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Graphs with maximal induced matchings of the same size

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ABSTRACT

A graph is *well-indumatched* if all its maximal induced matchings are of the same size. We first prove that recognizing whether a graph is well-indumatched is a co-NP-complete problem even for $(2P_5, K_{1,5})$-free graphs. We then show that decision problems Independent Dominating Set, Independent Set, and Dominating Set are NP-complete for the class of well-indumatched graphs. We also show that this class is a co-indumatching hereditary class, i.e., it is closed under deleting the end-vertices of an induced matching along with their neighborhoods, and we characterize well-indumatched graphs in terms of forbidden co-indumatching subgraphs. We prove that recognizing a co-indumatching subgraph is an NP-complete problem. We introduce a *perfectly well-indumatched* graph, in which every induced subgraph is well-indumatched, and characterize the class of these graphs in terms of forbidden induced subgraphs. Finally, we show that the weighted versions of problems Independent Dominating Set and Independent Set can be solved in polynomial time for perfectly well-indumatched graphs, but problem Dominating Set is NP-complete.

1. Introduction

In this paper, we use graph-theoretic terminology of Bondy and Murty [7] (unless noted otherwise), and computational complexity terminology of Garey and Johnson [29].

A *matching* in a graph is a set of edges with no two edges having a common vertex. An *induced matching* is a matching with an additional property that no two of its edges are joined by an edge. An induced matching M in a graph G is *maximal* if no other induced matching in G contains M, while an induced matching of maximum size is a *maximum induced matching*. The problem of deciding if there exists an induced matching of a given size or larger is called Maximum Induced Matching. Induced matchings have applications in communication network testing [47], concurrent transmission of messages in

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wireless ad hoc networks [2] and secure communication channels in broadcast networks [30]. The Maximum Induced Matching problem was also studied in [8–14,19,22,38,39,42,43], see [10,22] also for a survey and new results. In [39], Kobler and Rotics showed that the graphs where the sizes of a maximum matching and a maximum induced matching coincide, can be recognized in polynomial time. Later, Cameron and Walker [15] extended this result by giving a complete structural description of these graphs.

We call a graph *well-indumatched* if all its maximal induced matchings have the same size. For example, the graph obtained from a star $K_{1,n}$ by subdividing each edge by two vertices is well-indumatched. We denote by $\mathcal{WIM}$ the class of well-indumatched graphs. We study the problem of recognizing if a graph is well-indumatched, and examine computational complexity of problems Independent Dominating Set, Independent Set, and Dominating Set for such graphs.

The fact that any maximal induced matching of a well-indumatched graph has the maximum size implies that the problem Maximum Induced Matching can be solved by a greedy algorithm for well-indumatched graphs, or, in other words, class $\mathcal{WIM}$ forms a set of greedy instances in the sense of Caro et al. [18] with respect to the problem Maximum Induced Matching. Greedy instances of other combinatorial problems can be defined in a similar way (see, e.g., [18,23,32,40,50–52]).

The class of *well-covered* graphs is a class of greedy instances of the problem Independent Set [18]. A graph is well-covered if all its maximal independent sets are of the same size. This concept is introduced by Plummer [44], and applications exist in distributed computing systems [56]. The problem of recognizing well-covered graphs is proved to be co-NP-complete for general graphs, independently by Sankaranarayana and Stewart [46], and Chvátal and Slater [20]. It is co-NP-complete for $K_{1,4}$-free graphs [18], but solvable in polynomial time for $K_{1,3}$-free graphs [48,49]. Significant work has been done towards characterizing well-covered graphs such as trees and bipartite graphs (Ravindra [45]), graphs with a girth of at least 5 (Finbow et al. [24]), cubic graphs (Campbell et al. [16]), and plane triangulations (Finbow et al. [25–27]).

A graph is called *equimatchable* if all its maximal matchings have the same size, see Lovász and Plummer [41]. The class of equimatchable graphs and the class $\mathcal{WIM}$ of well-indumatched graphs can be viewed as edge analogues of well-covered graphs, because the property of maximal matchings (induced matchings) to have the same size correlates with that of all maximal independent vertex sets to have the same size. The problem of recognizing an equimatchable graph was first studied by Lesk et al. [40] who gave a characterization of equimatchable graphs in terms of Gallai–Edmonds structure theorem, see, e.g., [55], and showed that there exists a polynomial-time algorithm which decides whether a given graph is equimatchable.

In Section 2, we first show that recognizing a well-indumatched graph is a co-NP-complete problem even for $(2P_5, K_{1,5})$-free graphs. Then we prove that, for the same graphs, the problem of recognizing a graph that has maximal induced matchings of at most t distinct sizes is co-NP-complete for any given $t \geqslant 1$.

Let IMatch(G) be the set of all maximal induced matchings of the graph G. Define the *minimum maximal induced matching number* as $\sigma(G) = \min\{|M| : M \in \text{IMatch}(G)\}$, and the *maximum induced matching number* as $\Sigma(G) = \max\{|M| : M \in \text{IMatch}(G)\}$. Similar to the definition of a maximum induced matching that comprises $\Sigma(G)$ edges, we define a *min–max induced matching* as a maximal induced matching with $\sigma(G)$ edges. In a greedy way we can find both parameters $\sigma(G)$ and $\Sigma(G)$ in any well-indumatched graph G. It is well known that the problem of deciding "$\Sigma(G) \geqslant K$" is NP-complete [12,47]. Moreover, in general graphs with n vertices, the optimization version of the Maximum Induced Matching problem cannot be approximated within a factor of $n^{1/2-\varepsilon}$ for any constant $\varepsilon > 0$, unless P = NP [43]. In Section 2, we prove that the problem of deciding "$\sigma(G) \leqslant K$" is NP-complete even if the graph has maximal induced matchings of at most two sizes differing by one. Other results on the complexity and inapproximability of the problem associated with $\sigma(G)$ can be found in [43].

Ko and Shepherd [38] investigated relations between $\Sigma(G)$ and $\gamma(G)$, the domination number of a graph G. They write that they do not know any class of graphs for which exactly one of the values γ and Σ is computable in polynomial time. In Sections 3 and 5, we prove that problems Independent Dominating Set, Independent Set, and Dominating Set are NP-complete for well-indumatched graphs. Thus, for the class $\mathcal{WIM}$, calculating γ is an NP-hard problem (see Section 5), while Σ can be calculated in polynomial time by constructing an arbitrary maximal induced matching. Our proof for the Independent Set problem implies that the well-known problems Partition into Triangles, Chordal Graph Completion and Partition into Subgraphs Isomorphic to P_3 are NP-complete for well-indumatched graphs. NP-completeness of the latter problem implies that computing Σ is NP-hard even for Hamiltonian line graphs of well-indumatched graphs, which generalizes the result of Kobler and Rotics [39]. In Section 3, we also show that problems Graph k-Colorability and Clique are NP-complete for well-indumatched graphs.

A class of graphs is called *hereditary* if every induced subgraph of a graph in this class also belongs to the class. For a set $\mathcal{H}$ of graphs, a graph G is called $\mathcal{H}$*-free* if no induced subgraph of G is isomorphic to a graph in $\mathcal{H}$. In other words, $\mathcal{H}$-free graphs constitute a hereditary class defined by $\mathcal{H}$ as the set of forbidden induced subgraphs.

In Section 4, we observe that the class $\mathcal{WIM}$ of well-indumatched graphs is not hereditary, but we prove that $\mathcal{WIM}$ is a co-indumatching hereditary class, i.e., it is closed under deleting the end-vertices of an induced matching along with their neighborhoods. Also, in Section 4, we characterize well-indumatched graphs in terms of forbidden co-indumatching subgraphs. This means that we specify the minimal set $\mathcal{Z}_{\mathcal{WIM}}$ of graphs such that G is well-indumatched if and only if G does not contain any graph in $\mathcal{Z}_{\mathcal{WIM}}$ as a co-indumatching subgraph. We prove that recognizing a co-indumatching subgraph is an NP-complete problem. We conclude Section 4 by presenting a variant of the well-known result of Chvátal and Slater [20] and Sankaranarayana and Stewart [46] that recognizing well-covered graphs is co-NP-complete. Namely, we show that recognizing well-covered graphs is co-NP-complete even for $(2P_5, K_{1,10})$-free squares of line graphs.

Finally, in Section 5, we consider *perfectly well-indumatched* graphs, i.e., graphs in which every induced subgraph is well-indumatched. We characterize the class of perfectly well-indumatched graphs by a finite set of forbidden induced subgraphs, thus obtaining a new polynomial-time recognizable class, where both parameters σ and Σ are easy to compute. We show that problems Independent Set and Independent Dominating Set can be solved in polynomial time for perfectly well-indumatched graphs, even in their weighted versions, but problem Dominating Set remains NP-complete.

Let G be a graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The subgraph of G induced by a set $X \subseteq V$ is denoted by $G(X)$, and $G - X = G(V \setminus X)$. A vertex subset $U \subseteq V$ is called a *clique* in G if vertices in U are pairwise adjacent. The set of vertices adjacent to a vertex x in G is called the *neighborhood* of x and it is denoted as $N_G(x)$. The *degree* of x is defined as $\deg_G x = |N_G(x)|$. The *closed neighborhood* of x is $N_G[x] = N_G(x) \cup \{x\}$. The *neighborhood* of an edge $e = xy \in E$ is defined as $N_G(e) = N_G(x) \cup N_G(y)$. Clearly, $\{x, y\} \subseteq N_G(e)$. If $X \subseteq V$, then $N_G(X) = \bigcup_{x\in X} N_G(x)$, $N_G[X] = N_G(X) \cup X$, and $EN_G(X)$ is the set of all edges in G that have at least one end-vertex in X. Also, $N_G(M) = \bigcup_{e\in M} N_G(e)$ for a set $M \subseteq E$ of edges. The subscript G will be omitted whenever the context is clear.

The complete graph, the chordless path and the chordless cycle on n vertices are denoted by K_n, P_n and C_n, respectively. For the purposes of Section 4, we will need the notion of a *null graph* K_0 (in the terminology of Tutte [54]), i.e., the graph having no edges and no vertices. $K_4 - e$ is a graph obtained from the complete graph K_4 by deleting an arbitrary edge. The *star* $K_{1,n}$ is the complete bipartite graph with partition classes of cardinalities 1 and n. The star $K_{1,3}$ is known as a *claw*. The *complement* $\overline{G}$ of a graph G is that graph whose vertex set is $V(G)$ and where e is an edge of $\overline{G}$ if and only if e is not an edge of G. Let $G_1, G_2, \ldots, G_k$ be pairwise vertex-disjoint graphs. The *disjoint union* $G_1 \cup G_2 \cup \cdots \cup G_k$ denotes the graph with the vertex set $\bigcup_{i=1}^k V(G_i)$ and the edge set $\bigcup_{i=1}^k E(G_i)$. For a positive integer n, the disjoint union of n copies of a graph G is denoted by nG. The *join* of the graphs $G_1, G_2, \ldots, G_k$ is the graph $\sum_{i=1}^k G_i = \overline{\overline{G}_1 \cup \overline{G}_2 \cup \cdots \cup \overline{G}_k}$. We denote by G^2 the *square* of graph G, i.e., the graph on $V(G)$ in which two vertices are adjacent if and only if they have a distance of at most 2 in G. Finally, $L(G)$ is the *line graph* of a graph G, i.e., $V(L(G)) = E(G)$, and two vertices e and e' are adjacent in $L(G)$ if and only if the edges e and e' are adjacent in G.

2. Complexity of recognizing well-indumatched graphs

Consider the following decision problem.

Non-Well-Indumatched Graph
Instance: A graph G.
Question: Are there two maximal induced matchings of distinct sizes in G?

We first prove that Non-Well-Indumatched Graph is an NP-complete problem and thereby establish that recognizing the class $\mathcal{WIM}$ of well-indumatched graphs is a co-NP-complete problem. Then we extend this result by showing that recognizing a graph that has maximal induced matchings of at most t distinct sizes is a co-NP-complete problem for any $t \geqslant 2$. In our proof, we use a reduction from the NP-complete problem 3-Satisfiability (3-Sat), see Cook [21,29].

3-Sat
Instance: A collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over a set $X = \{x_1, x_2, \ldots, x_n\}$ of 0–1 variables such that $|c_j| = 3$ for $j = 1, 2, \ldots, m$.
Question: Is there a truth assignment for X that satisfies all the clauses in C?

Recall that a *truth assignment for* X is a mapping $\varphi : X \to \{0, 1\}$ that assigns a value $\varphi(x_i) \in \{0, 1\}$ to each variable $x_i \in X$. We extend φ to the set $X \cup \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ of the *literals* over X by setting $\varphi(\bar{x}_i) = \overline{\varphi(x_i)}$, $i = 1, 2, \ldots, n$. A literal $\ell \in X \cup \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ is *true under* φ if $\varphi(\ell) = 1$. A *clause* over X is a conjunction of some literals. A truth assignment φ for X *satisfies* a clause $c_j \in C$ if c_j involves at least one true literal under φ.

Theorem 1. *Problem* Non-Well-Indumatched Graph *is* NP-*complete.*

Proof. Obviously, the problem is in NP. To show that it is NP-complete, we construct a polynomial-time reduction from 3-Sat. Without loss of generality, we assume that no clause in C contains a variable x_i and its negation $\bar{x}_i$ because such a clause is satisfied by any truth assignment and can be eliminated. We also assume that $m = |C| \geqslant 2$.

Given an instance of 3-Sat, construct the following graph G. Some vertices of this graph are associated with the parameters of 3-Sat and use the same notation, and some vertices are auxiliary. Specifically, the vertex set of graph G is $C' \cup X'$, where $C' = \{c_1, c_2, \ldots, c_m\}$ and $X' = \{x_i, a_i, b_i, \bar{x}_i : i = 1, \ldots, n\}$. Edges of graph G are defined as follows.

- The set C' induces a clique.
- The set X' induces n paths $P^i = (x_i, a_i, b_i, \bar{x}_i)$ (with four vertices and the edges $x_i a_i$, $a_i b_i$ and $b_i \bar{x}_i$).
- For each clause $c_j = \left(\ell_j^1 \vee \ell_j^2 \vee \ell_j^3\right)$, introduce the three edges $c_j\ell_j^1, c_j\ell_j^2, c_j\ell_j^3$ between C' and X' in G.

An example of graph G constructed for $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $C = \{c_1 = (x_1 \vee x_2 \vee x_3), c_2 = (\bar{x}_2 \vee \bar{x}_3 \vee x_4), c_3 = (\bar{x}_1 \vee x_3 \vee x_5), c_4 = (\bar{x}_3 \vee \bar{x}_4 \vee x_5)\}$ is given in Fig. 1.

The graph G can obviously be constructed in time polynomial in m and n. Let us denote $X_i = \{x_i, a_i, b_i, \bar{x}_i\}$, $i = 1, \ldots, n$.

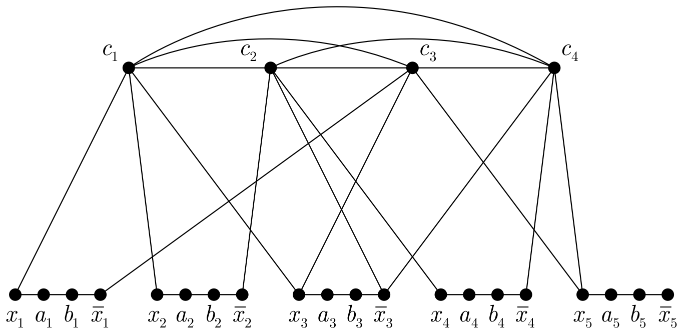

Fig. 1. Example of graph G.

Claim 1. *For any maximal induced matching M in G, the following statements hold:*

(i) $|EN_G(C') \cap M| \leqslant 1$,
(ii) $EN_G(X_i) \cap M \neq \emptyset$ *for* $i = 1, \dots, n$, *and*
(iii) $|M| \in \{n, n+1\}$.

Proof. (i) Since C' is a clique, the intersection of each induced matching of G with $EN_G(C')$ comprises at most one edge.

(ii) If M and $EN_G(X_i)$ are disjoint, then the edge a_ib_i can added to M, which contradicts the maximality of M.

(iii) By (i), $|EN_G(C') \cap M| \leqslant 1$. Also, M contains at most n edges of $G(X')$. Thus, $|M| \leqslant n+1$. The fact that $|M| \geqslant n$ follows from (ii). □

Note that G has always a maximum induced matching of size $n+1$, for example $\{c_1c_2\} \cup \{a_ib_i : i = 1, 2, \dots, n\}$. Recall that $m = |C| \geqslant 2$.

We now show that C is satisfiable if and only if the graph G has a maximal induced matching of size n. First, suppose that there exists a truth assignment φ satisfying C. We construct an induced matching $M \subseteq \{x_ia_i, b_i\overline{x}_i : i = 1, \dots, n\}$ by choosing n edges that correspond to the true literals. That is, if $\varphi(x_i) = 1$, then the edge x_ia_i is included in M, otherwise, the edge $b_i\overline{x}_i$ is included in M. Since φ satisfies C, each vertex of C' is adjacent to an end-vertex of an edge in M. It implies that we cannot extend M by adding an edge $e \notin E(G(X'))$. Indeed, such an edge is always incident to a vertex of C'. Thus, M is a maximal induced matching of size n.

Conversely, let M be a maximal induced matching in G of size n.

Claim 2. *The relation $M \subseteq E(G(X'))$ holds.*

Proof. Statement (ii) of Claim 1 and the equality $|M| = n$ imply that M contains exactly one edge from each set $EN_G(X_i)$. Moreover, M cannot have an edge connecting a vertex of X' with a vertex of C'. Indeed, suppose that an edge x_ic_j is in M. By statement (i) of Claim 1, $M \setminus \{x_ic_j\} \subseteq E(G(X'))$. The vertex $\overline{x}_i$ is non-adjacent to both x_i and c_j, since the clause c_j cannot contain both x_i and $\overline{x}_i$. Also, b_i is non-adjacent to both x_i and c_j. It follows that $M \cup \{b_i\overline{x}_i\}$ is an induced matching, which contradicts the maximality of M. □

We say that a vertex $c_j \in C'$ is *dominated* by M if c_j is adjacent to an end-vertex of an edge in M.

Claim 3. *At most one vertex in C' is not dominated by M.*

Proof. Suppose that c_1 and c_2 are not dominated by M. Then $M \cup \{c_1c_2\}$ is an induced matching, which contradicts the maximality of M. □

Claim 4. *There exists a maximal induced matching $M' \subseteq E(G(X'))$ such that $|M'| = |M|$ and M' dominates every vertex in C'.*

Proof. If M dominates every vertex in C', then we can set $M' = M$, since $M \subseteq E(G(X'))$ by Claim 2. Otherwise, by Claim 3, we may assume that c_1 is the only vertex in C' that is not dominated by M. Without loss of generality, let $c_1 = (x_1 \vee x_2 \vee x_3)$. By the maximality of M, we cannot add the edge $b_i\overline{x}_i$ to M for $i = 1, 2, 3$. Since $M \subseteq E(G(X'))$ and $x_ia_i \notin M$, the edges a_ib_i are in M, $i = 1, 2, 3$. Now, define $M' = (M \setminus \{a_1b_1\}) \cup \{x_1a_1\}$. Since both a_1 and b_1 dominate no vertex in C', the induced matching M' dominates every vertex in C'. Hence, M' is maximal. □

The matching M' in Claim 4 covers at most one of the vertices $x_i, \overline{x}_i$ for each $i = 1, \dots, n$. Therefore, we can define a partial truth assignment φ' satisfying C by deciding a literal to be true if the corresponding vertex of C' is covered by M'. It remains to extend φ' to a full assignment. This completes the proof of Theorem 1. □

A graph G is said to be *two-size indumatched* if there exists an integer $k \geqslant 1$ such that $|M| \in \{k, k+1\}$ for every maximal induced matching M in G. The proof of Theorem 1 implies the following corollaries.

Corollary 1. *Problem* Non-Well-Indumatched Graph *is* NP-*complete for two-size indumatched graphs.*

Corollary 2. Non-Well-Indumatched Graphs *is* NP-*complete for* $(2P_5, K_{1,5})$*-free graphs.*

Proof. Let G be the graph associated with an instance (C, X) of 3-Sat as in the proof of Theorem 1. It is easy to see that the graph G is a $(2P_5, K_{1,5})$-free graph, and the result follows from the proof of Theorem 1. □

Corollary 3. *The problem of deciding "*$\sigma(G) \leqslant K$*" is* NP-*complete for two-size indumatched graphs.*

Proof. The problem of deciding "$\sigma(G) \leqslant K$" belongs to NP. Let (C, X) be an instance of 3-Sat. Set $K = n = |X|$. For the graph G in the proof of Theorem 1, we have $\sigma(G) \geqslant n$ and $\Sigma(G) = n + 1$. Moreover, there exists a satisfying truth assignment for C if and only if $\sigma(G) = n$. □

Let $\mathcal{WIM}(t)$ be the class of graphs each of which has maximal induced matchings of at most t sizes. Note that $\mathcal{WIM}(1) = \mathcal{WIM}$ and $\mathcal{WIM}(t) \subset \mathcal{WIM}(t+1)$ for $t \geqslant 1$. Consider the following decision problem.

Non-t-Well-Indumatched Graph
Instance: A graph G and an integer number $t \geqslant 2$.
Question: Are there $t + 1$ maximal induced matchings of distinct sizes in G?

Based on the statements proved above, the following result is obtained.

Theorem 2. *For any* $t \geqslant 2$*, the problem* Non-t-Well-Indumatched Graph *is* NP-*complete even for* $(2P_5, K_{1,5})$*-free graphs.*

Proof. Clearly, the problem is in NP. To show that it is NP-complete, we establish a polynomial-time reduction from the problem Non-Well-Indumatched Graph which is NP-complete by Theorem 1. Let G be a $(2P_5, K_{1,5})$-free graph which constitutes an instance for the problem Non-Well-Indumatched Graph. Consider the graph $H = tG$, i.e., the disjoint union of t copies of G. It is easy to see that the graph H is $(2P_5, K_{1,5})$-free, and that H can be constructed in polynomial time in the number of vertices of G. To complete the proof, it suffices to show the following.

Claim 5. $G \in \mathcal{WIM}$ *if and only if* $H \in \mathcal{WIM}(t)$.

Proof. Firstly, suppose that $G \in \mathcal{WIM}$. Consider any two maximal induced matchings M_1 and M_2 of the graph H. Since the intersection of M_i, $i \in \{1, 2\}$, and the edge set of the jth copy of G in H is a maximal induced matching of the graph G, and since all maximal induced matchings in G have the same size, we obtain $|M_1| = |M_2|$. Hence, $H \in \mathcal{WIM}(1) \subset \mathcal{WIM}(t)$.

Conversely, suppose that $H \in \mathcal{WIM}(t)$. If $G \notin \mathcal{WIM}$, then there exist two maximal induced matchings N_1 and N_2 in G such that $|N_1| \neq |N_2|$. Without loss of generality, we may assume that $|N_1| > |N_2|$. For $i = 0, 1, \ldots, t$, let us define

$$M_i = \left(N_1^1 \cup N_1^2 \cup \cdots \cup N_1^i\right) \cup \left(N_2^{i+1} \cup N_2^{i+2} \cup \cdots \cup N_2^t\right),$$

where N_1^j and N_2^j are the copies of N_1 and N_2, respectively, in the jth copy of the graph G. Note that $M_0 = N_2^1 \cup N_2^2 \cup \cdots \cup N_2^t$ and $M_t = N_1^1 \cup N_1^2 \cup \cdots \cup N_1^t$. Obviously, the set M_i is a maximal induced matching in H and $|M_i| = i|N_1| + (t-i)|N_2|$ for $i = 0, 1, \ldots, t$. From $|N_1| > |N_2|$, we obtain $|M_t| > |M_{t-1}| > \cdots > |M_1| > |M_0|$, and we arrive at a contradiction to the assumption that $H \in \mathcal{WIM}(t)$. □

This completes the proof of the theorem. □

3. NP-completeness results for well-indumatched graphs

A set $I \subseteq V(G)$ is called *independent* or *stable* if no two vertices in I are adjacent. The *independence number* of a graph G, denoted $\alpha(G)$, is the maximum size of an independent set in G. A set $D \subseteq V(G)$ is a *dominating* set if each vertex in $V(G) \setminus D$ is adjacent to a vertex of D, i.e., $N[D] = V(G)$. The minimum size of a dominating set in G is the *domination number* of G, denoted by $\gamma(G)$. A set $I \subseteq V(G)$ is called an *independent dominating* set if I is an independent set and I is a dominating set. It is well known [5] that any independent dominating set is a maximal independent set, and vice versa. The minimum size of an independent dominating set of G is the *independent domination number*, and it is denoted by $i(G)$.

The following three decision problems are known to be NP-complete (see, e.g., [29]).

Independent Set
Instance: A graph G and an integer k.
Question: Is $\alpha(G) \geqslant k$?

Dominating Set
Instance: A graph G and an integer k.
Question: Is $\gamma(G) \leqslant k$?

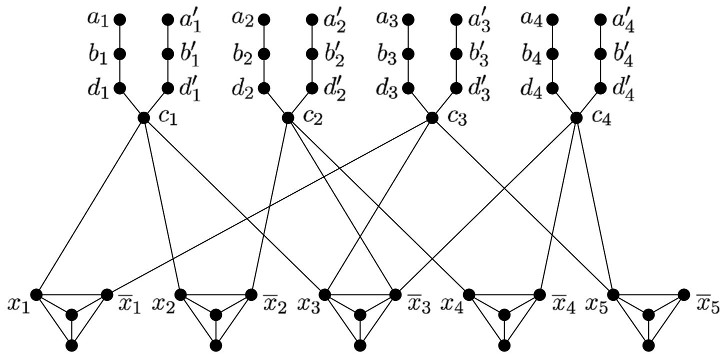

Fig. 2. An illustration of the construction.

INDEPENDENT DOMINATING SET
Instance: A graph G and an integer k.
Question: Is $i(G) \leqslant k$?

The complexity status of these problems on several classes of graphs has been surveyed in detail in [33]. Here we prove that the INDEPENDENT SET and the INDEPENDENT DOMINATING SET problems are NP-complete for well-indumatched graphs. Below, in Section 5, we show that the DOMINATING SET problem is NP-complete even in some hereditary subclass of well-indumatched graphs.

Theorem 3. INDEPENDENT DOMINATING SET *is an* NP-*complete problem for well-indumatched graphs.*

Proof. Clearly, the problem belongs to NP. To show that it is NP-complete, we establish a polynomial-time reduction from 3-SAT. For any instance (C, X) of 3-SAT with the clauses $C = \{c_1, c_2, \ldots, c_m\}$ and the variables $X = \{x_1, x_2, \ldots, x_n\}$, we construct a graph G as follows.

- For each variable x_i, we introduce a complete graph $F_i = K_4$ with two special vertices x_i and $\overline{x}_i$ called the *literal vertices*.
- For each clause c_j, we introduce a graph $H_j = P_7 = (a_j, b_j, d_j, c_j, d'_j, b'_j, a'_j)$, where c_j is called the *clause vertex*.
- The edges connecting $V(F_i)$ and $V(H_j)$ are defined as follows: a clause vertex c_j is connected to the three literal vertices corresponding to the literals in the clause c_j.

Fig. 2 illustrates this construction for the instance (C, X) of Fig. 1, see the proof of Theorem 1.
It is easy to see that the graph G can be constructed in time polynomial in $m = |C|$ and $n = |X|$.

Claim 6. *Each maximal induced matching M in G has exactly $2m + n$ edges.*

Proof. Clearly, M contains exactly one edge from each set $EN_G(V(F_i))$, $i = 1, 2, \ldots, n$. Also, M contains exactly two edges from each path $H_j, j = 1, 2, \ldots, m$. □

By Claim 6, G is a well-indumatched graph.

Claim 7. *There exists a satisfying truth assignment for C if and only if G has an independent dominating set of size $k = 2m + n$.*

Proof. First, suppose that C has a satisfying truth assignment. We construct an independent dominating set I in G as follows. If x_i is assigned value 1, then include the literal vertex x_i into I; otherwise, $\overline{x}_i$ is included into I. Finally, the set $\{b_j, b'_j : j = 1, 2, \ldots, m\}$ is included into I. It is straightforward to verify that I is an independent dominating set in G of cardinality $2m + n$, as required.

Conversely, suppose that I is an independent dominating set with $|I| = 2m+n$. Clearly, I contains exactly one vertex from each F_i. Also, I must contain at least two vertices from each set $S_j = \{a_j, b_j, a'_j, b'_j\}$ to dominate a_j and a'_j. Since $|I| = 2m + n$, I has exactly two vertices from each S_j. To dominate d_j and d'_j, we must have $I \cap S_j = \{b_j, b'_j\}$. However, the set S_j does not dominate c_j, so the vertex c_j must be dominated by some variable vertex. Thus, we can define a truth assignment $\varphi : X \to \{0, 1\}$ by $\varphi(x_i) = 1$ if $x_i \in I$, and $\varphi(x_i) = 0$ otherwise. □

Claims 6 and 7 imply that INDEPENDENT DOMINATING SET is NP-complete for well-indumatched graphs. □

We now prove that INDEPENDENT SET is NP-complete for well-indumatched graphs.

Theorem 4. INDEPENDENT SET *is an* NP-*complete problem for well-indumatched graphs.*

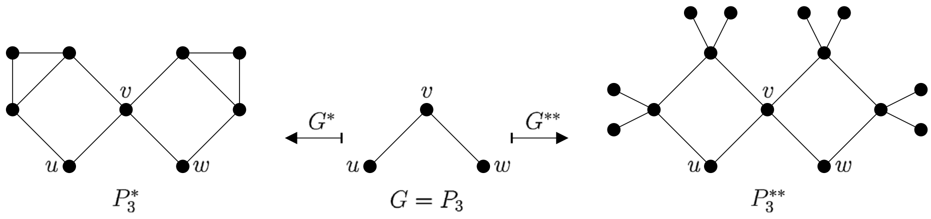

Fig. 3. An illustration of the transformations $G \mapsto G^*$ (on the left-hand side) and $G \mapsto G^{**}$ (on the right-hand side).

Proof. We give a polynomial-time reduction from the Independent Set problem for arbitrary graphs. Given a graph G with the edge set $E(G) = \{e_1, e_2, \dots, e_m\}$, $m \geqslant 1$, construct a new graph G^* as follows. First, for each edge $e_i = u_i v_i$, $i = 1, 2, \dots, m$, add a triangle (x_i, y_i, z_i, x_i), and join x_i to u_i and y_i to v_i, respectively. Thus, $V(G^*) = V(G) \cup \{x_i, y_i, z_i : i = 1, 2, \dots, m\}$ and $E(G^*) = \{e_i, x_i u_i, y_i v_i, x_i y_i, y_i z_i, z_i x_i : i = 1, 2, \dots, m\}$.

Fig. 3 illustrates the transformation of the graph G into the graph G^* (abbreviated as $G \mapsto G^*$), where G is the path $P_3 = (u, v, w)$.

Claim 8. *G^* is a well-indumatched graph.*

Proof. Let $G_i^* = G^*(\{u_i, v_i, x_i, y_i, z_i\})$, $i = 1, 2, \dots, m$. An arbitrary maximal induced matching M contains at most one edge of each G_i^*, therefore $|M| \leqslant m$. Suppose that M and $E(G_i^*)$ are disjoint for some $i \in \{1, 2, \dots, m\}$. By symmetry, we may assume that M does not contain any edge that is incident to u_i. It implies that $M \cup \{x_i z_i\}$ is an induced matching, which is a contradiction to the maximality of M. Thus, $|M| = m$. □

Claim 9. *$\alpha(G^*) = \alpha(G) + m$, where $m = |E(G)|$.*

Proof. Let $Z = \{z_1, z_2, \dots, z_m\}$. If I is a maximum independent set in G, then $I^* = I \cup Z$ is an independent set in G^*. Hence, $\alpha(G^*) \geqslant |I^*| = \alpha(G) + m$. Conversely, let I^* be a maximum independent set in G^*. Without loss of generality, we may assume that $Z \subseteq I^*$. The set $I = I^* \setminus Z$ is independent in G, therefore $\alpha(G) \geqslant |I| = \alpha(G^*) - m$. □

Claims 8 and 9 imply that the described mapping of an instance (G, k) to the instance $(G^*, k + |E(G)|)$ gives the required reduction, and the result follows. □

The transformation used in the proof of Theorem 4 yields some other interesting results for well-indumatched graphs. Consider the following decision problem.

Partition into Triangles
Instance: A graph G, with $|V(G)| = 3q$ for some positive integer q.
Question: Is there a partition $V_1 \cup V_2 \cup \dots \cup V_q = V(G)$ into sets V_i inducing triangles in the graph G?

Corollary 4. Partition into Triangles *is an* NP-*complete problem for well-indumatched graphs.*

Proof. Partition into Triangles is an NP-complete problem in general (see, e.g., [29]). The transformation $G \mapsto G^*$ described in the proof of Theorem 4 reduces the problem to well-indumatched graphs. □

Recall that a graph G is *chordal* or *triangulated* if G is $(C_n : n \geqslant 4)$-free. The following problem is also known as Minimum Fill-In problem.

Chordal Graph Completion
Instance: A graph G and an integer k.
Question: Is there a set E' containing $E(G)$ such that $|E' \setminus E(G)| \leqslant k$ and the graph $G' = (V(G), E')$ is chordal?

In other words, the problem asks whether a graph can be triangulated by adding at most k new edges. Chordal Graph Completion is a well-studied problem with a number of practical applications (see, e.g., Heggernes [34]). According to Kaplan et al. [35], the problem is fixed-parameter tractable.

Corollary 5. Chordal Graph Completion *is an* NP-*complete problem for well-indumatched graphs.*

Proof. Chordal Graph Completion is an NP-complete problem in general, see Yannakakis [57]. The transformation $(G, k) \mapsto (G^*, k + m)$, where $m = |E(G)|$, in the proof of Theorem 4 reduces the problem to well-indumatched graphs. □

Let H be a given graph. Consider the following decision problem.

Partition into Subgraphs Isomorphic to H
Instance: A graph G with $|V(G)| = q|V(H)|$ for some positive integer q.
Question: Is there a partition $V_1 \cup V_2 \cup \dots \cup V_q = V(G)$ such that $G(V_i)$ contains a subgraph isomorphic to H for all $i = 1, 2, \dots, q$?

It is well known that this problem is NP-complete for each H that contains a connected component of three or more vertices (Kirkpatrick and Hell [36,37]; see also [29]).

Theorem 5. Partition into Subgraphs Isomorphic to P_3 *is* NP-*complete for well-indumatched graphs.*

Proof. We give a polynomial-time reduction from the problem with $H = P_3$ for arbitrary graphs. Given a graph G with $|V(G)| = 3q$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$, $m \geqslant 1$, construct a new graph G^{**} as follows. First, for each edge $e_i = u_i v_i$, $i = 1, 2, \ldots, m$, add a new edge $x_i y_i$ and join x_i to u_i and y_i to v_i, respectively. Then attach two pendant vertices z' and z'' to each $z \in \{x_i, y_i\}$. More precisely,

$$V(G^{**}) = V(G) \cup \{x_i, x_i', x_i'', y_i, y_i', y_i'' : i = 1, 2, \ldots, m\}$$

and $E(G^{**})$ consists of

- e_i for all $i = 1, 2, \ldots, m$,
- $x_i u_i, y_i v_i$, for all $e_i = u_i v_i$, $i = 1, 2, \ldots, m$, and
- $x_i y_i, x_i x_i', x_i x_i'', y_i y_i', y_i y_i''$ for all $i = 1, 2, \ldots, m$.

For example, Fig. 3 illustrates the transformation of the graph G into the graph G^{**} (abbreviated as $G \mapsto G^{**}$), where G is the path $P_3 = (u, v, w)$.

It is easy to see that each maximal induced matching contains exactly one edge from each subgraph induced by $\{u_i, v_i, x_i, y_i, x_i', x_i'', y_i', y_i''\}$. Thus, G^{**} is a well-indumatched graph. Also, G has a partition into subgraphs P_3 if and only if G^{**} has a partition into subgraphs P_3. Indeed, all 3-paths (x_i', x_i, x_i'') and (y_i', y_i, y_i'') are always a partition of $G^{**} - V(G)$ into 3-paths. □

Kobler and Rotics [39] proved that computing Σ is NP-hard for line graphs and therefore, for claw-free graphs since every line graph is a claw-free graph [4]. One interesting special case of the problem is when the input line graph $L(G)$ is obtained from a graph G with $\Sigma(G)$ computable in polynomial time. Corollary 6 shows that this special case is also NP-complete even if restricted to Hamiltonian line graphs of well-indumatched graphs.

Remind that a graph G is *Hamiltonian* if G has a simple cycle containing all vertices of G. A subgraph H of G is *Eulerian* if it is connected and every vertex of H has an even degree, and H is *dominating* if every edge of G has at least one end-vertex in H.

Theorem 6 (*Harary and Nash-Williams [31]*)**.** *Let G be a graph with at least three edges. Then $L(G)$ is Hamiltonian if and only if G has a dominating Eulerian subgraph.*

We obtain the following result.

Corollary 6. *Computing Σ for Hamiltonian line graphs $L(G)$ is* NP-*hard even if G is a well-indumatched graph.*

Proof. As in the proof of Theorem 5, we consider the well-indumatched graph G^{**} associated with an arbitrary graph G, $|V(G)| = 3q$. Let $k = 2m + q$, where $m = |E(G)|$. Kobler and Rotics [39] noted that an induced matching in $L(H)$ corresponds precisely to a set of mutually vertex-disjoint 3-paths (not necessarily induced) in H. Thus, G^{**} has a partition into 3-paths if and only if $L(G^{**})$ has an induced matching of a size of at least k.

It is clear that $G^{**} - X$, where $X = \{x_i', x_i'', y_i', y_i'' : i = 1, 2, \ldots, m\}$, is a dominating Eulerian subgraph of G^{**}. Therefore, $L(G^{**})$ is Hamiltonian by Theorem 6. □

Kobler and Rotics [39] also proved that computing Σ is NP-hard for Hamiltonian graphs. This proof is not related to their proof for claw-free graphs. Corollary 7 gives a stronger result.

Corollary 7. *Computing Σ is* NP-*hard for Hamiltonian claw-free graphs.*

The reduction used in the proof of Theorem 5 also gives the following result (we refer the reader to [29] for the definitions of the Graph k-Colorability and Clique problems).

Corollary 8. Graph k-Colorability *and* Clique *are* NP-*complete problems for well-indumatched graphs.*

4. Co-indumatching subgraphs

The class $\mathcal{WIM}$ of well-indumatched graphs is not hereditary, therefore it does not have a forbidden induced subgraph characterization. For example, the path P_7 is a well-indumatched graph, while the path P_5 is not. In turn, P_7 contains P_5 as an induced subgraph. Moreover, for any graph H, there exists a well-indumatched graph G which contains H as an induced subgraph (as an example of such a graph G we can take the graph H^* from the proof of Theorem 4). In this section, we introduce a new hereditary system for well-indumatched graphs that is similar to that for well-covered graphs (Zverovich [58]; see also [17]).

A subgraph F of G is called *co-indumatching* if $F = G - N(M)$ for some induced matching M (possibly $M = \emptyset$) of G. We denote by CIMSub(G) the set of all co-indumatching subgraphs in G. For example, the cycle C_8 has the following co-indumatching subgraphs: C_8, P_4, K_1, K_0 and thus, CIMSub(C_8) $= \{C_8, P_4, K_1, K_0\}$.

Proposition 1. *Let G be a graph and H be a co-indumatching subgraph of G. Then the relation* $\mathrm{CIMSub}(H) \subseteq \mathrm{CIMSub}(G)$ *holds.*

Proof. Let $H \in \mathrm{CIMSub}(G)$ and let $F \in \mathrm{CIMSub}(H)$. Then there exist the induced matchings M' and M'' in G and H, respectively, such that $H = G - N_G(M')$ and $F = H - N_H(M'')$. Obviously, the set $M = M' \cup M''$ is an induced matching in G and $F = G - N_G(M)$, i.e., $F \in \mathrm{CIMSub}(G)$. □

A class $\mathcal{M}$ of graphs is *co-indumatching hereditary* if $\mathrm{CIMSub}(G) \subseteq \mathcal{M}$ for any $G \in \mathcal{M}$. In this section, we will consider well-indumatched graphs in view of this definition.

Proposition 2. *$\mathcal{WIM}$ is a co-indumatching hereditary class.*

Proof. Let G be a well-indumatched graph. It is sufficient to show that $G' = G - N(e)$ is also a well-indumatched graph for every edge $e \in E(G)$. Suppose it does not hold, i.e., there exists an edge e in G such that G' has two maximal induced matchings M' and N' with $|M'| \neq |N'|$. Then we can define in G two maximal induced matchings $M = M' \cup \{e\}$ and $N = N' \cup \{e\}$ with $|M| \neq |N|$, which is a contradiction to the condition that $G \in \mathcal{WIM}$. □

For a set $\mathcal{Z}$ of graphs, we put

$$\mathrm{FCIMS}(\mathcal{Z}) = \{G : \mathrm{CIMSub}(G) \cap \mathcal{Z} = \emptyset\}.$$

Let $\mathcal{M}$ be a co-indumatching hereditary class of graphs. If $\mathcal{M} = \mathrm{FCIMS}(\mathcal{Z})$ for a set $\mathcal{Z}$ of graphs, then $\mathcal{Z}$ is called a *set of forbidden co-indumatching subgraphs* for the class $\mathcal{M}$. A forbidden co-indumatching subgraph F for $\mathcal{M}$ is *minimal* if $\mathrm{CIMSub}(F) \setminus \{F\} \subseteq \mathcal{M}$. For example, P_5 is a minimal forbidden co-indumatching subgraph for $\mathcal{WIM}$. Indeed, it is not well-indumatched, but $P_5 - N(M)$ is well-indumatched for each non-empty induced matching M of P_5.

It is straightforward to prove that for an arbitrary co-indumatching hereditary class $\mathcal{M}$, there exists a set $\mathcal{Z}$ of graphs such that $G \in \mathcal{M}$ if and only if G does not contain each graph in $\mathcal{Z}$ as a co-indumatching subgraph. In fact, such a set $\mathcal{Z}$ must contain all minimal forbidden co-indumatching subgraphs for $\mathcal{M}$. Conversely, every set $\mathcal{Z}$ that contains all minimal forbidden co-indumatching subgraphs for the class $\mathcal{M}$ characterizes $\mathcal{M}$. For the sake of completeness, we provide the proof of these facts.

Proposition 3. *For a class $\mathcal{M}$ of graphs, the following statements hold.*

(i) *$\mathcal{M}$ is a co-indumatching hereditary class if and only if $\mathcal{M} = \mathrm{FCIMS}(\mathcal{Z})$ for some set $\mathcal{Z}$ of graphs.*
(ii) *A minimal (with respect to inclusion) set $\mathcal{Z}$ which satisfies* (i) *is uniquely defined, and it consists of all minimal forbidden co-indumatching subgraphs for $\mathcal{M}$.*

Proof. We begin with the necessity part of (i). Let $\mathcal{M}$ be a co-indumatching hereditary class of graphs and $G \in \mathcal{M}$. Put $\mathcal{Z} = \{H \text{ is a graph} : H \notin \mathcal{M}\}$. By the co-indumatching heredity of $\mathcal{M}$, we obtain $\mathrm{CIMSub}(G) \subseteq \mathcal{M}$, i.e., $\mathrm{CIMSub}(G) \cap \mathcal{Z} = \emptyset$. Thus, $G \in \mathrm{FCIMS}(\mathcal{Z})$ and $\mathcal{M} \subseteq \mathrm{FCIMS}(\mathcal{Z})$. Conversely, suppose $G \in \mathrm{FCIMS}(\mathcal{Z})$, i.e., $\mathrm{CIMSub}(G) \cap \mathcal{Z} = \emptyset$. Since $G \in \mathrm{CIMSub}(G)$, we have $G \notin \mathcal{Z}$. Thus, $G \in \mathcal{M}$ and $\mathrm{FCIMS}(\mathcal{Z}) \subseteq \mathcal{M}$.

The sufficiency part of statement (i) follows immediately from Proposition 1.

Now we verify statement (ii). Let $\mathcal{M} = \mathrm{FCIMS}(\mathcal{Z})$ for some $\mathcal{Z}$, and let $\mathcal{Z}^*$ be the set of all minimal co-indumatching subgraphs for $\mathcal{M}$. It is sufficient to prove that $\mathcal{Z}^* \subseteq \mathcal{Z}$ and $\mathcal{M} = \mathrm{FCIMS}(\mathcal{Z}^*)$. First we show that $\mathcal{Z}^* \subseteq \mathcal{Z}$. Suppose $G \in \mathcal{Z}^*$. Then $G \notin \mathcal{M}$ and since $\mathcal{M} = \mathrm{FCIMS}(\mathcal{Z})$, we obtain $\mathrm{CIMSub}(G) \cap \mathcal{Z} \neq \emptyset$. In turn, by the minimality of G, we have $\mathrm{CIMSub}(G) \setminus \{G\} \subseteq \mathcal{M}$. Finally, taking into consideration that $\mathcal{M} \cap \mathcal{Z} = \emptyset$, we obtain $\mathrm{CIMSub}(G) \cap \mathcal{Z} = \{G\}$, and hence $G \in \mathcal{Z}$.

Let us show that $\mathcal{M} = \mathrm{FCIMS}(\mathcal{Z}^*)$. Since $\mathcal{M} \cap \mathcal{Z}^* = \emptyset$ and $\mathrm{CIMSub}(G) \subseteq \mathcal{M}$, we have $\mathcal{M} \subseteq \mathrm{FCIMS}(\mathcal{Z}^*)$. Conversely, let $G \in \mathrm{FCIMS}(\mathcal{Z}^*)$, i.e., $\mathrm{CIMSub}(G) \cap \mathcal{Z}^* = \emptyset$. Assume, to the contrary, that $G \notin \mathcal{M}$. The finiteness of the set $\mathrm{CIMSub}(G)$ implies that there is a co-indumatching subgraph H of G such that $H \notin \mathcal{M}$ and $\mathrm{CIMSub}(H) \setminus \{H\} \subseteq \mathcal{M}$. Hence, $H \in \mathcal{Z}^*$, in contradiction to $\mathrm{CIMSub}(G) \cap \mathcal{Z}^* = \emptyset$. Thus, $G \in \mathcal{M}$ and, consequently, $\mathrm{FCIMS}(\mathcal{Z}^*) \subseteq \mathcal{M}$. This completes the proof. □

Now we address the problem of recognizing co-indumatching subgraphs. We show that it is hard to determine whether a given induced subgraph of a graph is a co-indumatching subgraph. To formalize our discussion, we introduce the following decision problem.

Co-Indumatching Subgraph
Instance: A graph G and a set $U \subseteq V(G)$ that induces a subgraph H.
Question: Is H a co-indumatching subgraph of G?

The following theorem shows that the problem is NP-complete.

Theorem 7. Co-Indumatching Subgraph *is an* NP-*complete problem.*

Proof. The problem is in NP. Indeed, if we guess an induced matching M of G and an isomorphism of H and $G - N(M)$, then we can check that H is a co-indumatching subgraph of G. To show that the problem is NP-complete, we use a polynomial-time reduction from 3-Sat. Let $C = \{c_1, c_2, \ldots, c_m\}$ and $X = \{x_1, x_2, \ldots, x_n\}$ be an instance of 3-Sat. We construct a graph G as follows:

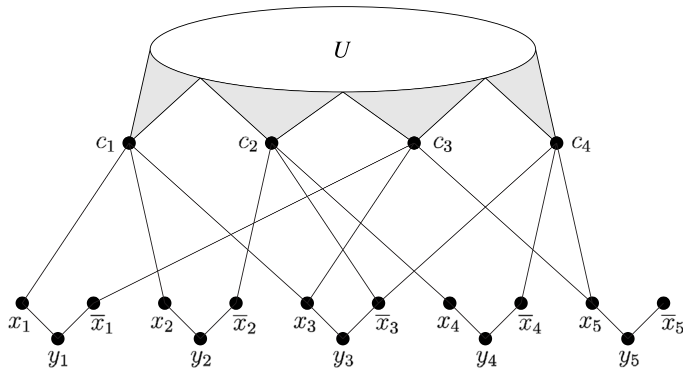

Fig. 4. An example of the graph G and the set $U \subseteq V(G)$.

- $V(G) = U \cup C \cup L \cup Y$, where $C = \{c_1, c_2, \ldots, c_m\}$, $L = \{x_1, \overline{x}_1, x_2, \overline{x}_2, \ldots, x_n, \overline{x}_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ and U are pairwise disjoint sets,
- the set U induces H,
- C, L and Y are independent sets,
- every vertex of U is adjacent to every vertex of C,
- each vertex of $\{x_i, \overline{x}_i\}$ is adjacent to the vertex y_i, $i = 1, 2, \ldots, n$, i.e., the set $\{x_i, y_i, \overline{x}_i\}$ induces P_3 in G with the edges $x_i y_i$ and $y_i \overline{x}_i$,
- a vertex $\ell \in L$ is adjacent to a vertex $c_j \in C$ if and only if the clause c_j contains the literal ℓ, and
- no edges between U and $L \cup Y$.

Fig. 4 gives an example of the construction for the instance (C, X) such as in Fig. 1, see the proof of Theorem 1. Clearly, G can be constructed in time polynomial in m, n and $|U|$. We consider G and $H = G(U)$ as an instance to Co-Indumatching Subgraph.

Let φ be a truth assignment satisfying C. The set $M = \{x_i y_i : \varphi(x_i) = 1\} \cup \{\overline{x}_i y_i : \varphi(x_i) = 0\}$ is an induced matching in the graph G. We have $G - N(M) = H$. Conversely, if $H = G(U)$ is a co-indumatching subgraph of G, then we have $H = G - N(M)$ for some induced matching M in G. Clearly, each edge in M is of the form ℓy_i, where $\ell \in L$. Since the set $\{\ell \in L : \ell y_i \in M\}$ dominates C, we can define a truth assignment satisfying C. Thus, there exists a satisfying truth assignment for C if and only if $H = G(U)$ is a co-indumatching subgraph of G. $\square$

Now we propose a characterization of the class $\mathcal{WIM}$ in terms of forbidden co-indumatching subgraphs.

Let I be an independent set in a graph G, including $I = \emptyset$. The subgraph $G - N[I]$ is called *co-stable*. We denote by $\mathrm{CSub}(G)$ the set of all co-stable subgraphs in G. A class of graphs $\mathcal{P}$ is *co-hereditary* if $\mathrm{CSub}(G) \subseteq \mathcal{P}$ for any $G \in \mathcal{P}$. For a co-hereditary class $\mathcal{P}$, a graph F is called a *forbidden co-stable subgraph* if $F \notin \mathcal{P}$. A forbidden co-stable subgraph F for $\mathcal{P}$ is *minimal* if $\mathrm{CSub}(F) \setminus \{F\} \subseteq \mathcal{P}$.

It is well known and easy to verify that the class of well-covered graphs is co-hereditary. We denote by $\mathcal{Z}_{\mathcal{W}}$ the set of all graphs of the form $\sum_{i=1}^{k} G_i$, where $k \geqslant 2$ and each graph G_i is well-covered and $\alpha(G_i) \neq \alpha(G_j)$ for $1 \leqslant i, j \leqslant k$, $i \neq j$. Obviously, the set $Z_{\mathcal{W}}$ consists of non-well-covered graphs.

Well-covered graphs were characterized in terms of forbidden co-stable subgraphs as follows.

Theorem 8 (*Zverovich [58]*). *$\mathcal{Z}_{\mathcal{W}}$ is the set of all minimal forbidden co-stable subgraphs for the class of well-covered graphs.*

Cameron [12] observed that a set $M \subseteq E(G)$ is an induced matching in a graph G if and only if M is an independent set in $(L(G))^2$. The following proposition is a straightforward extension of this result.

Proposition 4. *Let G be a graph and let M be an induced matching in G. Then*

$$(L(G - N_G(M)))^2 = (L(G))^2 - N_{(L(G))^2}[M^+],$$

where M^+ is the independent set which corresponds to M in the graph $(L(G))^2$.

Let $\mathcal{Z}_{\mathcal{WIM}}$ be the class of all graphs F such that $(L(F))^2 \in \mathcal{Z}_{\mathcal{W}}$. Here is our characterization of $\mathcal{WIM}$ in terms of forbidden co-indumatching subgraphs.

Theorem 9. *$\mathcal{Z}_{\mathcal{WIM}}$ is the set of all minimal forbidden co-indumatching subgraphs for the class $\mathcal{WIM}$.*

Proof. Let F be an arbitrary graph in $\mathcal{Z}_{\mathcal{WIM}}$.

Claim 10. *F is not well-indumatched.*

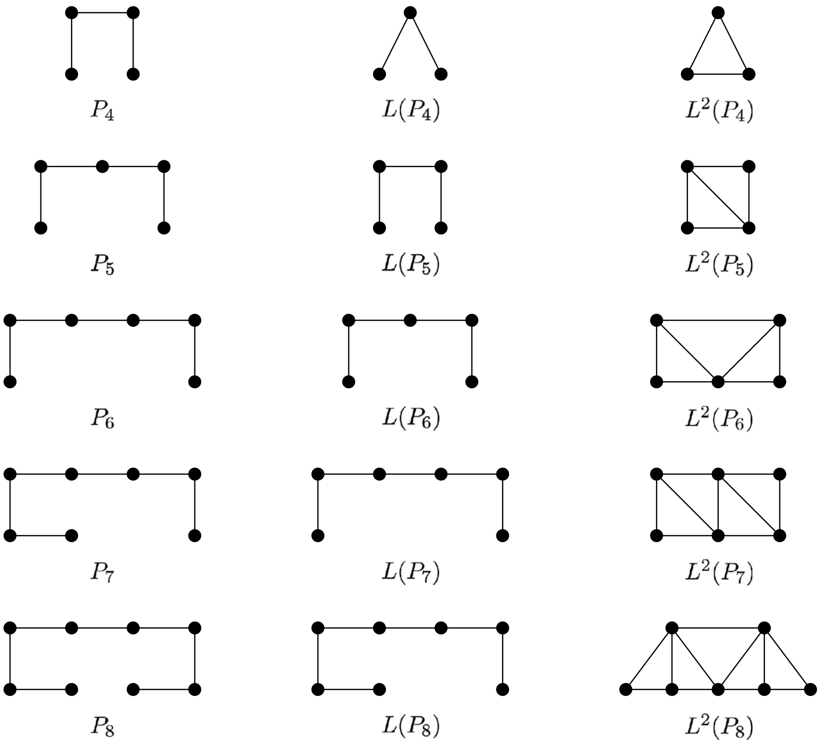

Fig. 5. An illustration of the transformation $G \mapsto (L(G))^2$.

Proof. Indeed, the graph $(L(F))^2 \in \mathcal{Z}_{\mathcal{W}}$ is not well-covered, i.e., it has maximal independent sets I and I' with $|I| \neq |I'|$. The sets I and I' correspond to maximal induced matchings M and M' in F. Since $|M| = |I| \neq |I'| = |M'|$, F is not well-indumatched. □

Claim 11. *F is a minimal forbidden co-indumatching subgraph for $\mathcal{WIM}$.*

Proof. By Claim 10, it is sufficient to show that $\mathrm{CIMSub}(F) \setminus \{F\} \subseteq \mathcal{WIM}$. Let F' be a proper co-indumatching subgraph of F, i.e., $F' = F - N_F(M)$ for a non-empty induced matching M in F. According to Proposition 4, $(L(F'))^2 = (L(F))^2 - N_{(L(F))^2}[M^+]$, i.e., $(L(F'))^2$ is a co-stable subgraph of $(L(F))^2$. Since $(L(F))^2 \in \mathcal{Z}_{\mathcal{W}}$, Theorem 8 implies that $(L(F'))^2$ is well-covered, and therefore F' is well-indumatched. □

Conversely, let H be a minimal forbidden co-indumatching subgraph for $\mathcal{WIM}$.

Claim 12. $(L(H))^2 \in \mathcal{Z}_{\mathcal{W}}$.

Proof. Since H contains maximal induced matchings M and M' with $|M| \neq |M'|$, the graph $G = (L(H))^2$ is not well-covered. Therefore, G contains a minimal non-well-covered subgraph G' as a co-stable subgraph. We have $G' = (L(H))^2 - N_{(L(H))^2}[M^+]$ for a non-empty independent set M^+ in $(L(H))^2$. Proposition 4 implies that $G' = (L(H'))^2$, where $H' = H - N_H(M)$. Since G' is not a well-covered subgraph, H' is not a well-indumatched graph. The minimality of H implies that $H = H'$ and $G = G'$. Thus, $(L(H))^2 = G \in \mathcal{Z}_{\mathcal{W}}$. □

The definition of $\mathcal{Z}_{\mathcal{WIM}}$ and Claims 11, 12 imply that $H \in \mathcal{Z}_{\mathcal{WIM}}$. □

The transformation $G \mapsto (L(G))^2$ is illustrated in Fig. 5 for the paths P_n, $n = 4, 5, 6, 7, 8$. The graphs $(L(P_4))^2$ and $(L(P_7))^2$ are well-covered, therefore both P_4 and P_7 are well-indumatched. The graphs $(L(P_5))^2$ and $(L(P_6))^2$ are minimal forbidden co-stable subgraphs for the class of well-covered graphs, implying that both P_5 and P_6 are minimal forbidden co-indumatching subgraphs for $\mathcal{WIM}$. Finally, $(L(P_8))^2$ is not well-covered, but it is not minimal. Accordingly, P_8 is a non-minimal non-well-indumatched graph.

We propose a variant of the well-known result of Chvátal and Slater [20] and Sankaranarayana and Stewart [46] that recognizing well-covered graphs is co-NP-complete.

Corollary 9. *Recognizing well-covered graphs is co-NP-complete even for $(2P_5, K_{1,10})$-free squares of line graphs.*

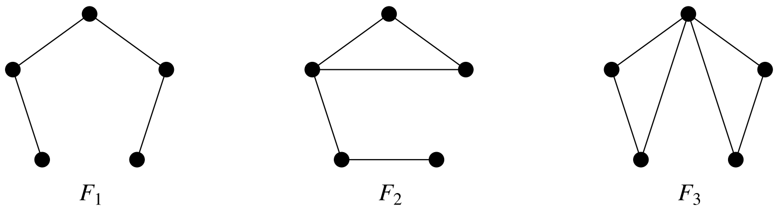

Fig. 6. The minimal forbidden induced subgraphs for perfectly well-indumatched graphs.

Proof. It is straightforward to verify the following result: If a graph G is $(2P_5, K_{1,5})$-free, then $(L(G))^2$ is $(2P_5, K_{1,10})$-free (see, e.g., Lozin and Rautenbach [42]). According to Corollary 2, recognizing well-indumatched graphs is co-NP-complete for $(2P_5, K_{1,5})$-free graphs. The proof of Theorem 9 shows that this problem can be reduced to the recognition of well-covered graphs in polynomial time within $(2P_5, K_{1,10})$-free squares of line graphs. □

5. Perfectly well-indumatched graphs

In this section, we consider a hereditary subclass of the class $\mathcal{WIM}$. A graph G is called *perfectly well-indumatched* if every induced subgraph of G is well-indumatched. We characterize perfectly well-indumatched graphs in terms of forbidden induced subgraphs.

Let us define some special graphs on five vertices (see Fig. 6): F_1 is the path P_5; F_2 is the graph with the vertex set $\{z_1, z_2, z_3, z_4, z_5\}$ and the edge set $\{z_1z_2, z_1z_3, z_2z_3, z_3z_4, z_4z_5\}$; F_3 is the graph obtained from F_2 by adding the edge z_3z_5. The graphs F_2 and F_3 are known in the literature under the names *kite* and *butterfly*, respectively.

We prove the following characterization theorem.

Theorem 10. *For a graph G, the following statements are equivalent:*

(i) *G is a perfectly well-indumatched graph.*
(ii) *G is a (F_1, F_2, F_3)-free graph.*
(iii) *$(L(G))^2$ is a $(K_4 - e)$-free graph.*

Proof. We will prove the equivalence (i) ⇔ (ii) and the equivalence (ii) ⇔ (iii) separately.

(i) ⇒ (ii). The graphs F_1, F_2 and F_3 are not perfectly well-indumatched graphs, and they are minimal. Therefore none of them can be an induced subgraph of a perfectly well-indumatched graph.

(ii) ⇒ (i). Suppose that G is a (F_1, F_2, F_3)-free graph, but G is not a perfectly well-indumatched graph. We may assume that G is minimal, i.e., each proper induced subgraph of G is perfectly well-indumatched. By definition, there are two maximal induced matchings M and N in G with $|M| > |N|$. For every edge $uv \in M \setminus N$, the set $N \cup \{uv\}$ is not an induced matching (by the maximality of N). It means that there exists an edge $f_{uv} \in N \setminus M$ such that $\{uv, f_{uv}\}$ is not an induced matching. Since $|M \setminus N| > |N \setminus M|$, there is an edge $wx \in M$, different from uv, such that $\{wx, f_{uv}\}$ is not an induced matching.

First suppose that f_{uv} is incident to vertex of $\{u, v, w, x\}$, say $f_{uv} = uy$. The vertex y is adjacent to a vertex of $\{w, x\}$. It follows that the set $\{u, v, w, x, y\}$ induces F_1 or F_2 or F_3, a contradiction.

Now $f_{uv} = yz$, where $y, z \notin \{u, v, w, x\}$. We may assume that y is adjacent to u. If y is adjacent to a vertex of $\{w, x\}$, then we delete z, and obtain essentially the previous situation. Thus, y is non-adjacent to both w and x. Similarly, we may assume that z is adjacent to w, and z is non-adjacent to both u and v. The set $\{u, v, w, x, y, z\}$ induces a graph containing either F_1 or F_2 as an induced subgraph, a contradiction.

(ii) ⇒ (iii). Let G be a (F_1, F_2, F_3)-free graph. We show that $H = (L(G))^2$ is a $(K_4 - e)$-free graph. Assume, to the contrary, that a set $\{u, v, a, b\} \subseteq V(H)$ induces in H a graph $K_4 - e$, where $e = ab$. There are edges u, v, a and b in G such that $\{a, b\}$ is an induced matching, but $\{u, v\}$, $\{u, a\}$, $\{u, b\}$, $\{v, a\}$ and $\{v, b\}$ are not induced matchings. Let $x = x_1x_2$, where $x \in \{u, v, a, b\}$. By symmetry, we have to consider two cases.

Case 1. The edge a is adjacent to some of the edges u or v. Without loss of generality, we may assume that the edges a and u are adjacent in G. By symmetry, let $a = u_1a_2$, i.e., $a_1 = u_1$. The edge b cannot be incident to each of the vertices u_1 or u_2, since the set $\{a, b\}$ is an induced matching in G. For the same reason, both b_1 and b_2 are non-adjacent to a_1 and a_2. Since the vertices u and b are adjacent in $K_4 - e$, there is an edge connecting u_2 and some vertex in $\{b_1, b_2\}$. Without loss of generality, we may assume that u_2 is adjacent to b_1. It follows that the subgraph of G induced by the set $\{u_2, a_1, a_2, b_1, b_2\}$ is one of F_1, F_2 or F_3, which is a contradiction. Note that we take into consideration the situation when the edge $v = v_1v_2$ coincides with one of the edges u_2a_2 or u_2b_2.

Case 2. The edge a is non-adjacent to the edges u and v. If the edge b is adjacent to some of the edges u or v, then we are in the same situation as in Case 1, which was shown to be impossible. Thus, b is non-adjacent to both u and v. Since $\{a, b\}$ is an induced matching in G, both a_1 and a_2 are non-adjacent to b_1 and b_2. Also, since u is adjacent to both a and b in $K_4 - e$, there are an edge f connecting $\{u_1, u_2\}$ and $\{a_1, a_2\}$ and an edge g connecting $\{u_1, u_2\}$ and $\{b_1, b_2\}$. Without loss of generality, we may assume that $f = u_1a_1$. Suppose that $g = u_1b_1$. As a result, the set $\{u_1, a_1, a_2, b_1, b_2\}$ induces one of F_1, F_2 or F_3, which is a contradiction. Hence, u_1 is non-adjacent to both b_1 and b_2. Now suppose that $g = u_2b_1$. By symmetry, u_2 is non-adjacent to both a_1 and a_2. It follows that the subgraph of G induced by the set $\{u_1, u_2, a_1, a_2, b_1\}$ is one of F_1 or F_2, which is again a contradiction.

Thus, we have contradictions in all possible cases, and the proof of the implication (ii) ⇒ (iii) is complete.

(iii) ⇒ (ii). This follows from the observation that each of the graphs $(L(F_1))^2$, $(L(F_2))^2$ and $(L(F_3))^2$ contains an induced subgraph which is isomorphic to $K_4 - e$. □

Theorem 10 implies that the class of perfectly well-indumatched graphs contains all $2K_2$-free graphs and therefore, all split graphs. Remind that a graph G is called a *split graph* if there is a partition $Q \cup J$ of its vertex set, so that Q induces a clique and J induces an independent set. Földes and Hammer [28] characterized split graphs as $(2K_2, C_4, C_5)$-free graphs. It is well known that Dominating Set is an NP-complete problem for split graphs [6]. Thus, we have the following corollary.

Corollary 10. Dominating Set *is an NP-complete problem for perfectly well-indumatched graphs.*

As we have shown, Independent Set and Independent Dominating Set are NP-complete problems for well-indumatched graphs. However, they can be solved in polynomial time for perfectly well-indumatched graphs.

Theorem 11. *The* Independent Set *problem and the* Independent Dominating Set *problem can be solved in polynomial time for perfectly well-indumatched graphs, even in their weighted versions.*

Proof. Let G be a perfectly well-indumatched graph. We show that each connected component H of G is a $2K_2$-free graph. Suppose that the set $S = \{a, b, c, d\} \subseteq V(H)$ induces $2K_2$ with the edges ab and cd. The connectivity of H implies that there exists a shortest path $P = (u_1, u_2, \ldots, u_k)$ such that $u_1 \in \{a, b\}$ and $u_k \in \{c, d\}$. Clearly, $k \geqslant 3$. If $k = 3$, then the set $S \cup \{u_2\}$ induces one of F_1, F_2, or F_3 (Fig. 6). If $k \geqslant 4$, then the set $\{a, b, u_2, u_3, u_4\}$ induces either F_1 or F_2. In any case, we have a contradiction to Theorem 10.

A general result of Balas and Yu [3] implies that each $2K_2$-free graph has polynomially many maximal independent sets, see also Alekseev [1]. Using an algorithm of Tsukiyama et al. [53], these sets for a graph H can be listed in $O(nmN)$ time, where $n = |V(H)|$, $m = |E(H)|$ and N is the total number of maximal independent sets in H. □

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