

Firstly, for a given $\mu \in M$, each partition of unity $G = \{g_1, \dots, g_k\}$, and any $n \in \mathbb{N}$ we set

$$\tau_n(\mu, G) := \sup_{m \in M} \sum_{g_i \in G} \mu(g_i) \ln \frac{m(A^n g_i)}{\mu(g_i)}. \tag{2}$$

Here, if $\mu(g_i) = 0$ for some $g_i \in G$ then the corresponding summand in (2) is assumed to be zero regardless of the value $m(A^n g_i)$; if $A^n g_i = 0$ for some $g_i \in G$ and at the same time $\mu(g_i) > 0$, then $\tau_n(\mu, G) = -\infty$.

Secondly, we put

$$\tau_n(\mu) := \inf_G \tau_n(\mu, G), \tag{3}$$

here, the infimum is taken over all partitions of unity G in $C(X)$.

Finally, the t-entropy $\tau(\mu)$ is defined as

$$\tau(\mu) := \inf_{n \in \mathbb{N}} \frac{\tau_n(\mu)}{n}. \tag{4}$$

Let A be a given transfer operator in $C(X)$. In what follows, we denote by A_φ the family of transfer operators in $C(X)$, where $\varphi \in C(X)$, given by the formula

$$A_\varphi f = A(e^\varphi f).$$

Next, we denote by $\lambda(\varphi)$ the *spectral potential* of A_φ, namely,

$$\lambda(\varphi) = \lim_{n \to \infty} \frac{1}{n} \ln \left\| A_\varphi^n \right\|.$$

The principal importance of t-entropy is clearly demonstrated by the following Variational Principle.

Theorem 1. ([5], Theorem 5.6) *Let $A : C(X) \to C(X)$ be a transfer operator for a continuous mapping $\alpha : X \to X$ of a compact Hausdorff space X. Then,*

$$\lambda(\varphi) = \max_{\mu \in M_\alpha} \big(\mu(\varphi) + \tau(\mu)\big), \qquad \varphi \in C(X).$$

The next principal result of the article presents a new definition of t-entropy.

Theorem 2. *For α-invariant measures $\mu \in M_\alpha$, the following formula is true*

$$\tau(\mu) = \inf_{n,G} \frac{1}{n} \sum_{g \in G} \mu(g) \ln \frac{\mu(A^n g)}{\mu(g)}. \tag{5}$$

In other words, in the definition of t-entropy, one should not calculate the supremum in (2) but can simply put $m = \mu$ there. Thus, expression (2) is changed for

$$\tau_n'(\mu, G) = \sum_{g \in G} \mu(g) \ln \frac{\mu(A^n g)}{\mu(g)}. \tag{6}$$

Remark 1. *In connection with Theorem 2, it is worth mentioning the results of [7], where for a special case of transfer operator similar formulae are obtained and their relation to thermodynamical formalism is studied.*

To prove Theorem 2, we need the next

Lemma 1. *Let G be a partition of unity in $C(X)$. Then, for any pair of numbers $n \in \mathbb{N}$, $\varepsilon > 0$ there exists a partition of unity E in $C(X)$ such that for each pair of functions $g \in G$ and $h \in E$ the oscillation of $A^n g$ over* $\operatorname{supp} h := \{x \in X \mid h(x) > 0\}$ *is less than ε:*

$$\sup\{A^n g(x) \mid h(x) > 0\} - \inf\{A^n g(x) \mid h(x) > 0\} < \varepsilon. \tag{7}$$

Proof. For any $g \in G$ and $n \in \mathbb{N}$, the function $A^n g$ belongs to $C(X)$. Therefore, its range is contained in a certain segment $[a, b]$.

Evidently, there exists a partition of unity $\{f_1, \dots, f_k\}$ in $C[a, b]$ such that the support of every one of its elements is contained in a certain interval of the length less than ε. Then, the family $E_g = \{f_1 \circ A^n g, \dots, f_k \circ A^n g\}$ forms a partition of unity in $C(X)$ and the oscillation of $A^n g$ is less than ε on the support of each of its elements. Now all the products $\prod_{g \in G} h_g$, where $h_g \in E_g$, form the desired partition of unity E. □

Now let us prove Theorem 2. Comparing (2) and (6), one sees that

$$\tau'_n(\mu, G) \le \tau_n(\mu, G).$$

Therefore, to prove (5), it is enough to verify the inequality

$$\tau_n(\mu) \le \tau'_n(\mu, G).$$

Since in the case when $\tau_n(\mu) = -\infty$ the latter inequality is trivial, in what follows we assume that $\tau_n(\mu) > -\infty$.

Let us fix some $n \in \mathbb{N}$, a partition of unity G in $C(X)$ and $\varepsilon > 0$. For these objects, there exists a continuous partition of unity E mentioned in Lemma 1. Consider one more partition of unity in $C(X)$ that consists of the functions $g \cdot h \circ \alpha^n$, here $g \in G$ and $h \in E$. For this partition, by the definition of $\tau_n(\mu)$ (see (2) and (3)), there exists a probability measure $m \in M$ for which the next inequality holds:

$$\tau_n(\mu) - \varepsilon \le \sum_{g \in G} \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln \frac{m\big(A^n(g \cdot h \circ \alpha^n)\big)}{\mu(g \cdot h \circ \alpha^n)}.$$

From the homological identity, it follows that $A^n(g \cdot h \circ \alpha^n) = hA^n g$. Therefore, the latter inequality is equivalent to

$$\tau_n(\mu) - \varepsilon \le \sum_{g \in G} \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln \frac{m(hA^n(g))}{\mu(g \cdot h \circ \alpha^n)}. \tag{8}$$

Now for each pair $g \in G$, $h \in E$ choose a number y_{gh} satisfying two conditions

$$m(hA^n g) = m(h) y_{gh}, \tag{9}$$

$$\inf\{A^n g(x) \mid h(x) > 0\} \le y_{gh} \le \sup\{A^n g(x) \mid h(x) > 0\}. \tag{10}$$

Then, inequality (8) takes the form

$$\tau_n(\mu) - \varepsilon \le \sum_{g \in G} \sum_{h \in E} \mu(g \cdot h \circ \alpha^n) \ln \frac{m(h) y_{gh}}{\mu(g \cdot h \circ \alpha^n)}, \tag{11}$$

which is equivalent to

$$\tau_n(\mu) - \varepsilon \le \sum_{g\in G}\sum_{h\in E} \mu(g\cdot h\circ\alpha^n)\ln\frac{y_{gh}}{\mu(g\cdot h\circ\alpha^n)} + \sum_{g\in G}\sum_{h\in E} \mu(g\cdot h\circ\alpha^n)\ln m(h). \tag{12}$$

Let us consider separately the second summand in the right-hand side of (12):

$$\sum_{g\in G}\sum_{h\in E} \mu(g\cdot h\circ\alpha^n)\ln m(h) = \sum_{h\in E}\mu(h\circ\alpha^n)\ln m(h) = \sum_{h\in E}\mu(h)\ln m(h). \tag{13}$$

Here, in the left-hand equality, we have exploited the fact that G is a partition of unity and in the right-hand equality we have used α-invariance of μ. If we treat $m(h)$ in (13) as independent nonnegative variables satisfying the condition $\sum_{h\in E} m(h) = 1$, then the routine usage of the Lagrange multipliers principle shows that the function $\sum_{h\in E}\mu(h)\ln m(h)$ attains its maximum when $m(h) = \mu(h)$. Evidently, the same is true for the right-hand sides in (12) and (11). Therefore,

$$\tau_n(\mu) - \varepsilon \le \sum_{g\in G}\sum_{h\in E} \mu(g\cdot h\circ\alpha^n)\ln\frac{\mu(h)y_{gh}}{\mu(g\cdot h\circ\alpha^n)}\,. \tag{14}$$

Observe that estimates (7) and (10) imply

$$\mu(h)y_{gh} \le \mu\big(h(A^n g + \varepsilon)\big). \tag{15}$$

Observing that the logarithm is a concave function, and using (14), (15), and the fact that E is a partition of unity in $C(X)$, we conclude that

$$\begin{aligned}
\tau_n(\mu) - \varepsilon &\le \sum_{g\in G}\sum_{h\in E} \mu(g\cdot h\circ\alpha^n)\ln\frac{\mu\big(h(A^n g+\varepsilon)\big)}{\mu(g\cdot h\circ\alpha^n)}\\
&= \sum_{g\in G}\mu(g)\sum_{h\in E}\frac{\mu(g\cdot h\circ\alpha^n)}{\mu(g)}\ln\frac{\mu\big(h(A^n g+\varepsilon)\big)}{\mu(g\cdot h\circ\alpha^n)}\\
&\le \sum_{g\in G}\mu(g)\ln\sum_{h\in E}\frac{\mu\big(h(A^n g+\varepsilon)\big)}{\mu(g)} = \sum_{g\in G}\mu(g)\ln\frac{\mu(A^n g+\varepsilon)}{\mu(g)}.
\end{aligned}$$

By the arbitrariness of ε, this implies

$$\tau_n(\mu) \le \sum_{g\in G}\mu(g)\ln\frac{\mu(A^n g)}{\mu(g)} = \tau'_n(\mu, G)$$

and finishes the proof of Theorem 2.

Now let us proceed to the general C^*-dynamical setting.

3. The General Case of C^*-Dynamical Systems

The general notion of t-entropy involves the so-called base algebra and a transfer operator for a C^*-dynamical system. Let us recall definitions of these objects (see [5]).

Let $\mathcal{B}$ be a commutative C^*-algebra with an identity $\mathbf{1}$ and $\mathcal{C}$ be its selfadjoint part, that is,

$$\mathcal{C} = \{\, b\in\mathcal{B} \mid b^* = b \,\}.$$

In this situation, we call $\mathcal{C}$ a *base algebra*.

A C^*-*dynamical system* is a pair $(\mathcal{C}, \delta)$, where δ is an endomorphism of $\mathcal{C}$ satisfying the equality $\delta(\mathbf{1}) = \mathbf{1}$.

Definition of a *transfer operator* A (for $(\mathcal{C},\delta)$) is given in the following way:

(a) A is a linear positive operator in $\mathcal{C}$ and
(b) the homological identity for A is valid:

$$A\big((\delta g)f\big) = gAf, \qquad g, f \in \mathcal{C}. \tag{16}$$

Let $M(\mathcal{C})$ be the family of all linear positive normalized functionals on $\mathcal{C}$. A functional $\mu \in M(\mathcal{C})$ is δ-*invariant* if $\mu(\delta f) = \mu(f)$ for all $f \in \mathcal{C}$. By $M_\delta(\mathcal{C})$, we denote the family of all δ-invariant functionals from $M(\mathcal{C})$.

By a *partition of unity* in the algebra $\mathcal{C}$, we mean any finite collection $G = \{g_1, \ldots, g_k\}$ consisting of nonnegative elements $g_i \in \mathcal{C}$ satisfying the identity $g_1 + \cdots + g_k = \mathbf{1}$.

The formulae (2)–(4) from the previous section naturally lead to a definition of t-entropy for C^*-dynamical systems. Namely, the t-*entropy* $\tau(\mu)$ for $\mu \in M(\mathcal{C})$ is defined in three steps as follows:

$$\tau_n(\mu, G) := \sup_{m \in M(\mathcal{C})} \sum_{g \in G} \mu(g) \ln \frac{m(A^n g)}{\mu(g)}, \tag{17}$$

$$\tau_n(\mu) := \inf_G \tau_n(\mu, G), \tag{18}$$

and

$$\tau(\mu) := \inf_{n \in \mathbb{N}} \frac{\tau_n(\mu)}{n}. \tag{19}$$

The infimum in (18) is taken over all the partitions of unity G in $\mathcal{C}$.

The t-entropy just defined is of principal importance in spectral analysis of abstract transfer and weighted shift operators in L^p-type spaces (see [5], Theorems 6.10, 11.2, 13.1 and 13.6).

The similarity and essential difference between the objects considered in this and the previous sections are discussed in ([5], Section 7).

We now present the C*-dynamical analogue to Theorem 2.

Theorem 3. *For δ-invariant functionals $\mu \in M_\delta(\mathcal{C})$, the following formula is true*

$$\tau(\mu) = \inf_{n,G} \frac{1}{n} \sum_{g \in G} \mu(g) \ln \frac{\mu(A^n g)}{\mu(g)}. \tag{20}$$

Proof. This theorem can be derived from Theorem 2.

By means of the Gelfand transform, one can establish an isomorphism between the algebra $\mathcal{C}$ and the algebra $C(X)$ of continuous functions on X with real values (where X is the compact space of maximal ideals in $\mathcal{C}$).

Moreover, under the identification of $\mathcal{C}$ and $C(X)$ the endomorphism δ mentioned in the definition of the C*-dynamical system $(\mathcal{C},\delta)$ takes the form

$$[\delta f](x) = f(\alpha(x))$$

(for details, see [5], Theorem 6.2). Thus, the C*-dynamical system $(\mathcal{C},\delta)$ is completely defined by the corresponding dynamical system (X,α).

In terms of (X,α), the homological identity (16) for the transfer operator A can be rewritten as (1).

By the Riesz theorem, the identification between measures μ on X and functionals $\mu \in \mathcal{C}$ is given by

$$\mu(g) = \int_X g\, d\mu, \qquad g \in \mathcal{C} = C(X). \tag{21}$$

Finally, if $\mu \in M_\delta(\mathcal{C})$ is a δ-invariant functional, then the corresponding measure μ in (21) is α-invariant, that is

$$\mu(g) = \mu(g \circ \alpha), \qquad g \in C(X).$$

In this manner, one identifies the set $M_\delta(\mathcal{C})$ with M_α mentioned in Section 2.

Under all these identifications, the desired result follows from Theorem 2. □

Author Contributions: All the results were obtained in collaboration of the authors.

Conflicts of Interest: The authors declare no conflict of interest.

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