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BLOW-UP PROBLEM FOR SEMILINEAR HEAT EQUATION WITH NONLINEAR NONLOCAL NEUMANN BOUNDARY CONDITION

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Abstract. In this paper, we consider a semilinear parabolic equation with nonlinear nonlocal Neumann boundary condition and nonnegative initial datum. We first prove global existence result. We then give some criteria on this problem which determine whether the solutions blow up in finite time for large or for all nontrivial initial data. Finally, we show that under certain conditions blow-up occurs only on the boundary.

1. Introduction.

We consider the initial boundary value problem for the following semilinear parabolic equation

$$u_t = \Delta u - c(x,t)u^p,\ x \in \Omega,\ t > 0, \tag{1}$$

with nonlinear nonlocal boundary condition

$$\frac{\partial u(x,t)}{\partial \nu} = \int_\Omega k(x,y,t)u^l(y,t)\,dy,\ x \in \partial\Omega,\ t > 0, \tag{2}$$

and initial datum

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{3}$$

where $p > 0$, $l > 0$, Ω is a bounded domain in $\mathbb{R}^n$ for $n \geq 1$ with smooth boundary $\partial\Omega$, ν is unit outward normal on $\partial\Omega$.

Throughout this paper we suppose that the functions $c(x,t),\ k(x,y,t)$ and $u_0(x)$ satisfy the following conditions:

$$c(x,t) \in C^\alpha_{loc}(\overline{\Omega} \times [0,+\infty)),\ 0 < \alpha < 1,\ c(x,t) \geq 0;$$

$$k(x,y,t) \in C(\partial\Omega \times \overline{\Omega} \times [0,+\infty)),\ k(x,y,t) \geq 0;$$

$$u_0(x) \in C^1(\overline{\Omega}),\ u_0(x) \geq 0 \text{ in } \Omega,\ \frac{\partial u_0(x)}{\partial \nu} = \int_\Omega k(x,y,0)u_0^l(y)\,dy \text{ on } \partial\Omega.$$

Many authors have studied blow-up problem for parabolic equations and systems with nonlocal boundary conditions (see, for example, [2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 19, 20, 21, 22, 23, 24, 25, 26, 27] and the references therein). In particular, the

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initial boundary value problem for equation (1) with nonlinear nonlocal boundary condition

$$u(x,t) = \int_{\Omega} k(x,y,t)u^l(y,t)\,dy,\ x \in \partial\Omega,\ t > 0,$$

was considered for $c(x,t) \leq 0$ and $c(x,t) \geq 0$ in [14] and [12], respectively. The problem (1)–(3) with $c(x,t) \leq 0$ was investigated in [13] and closed problem was analyzed in [21].

Local existence theorem, comparison and uniqueness results for problem (1)–(3) have been established in [11].

In this paper we obtain necessary and sufficient conditions for the existence of global solutions as well as for blow-up in finite time of solutions for problem (1)–(3). Our global existence and blow-up results depend on the behavior of the functions $c(x,t)$ and $k(x,y,t)$ as $t \to \infty$.

The paper is organized as follows. The global existence theorem for any initial data and blow-up in finite time of solutions for large initial data are proved in section 2. In section 3 we present finite time blow-up of all nontrivial solutions as well as the existence of global solutions for small initial data. Finally, in section 4 we show that under certain conditions blow-up occurs only on the boundary.

2. Global existence.

Let $Q_T = \Omega \times (0,T),\ S_T = \partial\Omega \times (0,T),\ \Gamma_T = S_T \cup \overline{\Omega} \times \{0\}$, $T > 0$.

Definition 2.1. We say that a nonnegative function $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a supersolution of (1)–(3) in Q_T if

$$u_t \geq \Delta u - c(x,t)u^p,\ (x,t) \in Q_T, \tag{4}$$

$$\frac{\partial u(x,t)}{\partial \nu} \geq \int_{\Omega} k(x,y,t)u^l(y,t)\,dy,\ (x,t) \in S_T, \tag{5}$$

$$u(x,0) \geq u_0(x),\ x \in \Omega, \tag{6}$$

and $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a subsolution of (1)–(3) in Q_T if $u \geq 0$ and it satisfies (4)–(6) in the reverse order. We say that $u(x,t)$ is a solution of problem (1)–(3) in Q_T if $u(x,t)$ is both a subsolution and a supersolution of (1)–(3) in Q_T.

To prove the main results we use the positiveness of a solution and the comparison principle which have been proved in [11].

Theorem 2.2. *Let u be the solution of (1)–(3) in Q_T, u_0 be a nontrivial function in Ω, $p \geq 1$ or $c(x,t) \equiv 0$. Then $u > 0$ in $Q_T \cup S_T$.*

Theorem 2.3. *Let $\overline{u}$ and $\underline{u}$ be a supersolution and a subsolution of problem (1)–(3) in Q_T, respectively. Suppose that $\underline{u}(x,t) > 0$ or $\overline{u}(x,t) > 0$ in $Q_T \cup \Gamma_T$ if $l < 1$. Then $\overline{u}(x,t) \geq \underline{u}(x,t)$ in $Q_T \cup \Gamma_T$.*

The proof of a global existence result relies on the continuation principle and the construction of a supersolution. We suppose that

$$c(x,t) > 0,\, x \in \overline{\Omega},\, t \geq 0. \tag{7}$$

Theorem 2.4. *Let $l \leq 1$ or $1 < l < p$ and (7) hold. Then every solution of (1)–(3) is global.*

Proof. In order to prove global existence of solutions we construct a suitable explicit supersolution of (1)–(3) in Q_T for any positive T. Suppose at first that $l \leq 1$. Since $k(x,y,t)$ is a continuous function there exists a constant $K > 0$ such that

$$k(x,y,t) \leq K \tag{8}$$

in $\partial\Omega \times Q_T$. Let λ_1 be the first eigenvalue of the following problem

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, \ x \in \Omega, \\ \varphi(x) = 0, \ x \in \partial\Omega, \end{cases}$$

and $\varphi(x)$ be the corresponding eigenfunction with $\sup\limits_{\Omega} \varphi(x) = 1$. It is well known $\varphi(x) > 0$ in Ω and $\max\limits_{\partial\Omega} \partial\varphi(x)/\partial\nu < 0$.

We now construct a supersolution of (1)–(3) in Q_T as follows

$$\overline{u}(x,t) = \frac{C\exp(\mu t)}{a\varphi(x) + 1},$$

where constants C, μ and a are chosen to satisfy the inequalities:

$$a \geq \max\left\{ K\int_\Omega \frac{dy}{(\varphi(y)+1)^l} \max_{\partial\Omega}\left(-\frac{\partial\varphi}{\partial\nu}\right)^{-1}, 1 \right\},$$

$$C \geq \max\{\sup_\Omega (a\varphi(x)+1)u_0(x), 1\}, \quad \mu \geq \lambda_1 + 2a^2 \sup_\Omega \frac{|\nabla\varphi|^2}{(a\varphi(x)+1)^2}.$$

It is easy to obtain

$$\overline{u}_t - \Delta\overline{u} + c(x,t)\overline{u}^p \geq \left(\mu - \frac{a\lambda_1\varphi}{(a\varphi(x)+1)^2} - 2a^2 \sup_\Omega \frac{|\nabla\varphi|^2}{(a\varphi(x)+1)^2}\right)\overline{u} \geq 0 \tag{9}$$

for $(x,t) \in Q_T$,

$$\begin{aligned} \frac{\partial\overline{u}}{\partial\nu} =& aC\exp(\mu t)\left(-\frac{\partial\varphi}{\partial\nu}\right) \geq KC^l \exp(l\mu t)\int_\Omega \frac{dy}{(\varphi(y)+1)^l} \\ \geq& \int_\Omega k(x,y,t)\overline{u}^l(y,t)\,dy \end{aligned} \tag{10}$$

for $(x,t) \in S_T$ and

$$\overline{u}(x,0) \geq u_0(x) \tag{11}$$

for $x \in \Omega$. By virtue of (9)–(11) the solution of (1)–(3) exists globally.

Suppose now that $1 < l < p$ and (7) holds. By (7) we have $c(x,t) \geq \underline{c}$ in Q_T, where $\underline{c}$ is some positive constant.

To construct a supersolution we use the change of variables in a neighborhood of $\partial\Omega$ as in [3]. Let $\overline{x}$ be a point in $\partial\Omega$. We denote by $\widehat{n}(\overline{x})$ the inner unit normal to $\partial\Omega$ at the point $\overline{x}$. Since $\partial\Omega$ is smooth it is well known that there exists $\delta > 0$ such that the mapping $\psi : \partial\Omega \times [0,\delta] \to \mathbb{R}^n$ given by $\psi(\overline{x}, s) = \overline{x} + s\widehat{n}(\overline{x})$ defines new coordinates $(\overline{x}, s)$ in a neighborhood of $\partial\Omega$ in $\overline{\Omega}$. A straightforward computation shows that, in these coordinates, Δ applied to a function $g(\overline{x}, s) = g(s)$, which is independent of the variable $\overline{x}$, evaluated at a point $(\overline{x}, s)$ is given by

$$\Delta g(\overline{x}, s) = \frac{\partial^2 g}{\partial s^2}(\overline{x}, s) - \sum_{j=1}^{n-1} \frac{H_j(\overline{x})}{1 - sH_j(\overline{x})} \frac{\partial g}{\partial s}(\overline{x}, s), \tag{12}$$

where $H_j(\overline{x})$ for $j = 1, ..., n-1$, denotes the principal curvatures of $\partial\Omega$ at $\overline{x}$.

For points in $Q_{\delta,T} = \partial\Omega \times [0,\delta] \times [0,T]$ of coordinates $(\overline{x}, s, t)$ define

$$\overline{u}(\overline{x}, s, t) = \left[(\alpha s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_+^{\frac{\beta}{\gamma}} + A, \tag{13}$$

where $\alpha > 0,\ 0 < \varepsilon < \omega < \alpha\delta,\ \max\{1/l, 2/(p-1)\} < \beta < 2/(l-1),\ 0 < \gamma < \beta/2$, $A \geq \sup_\Omega u_0(x),\ \sigma_+ = \max\{\sigma, 0\}$. For points in $\overline{Q_T} \setminus Q_{\delta,T}$ we put $\overline{u}(\overline{x}, s, t) = A$. As in [12], one can show

$$\overline{u}_t - \Delta\overline{u} + c(x,t)\overline{u}^p \geq 0, \quad (x,t) \in Q_T$$

for small ε and large A.

Now we prove the following inequality

$$\frac{\partial\overline{u}}{\partial\nu}(\overline{x}, 0, t) \geq \int_\Omega k(x,y,t)\overline{u}^l(y,t)\,dy, \quad (x,t) \in S_T \tag{14}$$

for a suitable choice of ε. To estimate the integral I in the right hand side of (14) we use the change of variables in a neighborhood of $\partial\Omega$ as above. Let

$$\overline{J} = \sup_{0<s<\delta} \int_{\partial\Omega} |J(\overline{y}, s)|\,d\overline{y},$$

where $J(\overline{y}, s)$ is Jacobian of the change of variables. Then we have

$$\begin{aligned} I \leq & 2^{l-1}K \int_\Omega \left[(\alpha s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_+^{\frac{\beta l}{\gamma}}\,dy + 2^{l-1}KA^l|\Omega| \\ \leq & 2^{l-1}K\overline{J} \int_0^{(\omega-\varepsilon)/\alpha} \left[(\alpha s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]^{\frac{\beta l}{\gamma}}\,ds + 2^{l-1}KA^l|\Omega| \\ \leq & \frac{2^{l-1}K\overline{J}}{\alpha(\beta l - 1)}\left[\varepsilon^{-(\beta l-1)} - \omega^{-(\beta l-1)}\right] + 2^{l-1}KA^l|\Omega|, \end{aligned}$$

where K was defined in (8), $|\Omega|$ is Lebesgue measure of Ω. On the other hand, since

$$\frac{\partial\overline{u}}{\partial\nu}(\overline{x}, 0, t) = -\frac{\partial\overline{u}}{\partial s}(\overline{x}, 0, t) = \alpha\beta\varepsilon^{-\gamma-1}\left[\varepsilon^{-\gamma} - \omega^{-\gamma}\right]_+^{\frac{\beta-\gamma}{\gamma}},$$

the inequality (14) holds if ε is small enough and hence by Theorem 2.3 we get

$$u(x,t) \leq \overline{u}(\overline{x}, s, t) \ \text{ in } \ \overline{Q}_T.$$

□

Remark 1. Let

$$\underline{\lambda} = \frac{\inf_{\Omega\times(0,+\infty)} c(x,t)}{\sup_{\partial\Omega\times\Omega\times(0,+\infty)} k(x,y,t)}.$$

Note that under $\beta = 2/(l-1)$ and a suitable choice of α in (13) the same proof holds if $l = p > 1$ and $\underline{\lambda}$ is large enough and consequently a solution of problem (1)–(3) is global.

To formulate finite time blow-up result we need that

$$k(x,y,t_0) > 0, \ \ x \in \partial\Omega, \ y \in \partial\Omega. \tag{15}$$

Theorem 2.5. *Let $l > \max\{1, p\}$ and (15) hold with $t_0 \geq 0$ if $p \leq 1$ and with $t_0 = 0$ if $p > 1$. Then there exist solutions of (1)–(3) with finite time blow-up.*

Proof. At first let $p \leq 1,\ l > 1$ and (15) hold with $t_0 \geq 0$. To prove the theorem we construct a subsolution of an auxiliary problem which blows up in finite time. First of all we get a lower bound for the solution of (1)–(3) with positive initial datum. We denote

$$\overline{c}(t) = \sup_{\Omega} c(x,t). \tag{16}$$

Then the function

$$w(t) = \begin{cases} \left[A^{1-p} - (1-p)\int_0^t \overline{c}(\tau)\,d\tau\right]^{1/(1-p)} & \text{for } 0 < p < 1, \\ A\exp\left[-\int_0^t \overline{c}(\tau)\,d\tau\right] & \text{for } p = 1 \end{cases}$$

is a subsolution of (1)–(3) in Q_T for any $T > 0$ if

$$u_0(x) \geq A > 0. \tag{17}$$

By Theorem 2.3 we have

$$u(x,t) \geq w(t) \ \text{ for } \ x \in \overline{\Omega} \ \text{ and } \ t \geq 0. \tag{18}$$

Consider the change of variables in a neighborhood of $\partial\Omega$ as in Theorem 2.4. Set $\Omega_\gamma = \{(\overline{x}, s) : \overline{x} \in \partial\Omega, 0 < s < \gamma\}$. By (15) we have

$$k(x,y,t) \geq k_1, \ \ x \in \partial\Omega,\ y \in \Omega_\gamma,\ t_0 < t < t_1 \tag{19}$$

for some positive $k_1,\ \gamma$ and $t_1 > t_0$.

Let us consider the following initial boundary value problem:

$$\begin{cases} v_t = \Delta v - c(x,t)v^p & \text{for } x \in \Omega_\gamma,\ \ t_0 < t < t_2, \\ \dfrac{\partial v(x,t)}{\partial \nu} = \displaystyle\int_{\Omega_\gamma} k(x,y,t)v^l(y,t)\,dy & \text{for } x \in \partial\Omega,\ \ t_0 < t < t_2, \\ v(x,t) = u(x,t) & \text{for } x \in \partial\Omega_\gamma \setminus \partial\Omega,\ \ t_0 < t < t_2, \\ v(x,t_0) = u(x,t_0) & \text{for } x \in \Omega_\gamma, \end{cases} \tag{20}$$

where ν is unit outward normal on $\partial\Omega$, $u(x,t)$ is the solution of (1)–(3), $t_2 \in (t_0, t_1)$ and will be chosen later. We can define the notions of a supersolution and a subsolution of (20) in a similar way as in Definition 2.1. We will use a comparison principle for a subsolution and a supersolution of (20) which can be proved analogously to Theorem 2.3. It is easy to see that $u(x,t)$ is a supersolution of (20) in $Q(\gamma, t_0, t_2) = \Omega_\gamma \times (t_0, t_2)$.

We define

$$\psi(s,t) = (t_2 + s - t)^{-\sigma}, \tag{21}$$

where $\sigma > 2/(l-1)$ and show that $\psi(s,t)$ is a subsolution of (20) in $Q(\gamma, t_0, t_2)$ under suitable choice of t_2 and γ. For $0 < s < \gamma$ and small γ we obtain

$$\left| \sum_{j=1}^{n-1} \frac{H_j(\overline{x})}{1 - sH_j(\overline{x})} \right| \leq C. \tag{22}$$

Using (12), (21), (22) we find

$$\begin{aligned} -\psi_t + \Delta\psi - c(x,t)\psi^p \geq & (t_2 + s - t)^{-\sigma-2}\,\{\sigma(\sigma+1) - \sigma(C+1)(t_2 - t_0 + \gamma) \\ & - \sup_{(t_0,t_2)} \overline{c}(t)(t_2 - t_0 + \gamma)^{\sigma+2-\sigma p}\ \} \geq 0 \end{aligned}$$

in $Q(\gamma, t_0, t_2)$ if we take γ and $t_2 - t_0$ small enough. Next we check the boundary condition. Let

$$\underline{J} = \inf_{0<s<\gamma} \int_{\partial\Omega} |J(\overline{y}, s)|\,d\overline{y}.$$

By virtue of (19), (21) we have

$$\frac{\partial\psi}{\partial\nu}(0,t) - \int_{\Omega_\gamma} k(x,y,t)\psi^l(s,t)\,dy \le \sigma(t_2-t)^{-\sigma-1} - k_1\underline{J}\int_0^\gamma (t_2+s-t)^{-\sigma l}ds$$

$$\le \sigma(t_2-t)^{-\sigma-1} - k_1\underline{J}\frac{(t_2-t)^{-\sigma l+1}}{\sigma l-1}\left[1-\left(\frac{t_2}{t_2+\gamma}\right)^{\sigma l-1}\right] \le 0$$

for $x\in\partial\Omega$, $t_0<t<t_2$ and small enough t_2-t_0.

As for γ and A we assume:

$$\gamma < t_2 - t_0, \tag{23}$$

$$A \ge \left[(1-p)\int_0^{t_1}\overline{c}(\tau)\,d\tau + \gamma^{-\sigma(1-p)}\right]^{1/(1-p)} \quad \text{for } 0<p<1, \tag{24}$$

$$A \ge \gamma^{-\sigma}\exp\left[\int_0^{t_1}\overline{c}(\tau)\right] \quad \text{for } p=1. \tag{25}$$

Making use of (17), (18), (23)–(25) we obtain

$$\psi(s,t)\le u(x,t) \text{ for } x\in\Omega_\gamma,\ t=t_0 \text{ and } x\in\partial\Omega_\gamma\setminus\partial\Omega,\ \ t_0\le t\le t_2.$$

Comparing $u(x,t)$ and $\psi(s,t)$ in $Q(\gamma,t_0,t_2)$ we prove the theorem for $p\le 1$, $l>1$, since $\psi(0,t)\to\infty$ as $t\to t_2$.

Let $l>p>1$ and (15) hold with $t_0=0$. We set $c_1=\sup_{Q_{t_1}}c(x,t)$ and suppose that

$$\max\left\{\frac{1}{p-1},\frac{2}{l-1}\right\}<\sigma<\frac{2}{p-1},\ u_0(x)\ge\max\left\{[t_2(p-1)c_1]^{-\frac{1}{p-1}},t_2^{-\sigma}\right\},$$

where $t_2\in(0,t_1)$ and will be chosen later. It can be directly checked that the function $w(t)=[(p-1)c_1(t+t_2)]^{-\frac{1}{p-1}}$ is a subsolution of (1)–(3) in Q_{t_2}. Then by Theorem 2.3 we have

$$w(t)\le u(x,t) \text{ for } x\in\overline{\Omega} \text{ and } 0\le t\le t_2.$$

In the same way as above we can show that $\psi(s,t)$ is a subsolution of (20) in $Q(\gamma,t_0,t_2)$ with $t_0=0$ for small values of γ and

$$t_2\le\min\left\{t_1,\frac{\gamma^{\sigma(p-1)}}{2(p-1)c_1}\right\}.$$

□

Remark 2. We put

$$\overline{\lambda}=\frac{\sup_{\partial\Omega}c(x,0)}{\inf_{\partial\Omega\times\partial\Omega}k(x,y,0)}$$

and consider

$$\psi(s,t)=(t_2+\omega s-t)^{-2/(p-1)},\ \omega>0 \tag{26}$$

instead of (21). Under a suitable choice of ω in (26) the same proof holds for $l=p>1$ if $\overline{\lambda}$ is small enough and hence there exist solutions of (1)–(3) with finite time blow-up.

3. **Blow-up of all nontrivial solutions.** In this section we find the conditions which guarantee blow-up in finite time of all nontrivial solutions of (1)–(3).

First we prove that for $p < 1$ and $l > 1$ no blow-up of all nontrivial solutions of (1)–(3) if

$$\inf_{\Omega} c(x,0) > 0. \tag{27}$$

Theorem 3.1. *Let $p < 1,\ l > 1$ and (27) hold. Then problem (1)–(3) has global solutions for small initial data.*

Proof. Thanks to the made assumptions we have $c(x,t) \geq c_0$ and $k(x,y,t) \leq K$ in Q_τ and $\partial\Omega \times Q_\tau$, respectively, where $c_0,\ K$ and τ are some positive constants.

Let $\psi(x)$ be a positive solution of the following problem

$$\Delta\psi = 1,\ x \in \Omega;\ \frac{\partial\psi(x)}{\partial\nu} = \frac{|\Omega|}{|\partial\Omega|},\ x \in \partial\Omega. \tag{28}$$

We put

$$b = \inf_{\Omega} \psi(x). \tag{29}$$

Defining the function $f(t)$ as follows

$$f(t) = \exp(t/b)\left\{f^{1-p}(0) - c_0 b^p\left(1 - \exp[(p-1)t/b]\right)\right\}_+^{1/(1-p)}$$

and assuming that

$$0 < f(0) < \left\{c_0 b^p\left(1 - \exp[(p-1)\tau/b]\right)\right\}^{1/(1-p)}$$

we deduce $f(t) \equiv 0$ for $t \geq \tau$.

To prove the theorem we construct a supersolution of (1)–(3) in the following form $v(x,t) = \psi(x)f(t)$. It is not difficult to check that

$$v_t - \Delta v + c(x,t)v^p \geq 0$$

for $x \in \Omega,\ t > 0$. On the parabolic boundary, we have that for $x \in \partial\Omega,\ t > 0$,

$$\begin{aligned}\frac{\partial v}{\partial\nu}(x,t) =& \frac{|\Omega|}{|\partial\Omega|}f(t) \geq \int_\Omega k(x,y,t)\psi^l(y)f^l(t)\,dy \\ =& \int_\Omega k(x,y,t)v^l(y,t)\,dy\end{aligned}$$

for small values of $f(0)$. Hence, $v(x,t)$ is indeed a supersolution of (1)–(3) in Q_T for any $T > 0$ if

$$u_0(x) \leq \psi(x)f(0), \quad x \in \Omega.$$

Now Theorem 2.3 guarantees the existence of global solutions of (1)–(3) for small initial data. □

The following two statements deal with the case $p = 1,\ l > 1$. Let us introduce the notations

$$\underline{c}(t) = \inf_{\Omega} c(x,t), \quad \overline{k}_c(t) = \sup_{\partial\Omega\times\Omega} k(x,y,t)\exp\left\{-(l-1)\int_0^t \underline{c}(\tau)\,d\tau\right\},$$

$$\underline{k}_c(x,t) = \inf_{\Omega} k(x,y,t)\exp\left\{-(l-1)\int_0^t \overline{c}(\tau)\,d\tau\right\},$$

where $\overline{c}(t)$ was defined in (16).

We prove that any nontrivial solution of (1)–(3) blows up in finite time if

$$\int_0^\infty \int_{\partial\Omega} \underline{k}_c(x,t)\,dS_x dt = \infty. \tag{30}$$

Conversely, problem (1)–(3) has bounded global solutions with small initial data, provided that

$$\int_0^\infty \overline{k}_c(t)\,dt < \infty, \tag{31}$$

and there exist positive constants α, t_0 and K such that $\alpha > t_0$ and

$$\int\limits_{t-t_0}^{t} \frac{\overline{k}_c(\tau)}{\sqrt{t-\tau}}\,d\tau \le K \text{ for any } t \ge \alpha. \tag{32}$$

Theorem 3.2. *Let $p = 1$, $l > 1$ and (30) hold. Then any nontrivial solution of (1)–(3) blows up in finite time and upper bound $t^\star$ of the blow-up time is given by*

$$\int_0^{t^\star}\int_{\partial\Omega} \underline{k}_c(x,t)\,dS_x dt = \frac{1}{(l-1)}\left\{|\Omega|\int_\Omega u_0(y)\,dy\right\}^{-(l-1)}.$$

Proof. Let $v(x,t)$ be the solution of the following problem

$$v_t = \Delta v \;\text{ for }\; x \in \Omega,\, t > 0, \tag{33}$$

$$\frac{\partial v(x,t)}{\partial \nu} = \underline{k}_c(x,t)\int_\Omega v^l(y,t)\,dy \;\text{ for }\; x \in \partial\Omega,\, t > 0, \tag{34}$$

$$v(x,0) = u_0(x) \;\text{ for }\; x \in \Omega. \tag{35}$$

By a direct computation we can check that

$$\underline{u}(x,t) = \exp\left(-\int_0^t \overline{c}(\tau)\,d\tau\right) v(x,t)$$

is a subsolution of (1)–(3) in Q_T for any $T > 0$. Then by Theorem 2.3 we get

$$\underline{u}(x,t) \le u(x,t),\; (x,t) \in Q_T$$

for any $T > 0$. To prove the theorem we show that any nontrivial solution of (33)–(35) blows up in finite time. We set

$$V(t) = \int_\Omega v(x,t)\,dx.$$

Integrating (33) over Ω and using Green's identity and Jensen's inequality, we have

$$\begin{aligned} V'(t) = &\int_\Omega \Delta v(x,t)\,dx = \int_{\partial\Omega} \frac{\partial v(x,t)}{\partial \nu}\,dS_x = \int_{\partial\Omega} \underline{k}_c(x,t)dS_x \int_\Omega v^l(y,t)\,dy \\ \ge &|\Omega|^{1-l}\int_{\partial\Omega} \underline{k}_c(x,t)dS_x V^l(t). \end{aligned}$$

Integrating last inequality, we obtain the desired result due to (30). □

Theorem 3.3. *Let $p = 1$, $l > 1$ and (31), (32) hold. Then problem (1)–(3) has bounded global solutions for small initial data.*

Proof. Let $w(x,t)$ be the solution of the following problem

$$\begin{cases} w_t = \Delta w & \text{for } x \in \Omega,\, t > 0, \\ \dfrac{\partial w(x,t)}{\partial \nu} = \overline{k}_c(t)\displaystyle\int_\Omega w^l(y,t)\,dy & \text{for } x \in \partial\Omega,\, t > 0, \\ w(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases} \tag{36}$$

It is easy to check that

$$\overline{u}(x,t) = \exp\left(-\int_0^t \underline{c}(\tau)\,d\tau\right) w(x,t)$$

is a supersolution of (1)–(3) in Q_T for any $T > 0$. To prove the theorem we show the existence of global bounded solutions of (36). Let us consider the following auxiliary problem

$$\begin{cases} h_t = \Delta h, \ x \in \Omega, \ t > 0 \\ \dfrac{\partial h(x,t)}{\partial \nu} = \overline{k}_c(t), \ x \in \partial\Omega, \ t > 0, \\ h(x,0) = h_0(x), \ x \in \Omega. \end{cases} \tag{37}$$

As it was proved in [13] any solution of (37) is a bounded function. Now we construct a supersolution of (36) in such a form that $g(x,t) = ah(x,t)$, where a is some positive constant. It is obvious,

$$g_t = \Delta g, \ x \in \Omega, \ t > 0.$$

Moreover,

$$\frac{\partial g(x,t)}{\partial \nu} = a\overline{k}_c(t) \geq a^l \overline{k}_c(t) \int_\Omega h^l(y,t)dy = \overline{k}_c(t) \int_\Omega g^l(y,t)dy, \ x \in \partial\Omega, \ t > 0$$

for small values of a. Then by the comparison principle for (36)

$$w(x,t) \leq g(x,t), \ (x,t) \in Q_T$$

for any $T > 0$ if $u_0(x) \leq ah_0(x)$, $x \in \Omega$. Since $g(x,t)$ is bounded function in $\Omega \times (0,\infty)$, so does the solution $u(x,t)$ of (1)–(3) □

Remark 3. By Theorem 3.2 and Theorem 3.3 the condition (31) is optimal for global existence of solutions of (1)–(3) with $c(x,t) = c(t)$ and $k(x,y,t) = k(t)$. Arguing in the same way as in the proof of Lemma 3.3 of [13] it is easy to show that (32) is optimal for the existence of nontrivial bounded global solutions of (1)–(3) with $c(x,t) = c(t)$ and $k(x,y,t) = k(t)$ under the condition

$$\int_0^\infty c(t)\,dt < \infty.$$

Now we prove finite time blow-up of all nontrivial solutions of (1)–(3) for $l > p > 1$. Let $m_0 = \inf\{\sup_\Omega \psi(x)\}$, where $\psi(x)$ was defined in (28). To formulate blow-up result we put

$$\underline{k}(t) = \inf_{\partial\Omega\times\Omega} k(x,y,t)$$

and suppose that

$$c(x,t) \leq c_1(t), \ c_1(t) \in C^1([t_0,\infty)), \ c_1(t) > 0 \text{ for } t \geq t_0, \tag{38}$$

where t_0 is some positive constant,

$$\liminf_{t\to\infty} \frac{c_1'(t)}{c_1(t)} > -\frac{p-1}{m_0} \tag{39}$$

and

$$\lim_{t\to\infty} \underline{k}(t)[c_1(t)]^{(1-l)/(p-1)} = \infty. \tag{40}$$

Theorem 3.4. *Let $l > p > 1$ and (38)–(40) hold. Then any nontrivial solution of (1)–(3) blows up in finite time.*

Proof. Let $u(x,t)$ be nontrivial global solution of (1)–(3). Then by Theorem 2.2

$$u(x,t) > 0 \text{ for } x \in \overline{\Omega},\ t > 0. \tag{41}$$

At first we get an universal lower bound for $u(x,t)$. From (39) we see that there exists a constant m satisfying $m > m_0$ and

$$\liminf_{t\to\infty} \frac{c_1'(t)}{c_1(t)} > -\frac{p-1}{m}. \tag{42}$$

We now let $f(t)$ be the solution of the equation

$$f'(t) = \frac{f(t)}{m} - m^{p-1}c_1(t)f^p(t),\ t \geq t_1 \geq t_0. \tag{43}$$

Then $f(t)$ can be written in an explicit form

$$\begin{aligned} f(t) =& \exp(t/m)\left\{[f(t_1)\exp(-t_1/m)]^{1-p}\right. \\ &\left. + (p-1)m^{p-1}\int_{t_1}^{t} \exp[(p-1)\tau/m]c_1(\tau)d\tau\ \right\}^{-\frac{1}{p-1}}. \end{aligned} \tag{44}$$

We rewrite (44) as following

$$\begin{aligned} &\left\{\frac{f(t)}{[c_1(t)]^{-1/(p-1)}}\right\}^{p-1} \\ =&\frac{\exp[(p-1)t/m]c_1(t)}{[f(t_1)\exp(-t_1/m)]^{1-p} + (p-1)m^{p-1}\int_{t_1}^{t}\exp[(p-1)\tau/m]c_1(\tau)d\tau} \end{aligned} \tag{45}$$

and prove that right hand side I of (45) is bounded below by some positive constant. Indeed, the numerator and the denominator of I tend to infinity as $t \to \infty$ by virtue of (39). Using (42) we can obtain

$$\liminf_{t\to\infty} I \geq \liminf_{t\to\infty} \frac{\exp[(p-1)t/m]\left\{(p-1)c_1(t)/m + c_1'(t)\right\}}{(p-1)m^{p-1}c_1(t)\exp[(p-1)t/m]} > 0. \tag{46}$$

By (44)–(46) we conclude

$$f(t) \geq d_1[c_1(t)]^{-\frac{1}{p-1}},\ t \geq t_1, \tag{47}$$

where $d_1 > 0$.

Let $\psi(x)$ satisfy (28) and

$$\sup_{\Omega} \psi(x) = m. \tag{48}$$

Now we define

$$\underline{u}(x,t) = \psi(x)f(t) \tag{49}$$

and show that $\underline{u}(x,t)$ is a subsolution of (1)–(3) in $\Omega \times (t_1, T)$ under suitable choice of t_1 and $T > t_1$. Due to (28), (43) we have

$$\underline{u}_t \leq \Delta\underline{u} - c(x,t)\underline{u}^p,\ \ x \in \Omega,\ t > t_1. \tag{50}$$

Using (28), (40), (47), (49) we find

$$\begin{aligned} \frac{\partial \underline{u}}{\partial \nu}(x,t) =& \frac{|\Omega|}{|\partial\Omega|}f(t) \leq d_1^{l-1}[c_1(t)]^{-\frac{l-1}{p-1}}\underline{k}(t)f(t)\int_\Omega \psi^l(y)\,dy \\ &\leq \int_\Omega k(x,y,t)\underline{u}^l(y,t)\,dy,\ x \in \partial\Omega,\ t > t_1 \end{aligned} \tag{51}$$

for large values of t_1. By (41), (47)–(51) and Theorem 2.3

$$u(x,t) \geq \underline{u}(x,t) \geq d_2[c_1(t)]^{-\frac{1}{p-1}},\ \ (x,t) \in \Omega \times (t_1, T) \tag{52}$$

for some $d_2 > 0$ and any $T > t_1$ if

$$f(t_1) \leq \frac{\inf_\Omega u(x,t_1)}{m}.$$

We set

$$U(t) = \int_\Omega u(x,t)\,dx. \tag{53}$$

Integrating (1) over Ω and using (38), (40), (52), (53) and Green's identity, we have

$$U'(t) = \int_\Omega (\Delta u(x,t) - c(x,t)u^p(x,t))\,dx \geq \int_\Omega \big(|\partial\Omega|\underline{k}(t)u^l(x,t) - c_1(t)u^p(x,t)\big)\,dx$$
$$\geq \frac{1}{2}|\partial\Omega|\underline{k}(t)\int_\Omega u^l(x,t)dx \geq \frac{1}{2}|\partial\Omega| d_2^{l-1}\underline{k}(t)[c_1(t)]^{-\frac{l-1}{p-1}}\int_\Omega u(x,t)dx = \xi(t)U(t), \tag{54}$$

where $t \geq t_2,\ t_2$ is large enough and $\lim_{t\to\infty}\xi(t)\,dt = \infty$. Integrating (54) over (t_2,t) we find

$$U(t) \geq U(t_2)\exp\left(\int_{t_2}^t \xi(\tau)\,d\tau\right). \tag{55}$$

Now we deduce lower bound for $\underline{k}(t)$. From (42) we can write

$$c_1(t) \geq c_1(t_3)\exp\left(-\frac{(p-1)t}{m}\right),\ t \geq t_3 \tag{56}$$

for some $t_3 \geq t_2$. By (40), (56) we have

$$\underline{k}(t) = \gamma_1(t)[c_1(t)]^{(l-1)/(p-1)} \geq \gamma_2(t)\exp\left(-\frac{(l-1)t}{m}\right) \text{ for } t \geq t_3, \tag{57}$$

where $\lim_{t\to\infty}\gamma_i(t) = \infty,\ i = 1, 2$.

Let us change unknown function

$$w(x,t) = \exp\left(-\frac{t}{m}\right)u(x,t). \tag{58}$$

It is easy to check that $w(x,t)$ is the solution of the following problem

$$w_t = \Delta w - c(x,t)\exp\left(\frac{(p-1)t}{m}\right)w^p - \frac{1}{m}w, x \in \Omega, t > 0, \tag{59}$$

$$\frac{\partial w(x,t)}{\partial\nu} = \exp\left(\frac{(l-1)t}{m}\right)\int_\Omega k(x,y,t)w^l(y,t)dy, x \in \partial\Omega, t > 0, \tag{60}$$

$$u(x,0) = u_0(x),\ x \in \Omega.$$

We put

$$W(t) = \int_\Omega w(x,t)\,dx. \tag{61}$$

From (53), (55), (58), (61) we conclude that

$$\lim_{t\to\infty} W(t) = \infty.$$

Integrating (59) over Ω and using (38), (40), (52), (54), (57), (60), (61), Green's identity and Jensen's inequality, we have

$$W'(t) \geq \sigma(t)W^l(t) - \frac{1}{m}W(t) \text{ for } t \geq t_3,$$

where $\lim_{t\to\infty}\sigma(t) = \infty$. Hence $W(t)$ blows in finite time. □

To show the optimality of (40) for blow-up of any nontrivial solution of (1)–(3) we put

$$\overline{k}(t) = \sup_{\partial\Omega\times\Omega} k(x,y,t)$$

and assume that

$$c(x,t) \geq c_2(t) \text{ for } t \geq 0,\ \ c_2(t) \in C([0,\infty)) \cap C^1([\sigma,\infty)),\ \ c_2(t) > 0 \text{ for } t \geq \sigma, \quad (62)$$

$$\limsup_{t\to\infty} \frac{c_2'(t)}{c_2(t)} \leq 0, \quad (63)$$

$$\overline{k}(t) \leq K_c [c_2(t)]^{(l-1)/(p-1)},\ \ t \geq 0, \quad (64)$$

where σ and K_c are some positive constants.

Theorem 3.5. *Let $l > p > 1$ and (62)–(64) hold. Then problem (1)–(3) has global solutions for small initial data.*

Proof. In order to prove the theorem we construct a supersolution of (1)–(3) in Q_T for any $T > 0$. Let us define $g(t)$ as a positive solution of the following equation

$$g'(t) = \frac{g(t)}{b} - b^{p-1} c_2(t) g^p(t), \quad (65)$$

where b was defined in (29). Specifically, $g(t)$ takes the form

$$g(t) = \exp(t/b) \left\{ [g(0)]^{1-p} + (p-1) b^{p-1} \int_0^t \exp[(p-1)\tau/b] c_2(\tau) d\tau \right\}^{-1/(p-1)}. \quad (66)$$

We rewrite (66) as following

$$\left\{ g(t) [c_2(t)]^{1/(p-1)} \right\}^{p-1} = \frac{\exp[(p-1)t/b] c_2(t)}{[g(0)]^{1-p} + (p-1) b^{p-1} \int_0^t \exp[(p-1)\tau/b] c_2(\tau) d\tau}. \quad (67)$$

Defining the functions

$$\alpha(t) = \exp[(p-1)t/b] c_2(t),$$

$$\beta(t) = [g(0)]^{1-p} + (p-1) b^{p-1} \int_0^t \exp[(p-1)\tau/b] c_2(\tau) d\tau$$

and using Cauchy's mean value theorem and (63), we obtain

$$\frac{\alpha(t)}{\beta(t)} - \frac{\alpha(a)}{\beta(t)} \leq \frac{\alpha(t) - \alpha(a)}{\beta(t) - \beta(a)} = \frac{\alpha'(\xi)}{\beta'(\xi)} = \frac{1}{b^p} + \frac{1}{(p-1) b^{p-1}} \frac{c_2'(\xi)}{c_2(\xi)} \leq \frac{2}{b^p} \quad (68)$$

for large values of a, $t > a$ and $\xi \in (a,t)$. From (67), (68) we get the estimate

$$\left\{ g(t) [c_2(t)]^{1/(p-1)} \right\}^{p-1} \leq \frac{3}{b^p},\ \ t \geq 0 \quad (69)$$

for small values of $g(0)$.

Now we define

$$\overline{u}(x,t) = \psi(x) g(t) \quad (70)$$

and show that $\overline{u}(x,t)$ is a supesolution of (1)–(3) in Q_T for any $T > 0$ if initial data are small. By (28), (65) we have

$$\overline{u}_t - \Delta \overline{u} + c(x,t) \overline{u}^p \geq 0, \quad x \in \Omega,\ t > 0. \quad (71)$$

We note that

$$\lim_{b\to\infty} \frac{m}{b} = 1, \quad (72)$$

where m was defined in (48). Using (28), (64), (69), (70), (72) we find

$$\begin{aligned}\frac{\partial \overline{u}}{\partial \nu}(x,t) =& \frac{|\Omega|}{|\partial\Omega|} g(t) \geq K_c |\Omega| m^l \left[\frac{3}{b^p}\right]^{\frac{l-1}{p-1}} g(t) \\ \geq& K_c \left\{ g(t)[c_2(t)]^{1/(p-1)} \right\}^{l-1} g(t) \int_\Omega \psi^l(y)dy \geq \overline{k}(t) g^l(t) \int_\Omega \psi^l(y)dy \\ \geq& \int_\Omega k(x,y,t)\overline{u}^l(y,t)dy,\ x \in \partial\Omega,\ t > 0\end{aligned} \tag{73}$$

for large values of b. Thus, by (71), (73) and Theorem 2.3 $\overline{u}(x,t)$ is a supersolution of (1)–(3) in Q_T for any $T > 0$ if

$$u_0(x) \leq g(0)\psi(x).$$

□

We will write $h(x,t) \sim s(t)$ and $z(x,y,t) \sim s(t)$ as $t \to \infty$ if there exist positive constants β_i, $(i = \overline{1,6})$ such that

$$\beta_1 h(x,t) \leq s(t) \leq \beta_2 h(x,t) \text{ for } x \in \Omega \text{ and } t \geq \beta_3$$

and

$$\beta_4 z(x,y,t) \leq s(t) \leq \beta_5 z(x,y,t) \text{ for } x \in \partial\Omega,\ y \in \Omega \text{ and } t \geq \beta_6,$$

respectively.

Remark 4. By Theorem 3.4 and Theorem 3.5 the condition (40) is optimal in a certain sense for blow-up in finite time of any nontrivial solution of (1)–(3). In particular, let $c(x,t) \sim t^\alpha \ln^\beta t$ as $t \to \infty$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$. Then there exist global solutions of (1)–(3) for $k(x,y,t) \sim \{t^\alpha \ln^\beta t\}^{(l-1)/(p-1)}$ as $t \to \infty$ and any nontrivial solution of (1)–(3) blows up in finite time for $k(x,y,t) \sim \gamma(t)\{t^\alpha \ln^\beta t\}^{(l-1)/(p-1)}$ as $t \to \infty$ if $\lim_{t\to\infty} \gamma(t) = \infty$.

4. Blow-up on the boundary.

In this section we show that for problem (1)–(3) under certain conditions blow-up cannot occur at the interior domain.

Lemma 4.1. *Let $l > \max\{p, 1\}$, $\inf\limits_{\partial\Omega \times Q_T} k(x,y,t) > 0$ and the solution $u(x,t)$ of (1)–(3) blows up in $t = T$. Then for $t \in [0,T)$*

$$\int_0^t \int_\Omega u^l(x,\tau)\,dx\,d\tau \leq s\,(T-t)^{-1/(l-1)},\ s > 0. \tag{74}$$

Proof. Integrating (1) over Q_t and using Green's identity, we have

$$\begin{aligned}\int_\Omega u(y,t)dy =& \int_\Omega u_0(y)\,dy - \int_0^t \int_\Omega c(y,\tau)u^p(y,\tau)\,dyd\tau \\ &+ \int_0^t \int_{\partial\Omega} \int_\Omega k(\xi,y,\tau)u^l(y,\tau)\,dydS_\xi d\tau \\ \geq& \int_0^t \int_\Omega \left(k|\partial\Omega|u^l(y,\tau) - Cu^p(y,\tau)\right)\,dyd\tau \\ \geq& \frac{k|\partial\Omega|}{2} \int_0^t \int_\Omega u^l(y,\tau)\,dyd\tau - M,\end{aligned} \tag{75}$$

where

$$k = \inf_{\partial\Omega\times Q_T} k(x,y,t),\ C = \sup_{Q_T} c(x,t),\ M = T|\Omega| \left\{ \frac{2C^{l/p}}{k|\partial\Omega|} \right\}^{\frac{p}{l-p}}.$$

Applying Hölder's inequality, we obtain

$$\int_\Omega u(y,t)\,dy \le |\Omega|^{(l-1)/l} \left\{ \int_\Omega u^l(y,t)\,dy \right\}^{1/l}. \tag{76}$$

Let us introduce

$$J(t) = \int_0^t \int_\Omega u^l(x,\tau)\,dx\,d\tau.$$

Now from (75),(76) we have

$$(J'(t))^{1/l} \ge c_0 J(t) - M_1,\ c_0 > 0, M_1 > 0.$$

We suppose there exists $t_0 \in (0,T)$ such that $J(t_0) = 2M_1/c_0$ since otherwise (74) holds. Then $J(t) \le 2M_1/c_0$ for $0 \le t \le t_0$ and

$$J'(t) \ge \left(\frac{c_0}{2} J(t) \right)^l \text{ for } t \ge t_0. \tag{77}$$

Integrating (77) over $(t;T)$, we obtain (74). □

Theorem 4.2. *Let the conditions of Lemma 4.1 hold. Then for problem (1)–(3) blow-up can occur only on the boundary.*

Proof. In the proof we will use some arguments of [7], [17]. Let $G_N(x,y;t-\tau)$ be the Green function of the heat equation with homogeneous Neumann boundary condition. Then we have the representation formula:

$$\begin{aligned} u(x,t) = &\int_\Omega G_N(x,y;t)u_0(y)dy - \int_0^t \int_\Omega G_N(x,y;t-\tau)c(y,\tau)u^p(y,\tau)\,dyd\tau \\ &+ \int_0^t \int_{\partial\Omega} G_N(x,\xi;t-\tau) \int_\Omega k(\xi,y,\tau)u^l(y,\tau)\,dydS_\xi d\tau \end{aligned} \tag{78}$$

for $(x,t) \in Q_T$. We now take an arbitrary $\Omega' \subset\subset \Omega$ with $\partial\Omega' \in C^2$ such that $\mathrm{dist}(\partial\Omega, \Omega') = \varepsilon > 0$. It is well known (see, for example, [16],[18]) that

$$G_N(x,y;t-\tau) \ge 0,\ x,y \in \Omega,\ 0 \le \tau < t < T, \tag{79}$$

$$\int_\Omega G_N(x,y;t-\tau)\,dy = 1,\ x \in \Omega,\ 0 \le \tau < t < T. \tag{80}$$

$$0 \le G_N(x,y;t-\tau) \le c_\varepsilon,\ x \in \Omega',\ y \in \partial\Omega,\ 0 < \tau < t < T, \tag{81}$$

where c_ε is a positive constant depending on ε. By (74), (78)–(81) we have

$$\begin{aligned} \sup_{\Omega'} u(x,t) \le &\sup_\Omega u_0(x) + c_\varepsilon |\partial\Omega| \sup_{\partial\Omega\times Q_T} k(x,y,t) \int_0^t \int_\Omega u^l(y,\tau)\,dyd\tau \\ \le &c_1(T-t)^{-1/(l-1)}. \end{aligned}$$

As it is shown in [17], there exist a function $f(x) \in C^2(\overline{\Omega'})$ and positive constant c_2 such that

$$\Delta f - \frac{l}{l-1}\frac{|\nabla f|^2}{f} \ge -c_2 \text{ in } \Omega',\ f(x) > 0 \text{ in } \Omega',\ f(x) = 0 \text{ on } \partial\Omega'. \tag{82}$$

Now we compare $u(x,t)$ with

$$w(x,t) = c_3 \left(f(x) + c_2(T-t) \right)^{-1/(l-1)}$$

in $\Omega' \times (0,T)$, where a positive constant c_3 will be defined below. By (82) for $x \in \Omega'$ and $t \in [0,T)$ we get

$$w_t - \Delta w + c(x,t)w^p \geq \frac{w}{(l-1)[f(x)+c_2(T-t)]} \Big(c_2 + \Delta f - \frac{l|\nabla f|^2}{(l-1)[f(x)+c_2(T-t)]} \Big) \geq 0.$$

Choosing c_3 such that $c_3 \geq c_2^{1/(l-1)} c_1$ and $w(x,0) \geq u_0(x)$ for $x \in \Omega'$, by comparison principle we conclude

$$u(x,t) \leq w(x,t) \text{ in } \overline{\Omega'} \times [0,T).$$

Hence, $u(x,t)$ cannot blow up in $\Omega' \times [0,T]$. Since Ω' is an arbitrary subset of Ω, the proof is completed. □

From [1] it is easy to get the following result.

Theorem 4.3. *Let $p > 1$, $\inf_{Q_T} c(x,t) > 0$ and the solution of (1)–(3) blows up in $t = T$. Then blow-up occurs only on the boundary.*

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