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Analysis of second order difference schemes on non-uniform grids for quasilinear parabolic equations

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A B S T R A C T

On the base of the maximum principles two-sided estimates for solutions of difference schemes are proved without any assumption of sign-definiteness of input data. Second order unconditional monotone difference scheme for quasilinear convection–diffusion equation on uniform grids is constructed. A priori estimates of the difference solution on uniform norm C are established. The obtained results are generalized for the case of non-uniform spatial grids. Numerical experiments confirming theoretical results are presented.

1. Introduction

One of the actual problems of the contemporary computational mathematics is the increasing of the accuracy of the difference schemes on minimal stencil. A traditional approach for increasing of the accuracy of a method is to use non-uniform grids. Generally, in this case the second order of approximation is lost. For example, at the approximation of the second order derivative the relation

$$u''(x_i) - u_{\bar{x}\hat{x},i} = O\left(h_{i+1} - h_i + \hbar_i^2\right),$$

holds, where as usual $u_{\bar{x}\hat{x},i} = \left(u_{x,i} - u_{\bar{x},i}\right)/\hbar_i, u_{x,i} = (u_{i+1} - u_i)/h_{i+1}, u_{\bar{x},i} = (u_i - u_{i-1})/h_i, \hbar_i = 0.5\,(h_{i+1} + h_i), h_i$ is the step of the non-uniform grid. It was found that at the non-computational point $\bar{x}_i = (x_{i-1} + x_i + x_{i+1})/3 = x_i + (h_{i+1} - h_i)/3$ the standard approximation of second difference derivative preserves second order

$$u''(\bar{x}_i) - u_{\bar{x}\hat{x},i} = O\left(\hbar_i^2\right). \tag{1}$$

It turned out that the using of such simple idea is a fruitful approach for construction of difference schemes of second order of accuracy on non-uniform spatial grids for one-dimensional and multidimensional elliptic and parabolic equations [1–11]. Similar construction in the case of variable coefficients is more complicated since already does not exist a point x^*, such that the relation

$$\left(k\left(x^*\right)u'\left(x^*\right)\right)' - (au_{\bar{x}})_{\hat{x}} = O\left(\hbar_i^2\right),$$

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is fulfilled. Nevertheless, on the base of the equality

$$\left(ku'\right)' = 0,5\left((ku)'' + ku'' - k''u\right), \tag{2}$$

difference schemes of such type were constructed for the inhomogeneous parabolic equations with variable coefficients in [12].

In the present paper the previous results are generalized for construction of unconditional monotone difference scheme of second-order of local approximation on non-uniform grids in space for the quasilinear parabolic equation with unbounded nonlinearity [13,14]

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + f(x,t).$$

The finite-difference methods satisfying the grid maximum principle usually are called monotone [15]. Since a difference scheme can be written in the canonical form, then one understands the monotonicity as the fulfilment of positivity conditions on the coefficients of equations [15].

The monotone schemes play important role in the computational practice since the corresponding discrete problems are well-posed [16]. Moreover they provide numerical solution without oscillations even in the case of non-smooth solutions [17].

Usually the maximum principle is applied to prove existence and uniqueness of solutions of initial boundary value problems for parabolic and elliptic equations. In contrast to the energetic inequalities it is a powerful tool for establishing of a priori estimates of the solution in the strong uniform norm for arbitrary dimensional problems with non-selfadjoint elliptic operators [18].

It is non-less important that one can obtain lower estimates of the solutions to differential–difference problems or in the general case two-sided estimates for the solution of the problems. This is especially important for investigation of theoretical properties of the computational methods approximating problems with unbounded nonlinearities, where it is necessary to prove that discrete solution belongs to a neighbourhood of the exact solution. As an example we investigate the Gamma equation modelling pricing of options in financial mathematics [19]. In this context, it is interesting to note the paper [20], in which two-sided estimates for solution of difference schemes approximating Dirichlet problem for linear parabolic equation are obtained in the discrete and continuous cases.

In the present paper on the base of maximum principle under positivity conditions for coefficients the important two-sided estimate for the solution of the difference problem is proved without any assumption of sign-definiteness of the input data. This inequality is often used for obtaining a priori estimates in the norm $C\,(L_\infty)$.

The paper outline is as follows. In Section 2 the two-sided estimate of the solution of the difference problem is established and conditions for correctness of a difference scheme approximating IBVP for quasilinear Gamma equation are formulated and proved.

In Section 3 unconditional monotone difference scheme of second order of approximation for the quasilinear convection–diffusion equation on standard uniform grids is investigated. A priori estimates of the difference solution in uniform norm C are established. In Sections 4–6 the obtained results are generalized for the case of non-uniform spatial grids and some numerical experiments confirming theoretical results are presented.

2. Maximum principle for difference schemes with variable sign input data

Let in the n-dimensional Euclidian space a finite number of points of the grid Ω_h is given. To each point $x \in \Omega_h$ we associate one and only one stencil $\mathcal{M}(x)$—a subset of Ω_h, containing this point. The set $\mathcal{M}'(x) = \mathcal{M}(x) \setminus x$ is called *neighbourhood* of the point x. Let the functions $A(x), B(x,\xi), F(x)$ be given at $x \in \Omega_h, \xi \in \Omega_h$ and they take real values. Next, to each point $x \in \Omega_h$ corresponds one and only one equation of the form [15]

$$A(x)\,y(x) = \sum_{\xi\in\mathcal{M}'(x)} B(x,\xi)\,y(\xi) + F(x), \quad x \in \Omega_h, \tag{3}$$

which is called a *canonical form* of the difference scheme. Note that the set $\mathcal{M}'(x)$ could be empty, for example, in the case of Dirichlet boundary condition. As a result we get a system of linear algebraic equations. This system is often called *difference scheme*. Together with the grid Ω_h, we will consider its arbitrary subset $\overline{\omega}_h$ and we will denote

$$\overline{\Omega}_h = \bigcup_{x\in\omega_h} \mathcal{M}(x).$$

For example, let Ω_h be the set of the internal nodes at the approximation of the Poisson equation. Obviously, in this case $\overline{\omega}_h = \Omega_h$. According to [15], the point x is called a *grid boundary node*, if Dirichlet condition is posed

$$y(x) = \mu(x), \quad x \in \gamma,$$

where γ is the set of the boundary nodes. We know that for approximation boundary conditions of second or third kind the grid may not contain boundary nodes, i.e. all grid nodes will be only internal nodes. We will assume the fulfilment of the usual positivity conditions for the coefficients

$$A(x) > 0, \qquad B(x,\xi) > 0 \quad \text{for all } \xi \in \mathcal{M}'(x), \tag{4}$$

$$D(x) = A(x) - \sum_{\xi\in\mathcal{M}'(x)} B(x,\xi) > 0 \quad \text{for all } \xi \in \mathcal{M}'(x). \tag{5}$$

They guarantee the unique solvability of the difference scheme (3), as well as its monotonicity and stability in the uniform norm with respect to small perturbation of input data.

We now formulate the basic results that allow us to establish two-sided estimates for the discrete solution via input data at non-sign-definite input data of problem $F(x)$.

Theorem 1. *Suppose that the positivity conditions for coefficients (4)–(5) are fulfilled. Then the maximal and minimal values of the solution of the difference scheme (3) belong to the value interval of the input data*

$$\min_{x\in\Omega_h} \frac{F(x)}{D(x)} \leqslant y(x) \leqslant \max_{x\in\Omega_h} \frac{F(x)}{D(x)}. \tag{6}$$

Proof. Suppose the maximum of the solution $y(x)$ of the difference problem (3) is reached at the point $x_0 \in \Omega_h$

$$\max_{x\in\Omega_h} y(x) = y(x_0).$$

Then from Eq. (3) we have

$$A(x_0)\,y(x_0) = \sum_{\xi\in\mathcal{M}'(x_0)} B(x_0,\xi)\,y(\xi) + F(x_0) \leqslant \sum_{\xi\in\mathcal{M}'(x_0)} B(x_0,\xi)\,y(x_0) + F(x_0).$$

The assumption of the theorem implies that

$$A(x_0) - \sum_{\xi\in M'(x_0)} B(x_0,\xi) = D(x_0) > 0.$$

Therefore,

$$y(x) \leqslant y(x_0) \leqslant \frac{F(x_0)}{D(x_0)} \leqslant \max_{x\in\Omega_h} \frac{F(x)}{D(x)} \quad \text{for all } x \in \Omega_h.$$

So the first estimate (6) is proved. In a similar way, the second estimate can be proved. In fact, let minimum of the grid function $y(x)$ be reached at the point $x_1 \in \Omega_h$

$$\min_{x\in\Omega_h} y(x) = y(x_1).$$

Then from Eq. (3) it follows

$$A(x_1)\,y(x_1) = \sum_{\xi\in\mathcal{M}'(x_1)} B(x_1,\xi)\,y(\xi) + F(x_1) \geqslant \sum_{\xi\in\mathcal{M}'(x_1)} B(x_1,\xi)\,y(x_1) + F(x_1).$$

On the base of condition (5) we conclude that

$$y(x) \geqslant y(x_1) \geqslant \min_{x\in\Omega_h} \frac{F(x)}{D(x)}. \quad \square$$

Corollary 1 ([15]). *Let conditions of Theorem 1 be fulfilled. Then for the solution of the difference problem (3) the following estimate holds*

$$\|y\|_{L_\infty} = \|y\|_C = \max_{x\in\Omega_h} |y(x)| \le \left\| \frac{F}{D} \right\|_C.$$

Example 1. In the rectangle $\bar{Q}_T = \{(x,t) : 0 \le x \le l,\ 0 \le t \le T\}$ we consider the following initial boundary value problem for the quasilinear convection–diffusion equation with non-divergent convection term

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + r(x)\frac{\partial u}{\partial x} + f(x,t), \quad 0 < x < l,\ 0 < t \le T, \tag{7}$$

$$u(x,0) = u_0(x), \qquad u(0,t) = \mu_1(t), \qquad u(l,t) = \mu_2(t). \tag{8}$$

Following A. Friedman [21] Eq. (7) is called parabolic, if there exist two numbers k_1, k_2, such that

$$0 < k_1 \le k(u) \le k_2, \quad \forall u \in \bar{D}_u, \; k_1, k_2 = \text{const},$$
$$\bar{D}_u = \left\{ u(x,t) : \; m_1 \le u(x,t) \le m_2, \;\; (x,t) \in \bar{Q}_T \right\}. \tag{9}$$

Now we consider the particular case of Gamma equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \beta}{\partial x^2} + r(x) \frac{\partial u}{\partial x}, \tag{10}$$

obtained by transformation of the nonlinear Black–Scholes equation [19]. For the case $\beta = u/(1-\rho u)^2$, $\rho > 0$, we have the coefficient $k(u)$ of the form

$$k(u) = \frac{1+\rho u}{(1-\rho u)^3}.$$

Then in view of (9), Eq. (10) is parabolic, if $k(u) > 0, \; \forall u \in \bar{D}_u$, i.e. if

$$-\frac{1}{\rho} < u < \frac{1}{\rho}. \tag{11}$$

For the difference scheme of the form (3) approximating IBVP for Eq. (10) the positivity conditions for coefficients (4), (5) are fulfilled. By Theorem 1 the condition (11) will be satisfied, if input data satisfies conditions

$$-\frac{1}{\rho} < \min_{x_i \in \Omega_h} \frac{F^n(x_i)}{D^n(x_i)} \le y_i^n \le \max_{x_i \in \Omega_h} \frac{F^n(x_i)}{D^n(x_i)} < \frac{1}{\rho}.$$

3. Finite difference scheme of second order of approximation on uniform grids

In this section two-sided estimates for an unconditional monotone second order difference scheme approximated equations (7)–(9) on uniform grid are obtained.

3.1. Difference scheme for the convection–diffusion equation

We consider the non-stationary convection–diffusion problem (7)–(9) [22] with constants m_1, m_2 defined by the conditions

$$m_1 = \min_{(x,t)\in\bar{Q}_T} \{\mu_1(t), \mu_2(t), u_0(x)\} + T \min\left\{0, \min_{(x,t)\in Q_T} f(x,t)\right\},$$
$$m_2 = \max_{(x,t)\in\bar{Q}_T} \{\mu_1(t), \mu_2(t), u_0(x)\} + T \max\left\{0, \max_{(x,t)\in Q_T} f(x,t)\right\}.$$

In order to construct a monotone scheme for (7) which satisfies the maximum principle for arbitrary h and τ, we consider the equation with perturbed operator $\tilde{L}$ [15]

$$\frac{\partial u}{\partial t} = \tilde{L}u + f, \quad \tilde{L}u = \kappa(x,u) \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + r(x)\frac{\partial u}{\partial x}, \tag{12}$$

where

$$\kappa(x,u) = \frac{1}{1+R(x,u)}, \qquad R(x,u) = \frac{h\,|r(x)|}{2k(u)}.$$

We write $r(x)$ as a sum

$$r = r^+ + r^-, \quad r^+ = \frac{1}{2}(r+|r|) \ge 0, \; r^- = \frac{1}{2}(r-|r|) \le 0,$$

and approximate $r\frac{\partial u}{\partial x}$ by the expression

$$\left(r\frac{\partial u}{\partial x}\right)_i = \left(\frac{r}{k}\left(k\frac{\partial u}{\partial x}\right)\right)_i \sim b_i^+ a_{i+1}(u)\, u_{x,i} + b_i^- a_i(u)\, u_{\bar{x},i},$$

where

$$b_i^+ = \frac{r_i^+}{k(u_i)} \ge 0, \qquad b_i^- = \frac{r_i^-}{k(u_i)} \le 0, \qquad a_i(u) = \frac{1}{2}\left(k(u_{i-1}) + k(u_i)\right),$$
$$u_{x,i} = (u_{i+1} - u_i)/h, \qquad u_{\bar{x},i} = (u_i - u_{i-1})/h.$$

We replace the operator $\tilde{L}$ at a fixed $t = t_j$ by the difference operator

$$\tilde{\Lambda}\widehat{y} = \kappa\,(a(y)\widehat{y}_{\bar{x}})_x + b^+ a(y)^{(+1)}\widehat{y}_x + b^- a(y)\widehat{y}_{\bar{x}}, \quad \text{where } a(y)^{(+1)} = a_{i+1}(y).$$

Then for Eq. (12) on the uniform spatial and time grid

$$\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau, \quad \bar{\omega}_h = \left\{x_i = ih,\ i = \overline{0,N},\ hN = l\right\},\ \bar{\omega}_h = \omega_h \cup \{x_0 = 0,\ x_N = l\},$$
$$\bar{\omega}_\tau = \left\{t_n = n\tau,\ n = \overline{0,N_0},\ \tau N_0 = T\right\}, \quad \bar{\omega}_\tau = \omega_\tau \cup \left\{t_{N_0} = T\right\},$$

we construct the difference scheme

$$\frac{y_i^{n+1} - y_i^n}{\tau} = \frac{\kappa_i^n}{h}\left(a_{i+1}^n(y)\frac{y_{i+1}^{n+1} - y_i^{n+1}}{h} - a_i^n(y)\frac{y_i^{n+1} - y_{i-1}^{n+1}}{h}\right) + b_i^+ a_{i+1}^n(y)\frac{y_{i+1}^{n+1} - y_i^{n+1}}{h} + b_i^- a_i^n(y)\frac{y_i^{n+1} - y_{i-1}^{n+1}}{h} + f_i^{n+1}, \tag{13}$$
$$y_0^{n+1} = \mu_1(t_{n+1}), \qquad y_N^{n+1} = \mu_2(t_{n+1}), \quad \text{where } \kappa_i^n = \kappa\left(x_i, y_i^n\right).$$

3.2. Approximation error

The approximation error of the difference scheme (13) has the form

$$\psi = -u_t + \kappa\,(a(u)\widehat{u}_{\bar{x}})_x + b^+ a^{(+1)}(u)\widehat{u}_x + b^- a(u)\widehat{u}_{\bar{x}} + f. \tag{14}$$

Taking into account

$$b^+ = r^+/k, \qquad b^- = r^-/k, \qquad r^+ + r^- = r, \qquad r^+ - r^- = |r|,$$
$$u_t = \frac{\partial u}{\partial t} + O(\tau), \qquad (a(u)\widehat{u}_{\bar{x}})_x = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + O\left(h^2 + \tau\right),$$
$$a^{(+1)}(u)\widehat{u}_x = k(u)\frac{\partial u}{\partial x} + 0.5h\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + O\left(h^2 + \tau\right),$$
$$a(u)\widehat{u}_{\bar{x}} = k(u)\frac{\partial u}{\partial x} - 0.5h\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + O\left(h^2 + \tau\right),$$

we get

$$b^+ a^{(+1)}(u)\widehat{u}_x + b^- a(u)\widehat{u}_{\bar{x}} = r(x)\frac{\partial u}{\partial x} + R\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + O\left(h^2 + \tau\right).$$

It follows from (14) that

$$\psi = \frac{R^2}{1+R}\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + O\left(h^2 + \tau\right) = O\left(h^2 + \tau\right).$$

Therefore the difference scheme (13) has second order of approximation with respect to space and first order with respect to time.

3.3. Monotonicity, two-sided estimates and a priori estimates in the norm C

We write the difference scheme (13) in the canonical form (3)

$$A_i^n y_i^{n+1} = B_{1i}^n y_{i-1}^{n+1} + B_{2i}^n y_{i+1}^{n+1} + F_i^n, \quad i = 1, 2, \dots, N-1,\ n = \overline{0, N_0 - 1},$$
$$A_0^n y_0^{n+1} = F_0^n, \qquad A_N^n y_N^{n+1} = F_N^n,$$

with coefficients

$$B_{1i}^n = \frac{\tau}{h^2}a_i^n(y)\left(\kappa_i^n - hb_i^-\right), \qquad B_{2i} = \frac{\tau}{h^2}a_{i+1}^n(y)\left(\kappa_i^n + hb_i^+\right), \quad i = \overline{1, N-1},$$
$$A_i^n = 1 + B_{1i}^n + B_{2i}^n, \quad i = \overline{1, N-1}, \qquad A_0^n = A_N^n = 1,$$
$$F_i^n = y_i^n + \tau f_i^{n+1}, \quad i = \overline{1, N-1}, \qquad F_0^n = \mu_1^{n+1}, \qquad F_N^n = \mu_2^{n+1},$$
$$D_i^n = A_i^n - B_{1i}^n - B_{2i}^n = 1, \quad i = \overline{0, N}.$$

We need to prove that $a_i^n(y) > 0$ for all i, n. In fact, when $n = 0$, it is obvious that $a(u_0) > 0$. Assume that, for any arbitrary n, $a_i^n(y) > 0$ is also true. From this assumption we have $B_{1i}^n > 0, B_{2i}^n > 0, A_i^n > 0$. According to Theorem 1 on the base of the estimate (6) for arbitrary $t = t_n \in \omega_\tau$ and all $i = 0, 1, \dots, N$, we have

$$\min\left\{\mu_1^{n+1}, \mu_2^{n+1}, \min_{1\le i\le N-1}\left(y_i^n + \tau f_i^{n+1}\right)\right\} \le y_i^{n+1} \le \max\left\{\mu_1^{n+1}, \mu_2^{n+1}, \max_{1\le i\le N-1}\left(y_i^n + \tau f_i^{n+1}\right)\right\}. \tag{15}$$

Using induction on n, from (15) we obtain the two-sided estimate via the input data without assumption for sign-definiteness of input data

$$m_1^{n+1} \le y_i^{n+1} \le m_2^{n+1}, \quad i = \overline{0, N}, \tag{16}$$

where

$$m_1^{n+1} = \min_{(x,t)\in \bar{Q}_T} \{\mu_1(t), \mu_2(t), u_0(x)\} + t_{n+1} \min\left\{0, \min_{(x,t)\in Q_T} f(x,t)\right\} \ge m_1,$$
$$m_2^{n+1} = \max_{(x,t)\in \bar{Q}_T} \{\mu_1(t), \mu_2(t), u_0(x)\} + t_{n+1} \max\left\{0, \max_{(x,t)\in Q_T} f(x,t)\right\} \le m_2.$$

In view of (16) we obtain $y_i^{n+1} \in \bar{D}_u$, i.e. $a_i^{n+1}(y) > 0$. Since all positivity conditions for the coefficients (4)–(5) are satisfied, then the difference scheme (13) is monotone for all h and τ (i.e. *unconditionally monotone*). Therefore, the following theorem is proved.

Theorem 2. *Suppose that the conditions (9) are fulfilled. Then the finite difference scheme (13) is unconditionally monotone and for its solution $y \in \bar{D}_u$ the above two-sided estimates (16) hold.*

On the basis of the maximum principle in a standard way we obtain the a priori estimate in the norm C

$$\|y(t_n)\|_{C(\bar{\omega}_h)} \le \max\left\{\max_{t\in[0,t_n]} \{|\mu_1(t)|, |\mu_2(t)|\}, \|u_0\|_{C(0,l)}\right\} + t_n \max_{t\in[0,t_n]} \|f(t)\|_{C(0,l)},$$

where as usually

$$\|\upsilon\|_{C(\bar{\omega}_h)} = \max_{x\in\bar{\omega}_h} |\upsilon(x)|, \qquad \|g\|_{C(0,l)} = \max_{0\le x\le l} |g(x)|.$$

Remark 1. It is interesting to note that the maximal and minimal values of the difference solution do not depend on the diffusion coefficient $k(u)$ and the convection coefficient $r(x)$.

Example 2. We consider the particular case of Gamma Eq. (10) with initial and boundary conditions (8). In view of (16) for the difference scheme (13), approximating the problem (8), (10), the condition (11) will be fulfilled, if for all i, n

$$-\frac{1}{\rho} < \min_{(x,t)\in \bar{Q}_T} \{\mu_1(t), \mu_2(t), u_0(x)\} \le y_i^n \le \max_{(x,t)\in \bar{Q}_T} \{\mu_1(t), \mu_2(t), u_0(x)\} < \frac{1}{\rho}.$$

4. Finite difference scheme of second order of approximation on non-uniform grids

4.1. Difference schemes on non-uniform grids

We consider the initial boundary value problem with inhomogeneous boundary conditions for quasilinear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + f(x,t), \quad 0 < x < l, \ 0 < t \le T, \tag{17}$$

$$u(x,0) = u_0(x), \qquad u(0,t) = \mu_1(t), \qquad u(l,t) = \mu_2(t). \tag{18}$$

We suppose that there exist two real numbers k_1 and k_2 such that

$$0 < k_1 \le k(u) \le k_2, \quad \forall u \in [m_1, m_2]. \tag{19}$$

We introduce the non-uniform spatial grid $\bar{\widehat{\omega}}_h = \widehat{\omega}_h \cup \gamma_h$,

$$\widehat{\omega}_h = \{x_i = x_{i-1} + h_i, \ i = 1, 2, \dots, N-1\}, \qquad \gamma_h = \{x_0 = 0, \ x_N = l\},$$

and the uniform grid in time

$$\bar{\omega}_\tau = \{t_n = n\tau, \ n = 0, 1, \dots, N_0, \ \tau N_0 = T\} = \omega_\tau \cup \{T\}.$$

On the standard six-point stencil using the identity (2) we construct the difference scheme of second order of approximation on the non-uniform grid $\omega = \widehat{\omega}_h \times \omega_\tau$

$$y_{t(\beta_1\beta_2)} = 0.5\left[(k(y)\widehat{y})_{\bar{x}\widehat{x}} + k_{(\beta_1\beta_2)}(y)\widehat{y}_{\bar{x}\widehat{x}} - k_{\bar{x}\widehat{x}}(y)\widehat{y}_{(\beta_3\beta_4)}\right] + \varphi, \tag{20}$$

$$y_i^0 = u_0(x_i), \qquad y_0^{n+1} = \mu_1(t_{n+1}), \qquad y_N^{n+1} = \mu_2(t_{n+1}), \tag{21}$$

where

$$v_t = (\widehat{v} - v)/\tau, \quad \widehat{v} = v(x_i, t_{n+1}), \qquad \varphi = f(\bar{x}_i, t_{n+1}),$$
$$v_{\widehat{x}\bar{x}} = (v_x - v_{\bar{x}})/\hbar_i, \quad v_x = (v_{i+1} - v_i)/h_{i+1}, \ v_{\bar{x}} = (v_i - v_{i-1})/h_i,$$
$$v_{(\beta_k\beta_{k+1})} = \beta_{ki}v_{i+1} + \left(1 - \beta_{ki} - \beta_{k+1,i}\right)v_i + \beta_{k+1,i}v_{i-1},$$
$$\beta_{1i} = 0.5\left(\left|\tilde{h}_i\right| + \tilde{h}_i\right)/h_{i+1}, \qquad \beta_{2i} = 0.5\left(\left|\tilde{h}_i\right| - \tilde{h}_i\right)/h_i, \tag{22}$$
$$\beta_{3i} = 0.5\left(\tilde{h}_i k_{\widehat{x}\bar{x}} - \left|\tilde{h}_i k_{\widehat{x}\bar{x}}\right|\right)/(h_{i+1}k_{\widehat{x}\bar{x}}), \qquad \beta_{4i} = -0.5\left(\tilde{h}_i k_{\widehat{x}\bar{x}} + \left|\tilde{h}_i k_{\widehat{x}\bar{x}}\right|\right)/(h_i k_{\widehat{x}\bar{x}}),$$
$$\bar{x}_i = x_i + \tilde{h}_i, \quad \tilde{h}_i = (h_{i+1} - h_i)/3, \qquad \hbar_i = 0.5(h_{i+1} + h_i).$$

The variable in space weights $\beta_1, \beta_2, \beta_3, \beta_4$ are chosen in order to fulfil the following condition

$$v_{(\beta_k\beta_{k+1})} - v(\bar{x}_i) = O\left(\hbar_i^2\right), \quad k = 1, 3. \tag{23}$$

In view of (23) we obtain the condition for choice of spatial weights $\beta_1, \beta_2, \beta_3, \beta_4$ in a similar way as in (22)

$$\beta_{ki}h_{i+1} - \beta_{k+1,i}h_i = \frac{h_{i+1} - h_i}{3} = \tilde{h}_i, \quad k = 1, 3. \tag{24}$$

4.2. Approximation error

We show that the difference scheme (20)–(21) approximates the problem (17)–(19) with second order with respect to node $\bar{x}_i$, i.e.

$$\Psi(\bar{x}_i) = 0.5\left[(k(u)\widehat{u})_{\widehat{x}\bar{x}} + k_{(\beta_1\beta_2)}(u)\widehat{u}_{\widehat{x}\bar{x}} - k_{\widehat{x}\bar{x}}(u)\widehat{u}_{(\beta_3\beta_4)}\right] - u_{t(\beta_1\beta_2)} + \varphi = O\left(\hbar_i^2\right).$$

Using (1) we get

$$(k(u)\widehat{u})_{\widehat{x}\bar{x},i} - \frac{\partial^2(k(u)u)(\bar{x}_i, t_{n+1})}{\partial x^2} = O\left(\hbar_i^2 + \tau\right), \tag{25}$$
$$k_{\widehat{x}\bar{x},i}(u) - \frac{\partial^2 k(\bar{x}_i)}{\partial x^2} = O\left(\hbar_i^2\right). \tag{26}$$

In view of (23) we obtain

$$u_{t(\beta_1\beta_2)} - \frac{\partial u(\bar{x}_i, t_{n+1})}{\partial t} = O\left(\hbar_i^2 + \tau\right), \tag{27}$$
$$k_{(\beta_1\beta_2)}(u) - k(\bar{u}_i) = O\left(\hbar_i^2\right), \quad \bar{u}_i = u(\bar{x}_i). \tag{28}$$

Finally, from (1), (25)–(28) we find out that the approximation error is of second order in space

$$\Psi(\bar{x}_i, t_{n+1}) = O\left(\hbar_i^2 + \tau\right).$$

Therefore, the difference scheme (20)–(21) on arbitrary non-uniform spatial grid approximates the original differential problem with second order, so that

$$\max_{t\in\omega_\tau}\|\Psi\|_C \le M\left(\bar{h}^2 + \tau\right), \quad \bar{h} = \max_i h_i,$$

where as usual

$$\|.\|_C = \max_{x\in\widehat{\omega}_h}|.|, \qquad M = \text{const} > 0.$$

4.3. Monotonicity, two-sided estimates and a priori estimates in the norm C

We rewrite the difference scheme (20)–(21) in the canonical form

$$A_i^n y_{i-1}^{n+1} - C_i^n y_i^{n+1} + B_i^n y_{i+1}^{n+1} = -F_i^n, \quad i = 1, 2, \ldots, N-1, \tag{29}$$
$$y_0^{n+1} = \mu_1(t_{n+1}), \qquad y_N^{n+1} = \mu_2(t_{n+1}), \tag{30}$$

where

$$A_i^n = -\beta_{2i} + 0.5\tau\left[\left(k_{(\beta_1\beta_2)}\left(y^n\right) + k\left(y_{i-1}^n\right)\right)/(\hbar_i h_i) - \beta_{4i}k_{\widehat{x}\bar{x},i}\left(y^n\right)\right],$$
$$B_i^n = -\beta_{1i} + 0.5\tau\left[\left(k_{(\beta_1\beta_2)}\left(y^n\right) + k\left(y_{i+1}^n\right)\right)/(\hbar_i h_{i+1}) - \beta_{3i}k_{\widehat{x}\bar{x},i}\left(y^n\right)\right],$$
$$C_i^n = 1 + A_i^n + B_i^n, \qquad F_i^n = y_{(\beta_1\beta_2)}^n + \tau\varphi_i^{n+1}, \quad \varphi_i^{n+1} = f(\bar{x}_i, t_{n+1}), \ \bar{x}_i = x_i + \tilde{h}_i.$$

The difference scheme (29)–(30) will be monotone, if the coefficients of (4)–(5) are positive, i.e. if

$$A_i^n > 0, \qquad B_i^n > 0, \qquad D_i^n = C_i^n - A_i^n - B_i^n > 0. \tag{31}$$

For simplicity we denote $h_+ = h_{i+1}, h = h_i, \hbar = (h_+ + h)/2, \tilde{h} = (h_+ - h)/3, \upsilon = \upsilon_i = \upsilon(x_i), \upsilon_\pm = \upsilon_{i\pm 1} = \upsilon(x_{i\pm 1})$. Let us consider the case $\tilde{h} > 0$, $k_{\bar{x}\widehat{x}} > 0$ (we do not consider the trivial cases $\tilde{h} = 0$ and $k_{\bar{x}\widehat{x}} = 0$), then we get the concrete values of the weights

$$\beta_1 = \tilde{h}/h_+ > 0, \qquad \beta_2 = \beta_3 = 0, \qquad \beta_4 = -\tilde{h}/h < 0,$$
$$k_{(\beta_1\beta_2)}(y) = \frac{\tilde{h}}{h_+}k(y_+) + \left(1 - \frac{\tilde{h}}{h_+}\right)k(y) = \frac{\tilde{h}}{h_+}k(y_+) + \frac{2h_+ + h}{3h_+}k(y) > 0,$$
$$-\beta_4 k_{\bar{x}\widehat{x}}(y) = \frac{\tilde{h}}{h}k_{\bar{x}\widehat{x}}(y) > 0.$$

Hence $A > 0$ and

$$B = -\frac{\tilde{h}}{h_+} + 0.5\tau\frac{\left(1 + \frac{\tilde{h}}{h_+}\right)k(y_+) + \left(1 - \frac{\tilde{h}}{h_+}\right)k(y)}{\hbar h_+}.$$

Since $\left|\frac{\tilde{h}}{h_+}\right| < 1$, we have $B > -\frac{h_+ - h}{3h_+} + \frac{2\tau k_1}{h_+(h+h_+)}$. This implies $B_i^n > 0$ at $\tau \ge \left|h_{i+1}^2 - h_i^2\right| / (6k_1)$. In a similar way we can investigate all the other cases.

Therefore, the inequality

$$\tau \ge \frac{\left\|h_+^2 - h^2\right\|_C}{6k_1} \tag{32}$$

guarantees the fulfilment of the positivity of the coefficients (4), (5) and (31) (i.e. the difference scheme (20)–(21) is monotone) and on the basis of the estimate (6) for the arbitrary $t = t_n \in \omega_\tau$ and all $i = 0, 1, \dots, N$ we have

$$\min\left\{\mu_1^{n+1}, \mu_2^{n+1}, \min_{1\le i\le N-1}\left(y_{(\beta_1\beta_2)}^n + \tau\varphi_i^{n+1}\right)\right\} \le y_i^{n+1} \le \max\left\{\mu_1^{n+1}, \mu_2^{n+1}, \max_{1\le i\le N-1}\left(y_{(\beta_1\beta_2)}^n + \tau\varphi_i^{n+1}\right)\right\}. \tag{33}$$

Using induction in n from (33) and the inequalities

$$\min_{1\le i\le N-1} y_{(\beta_1\beta_2)}^n \ge \min_{1\le i\le N-1} y_i^n, \qquad \max_{1\le i\le N-1} y_{(\beta_1\beta_2)}^n \le \max_{1\le i\le N-1} y_i^n,$$

(since the variable weights $\beta_1, \beta_2 \ge 0$ are non-negative) we get the two-sided estimate via the input data without sign-definiteness of input data:

Theorem 3. *Suppose that the conditions (32) are fulfilled. Then for the solution $y \in \bar{D}_u$ of the difference scheme (20)–(21) the double-sided estimate*

$$\bar{m}_1^n \le y_i^n \le \bar{m}_2^n, \quad i = \overline{0, N},\ n = \overline{0, N_0},$$

holds, where

$$\bar{m}_1^n = \min_{(x,t)\in\bar{Q}_T}\{\mu_1(t), \mu_2(t), u_0(x)\} + t_n\min\left\{0, \min_{(x,t)\in Q_T} f(x,t)\right\} \ge m_1,$$
$$\bar{m}_2^n = \max_{(x,t)\in\bar{Q}_T}\{\mu_1(t), \mu_2(t), u_0(x)\} + t_n\max\left\{0, \max_{(x,t)\in Q_T} f(x,t)\right\} \le m_2.$$

In a standard way on the base of the maximum principle the a priori estimate in the norm C can be established:

Theorem 4. *Let the condition (32) be fulfilled. Then for the solution $y \in \bar{D}_u$ of the difference problem (20)–(21) the following a priori estimate holds*

$$\max_{t_n\in\omega_\tau}\|y(t_n)\|_{C(\widehat{\omega}_h)} \le \max\left\{\max_{t\in[0,t_n]}\{|\mu_1(t)|, |\mu_2(t)|\}, \|u_0\|_{C(0,l)}\right\} + t_n\max_{t\in[0,t_n]}\|f(t)\|_{C(0,l)}, \tag{34}$$

where as usually

$$\|\upsilon\|_{C(\widehat{\omega}_h)} = \max_{x\in\widehat{\omega}_h}|\upsilon(x)|, \quad \|g\|_{C(0,l)} = \max_{0\le x\le l}|g(x)|.$$

Remark 2. If the spatial grid is uniform ($h_+ = h$), then the difference scheme (20) transforms to implicit scheme

$$y_t = (a(y)\widehat{y}_{\bar{x}})_x + \varphi, \quad a(y) = 0.5[k(y) + k(y_-)],$$

for which the a priori estimate (34) holds without the restriction (32) on the relation between the steps h and τ (unconditional stability).

5. The non-stationary quasilinear convection–diffusion problem on non-uniform grids

In this section monotone difference schemes for the convection–diffusion problem (7)–(9) are constructed and analysed on non-uniform grids. For simplicity we consider the linear case, when the convective term is non-divergent [22]

$$Lu = \left(k(x)u'\right)' + r(x)u' - d(x)u = -f(x), \quad 0 < x < l, \tag{35}$$

$$u(0) = \mu_1, \qquad u(l) = \mu_2, \qquad k(x) \ge k_1 > 0, \qquad d(x) \ge d_1 > 0. \tag{36}$$

We develop a second-order difference scheme for which the maximum principle on arbitrary non-uniform grid $\widehat{\omega}_h$ holds. For example, in the work [11] second-order monotone schemes are constructed using the methodology of A.A. Samarskii, known as regularization principle [15]. Now we consider another approach for construction of similar schemes using the identities (2) and $ku' = 0.5\left((ku)' + ku' - k'u\right)$. Denoting $b(x) = r(x)/k(x)$ and $L_1 v = v'' + bv'$ we rewrite Eq. (35) in the form

$$Lu = 0.5L_1(ku) + 0.5kL_1u - 0.5uL_1k - du = -f.$$

We approximate the differential operator L_1 on the grid $\widehat{\omega}_h$ by the second order difference operator L_{1h}, i.e

$$L_{1h}v = \kappa v_{\bar{x}\widehat{x}} + b^+ v_x + b^- v_{\bar{x}} = L_1 v(\bar{x}) + O\left(\hbar^2\right). \tag{37}$$

Also, on the grid $\widehat{\omega}_h$ we replace the differential operator L by the difference one L_h

$$L_h u = 0.5L_{1h}(ku) + 0.5k_{(\beta_1\beta_2)}L_{1h}u - 0.5u_{(\bar{\beta}_3\bar{\beta}_4)}L_{1h}k - \bar{d}u_{(\beta_5\beta_6)}, \tag{38}$$

where

$$b^+ = \frac{r^+}{k}(\bar{x}) \ge 0, \qquad b^- = \frac{r^-}{k}(\bar{x}) \le 0, \qquad r^\pm = 0.5(r \pm |r|),$$
$$\kappa = \frac{1}{1+R}, \quad R = \frac{h_+ + 2h}{6}b^+ - \frac{2h_+ + h}{6}b^- \ge 0, \qquad \bar{d} = d(\bar{x}).$$

The variable in space weights $\bar{\beta}_3$, $\bar{\beta}_4$, β_5, β_6 are chosen from requirement of second order of approximation (24) in the following way

$$\bar{\beta}_3 = 0.5\left(\tilde{h}L_{1h}k - \left|\tilde{h}L_{1h}k\right|\right) / (h_+ L_{1h}k),$$
$$\bar{\beta}_4 = -0.5\left(\tilde{h}L_{1h}k + \left|\tilde{h}L_{1h}k\right|\right) / (hL_{1h}k),$$
$$\beta_5 = 0.5\left(\tilde{h} - \left|\tilde{h}\right|\right)/h_+, \qquad \beta_6 = -0.5\left(\tilde{h} + \left|\tilde{h}\right|\right)/h.$$

Therefore, the difference scheme

$$L_h y = -\varphi, \qquad \varphi = f(\bar{x}), \qquad y_0 = \mu_1, \qquad y_N = \mu_2, \tag{39}$$

approximates the differential problem (35)–(36) with second order on arbitrary non-uniform grid. It is interesting to mention that in the case of uniform grid $R = 0.5h|r|/k$, and the difference scheme (39) reduces to the well-known monotone scheme of second-order of approximation of A. A. Samarskii [15]. The difference scheme (39) can be written in the canonical form (29)–(30)

$$A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i, \quad i = 1, 2, \ldots, N-1,$$
$$y_0 = \mu_1, \qquad y_N = \mu_2$$

with coefficients

$$A_i = 0.5\left[\left(k_{(\beta_1\beta_2)} + k_{i-1}\right)\left(\kappa_i - b^- \hbar_i\right)/(\hbar_i h_i) - \bar{\beta}_{4i}L_{1h,i}k\right] - \beta_{6i}\bar{d}_i,$$
$$B_i = 0.5\left[\left(k_{(\beta_1\beta_2)} + k_{i+1}\right)\left(\kappa_i + b^+ \hbar_i\right)/(\hbar_i h_{i+1}) - \bar{\beta}_{3i}L_{1h,i}k\right] - \beta_{5i}\bar{d}_i, \tag{40}$$
$$C_i = \bar{d}_i + A_i + B_i, \qquad F_i = f(\bar{x}_i).$$

It is clear that $A_i > 0, B_i > 0, D_i = \bar{d}_i > 0$. So, for arbitrary non-uniform grid refinement the coefficients (40) of the difference scheme (39) satisfy the conditions (31) (unconditional monotonicity). By Theorem 1 we obtain two-sided estimates for the solution of the difference scheme (39)

$$\min\left\{\mu_1, \mu_2, \min_{1\le i\le N-1}\left(\bar{f}_i/\bar{d}_i\right)\right\} \le y_i \le \max\left\{\mu_1, \mu_2, \max_{1\le i\le N-1}\left(\bar{f}_i/\bar{d}_i\right)\right\}.$$

Also, by Corollary 1 the difference scheme (39) is stable with respect to right-hand side and the boundary conditions and for the solution the following a priori estimate holds

$$\|y\|_{\bar{C}} \le \max\left\{|\mu_1|, |\mu_2|, \left\|\bar{f}_i/\bar{d}_i\right\|_C\right\}. \tag{41}$$

Substituting $y = z + u$ in Eq. (39), we get the problem for the method error

$$L_h z = -\psi, \quad \psi = L_h u + \varphi, \qquad z|_{\gamma_h} = 0. \tag{42}$$

It is obvious that $\psi = O\left(\bar{h}^2\right), \bar{h} = \max_{1\le i\le N} h_i$. Since for the problem (42) all conditions (31) of the maximum principle are fulfilled, then from (41) we find out that $\|z\|_{\bar{C}} \le \|\psi\|_C \le c\bar{h}^2$, i.e. the difference scheme (39) converges to the exact solution with second order of convergence.

In a similar way we use formulas (27), (28), (37), (38) for construction on the usual six-point stencil of monotone difference scheme of second order of approximation of the non-stationary convection–diffusion equations (7)–(9) on the non-uniform grid $\omega = \widehat{\omega}_h \times \omega_\tau$ with the help of the change $g(x,u) = r(x)/k(u)$ and the operator $\Lambda v = v'' + g v', v = v(u)$

$$y_{t(\beta_1\beta_2)} = 0.5\left[\Lambda_h\left(k(y)\widehat{y}\right) + k_{(\beta_1\beta_2)}(y)\Lambda_h\widehat{y} - \widehat{y}_{(\tilde{\beta}_3\tilde{\beta}_4)}\Lambda_h k(y)\right] + \varphi, \tag{43}$$

$$y_i^0 = u_0(x_i), \qquad y_0^{n+1} = \mu_1(t_{n+1}), \qquad y_N^{n+1} = \mu_2(t_{n+1}), \tag{44}$$

where

$$\Lambda_h v = \bar{\kappa} v_{\bar{x}\widehat{x}} + g^+ v_x + g^- v_{\bar{x}}, \qquad \Lambda_h\widehat{v} = \bar{\kappa}\widehat{v}_{\bar{x}\widehat{x}} + g^+\widehat{v}_x + g^-\widehat{v}_{\bar{x}},$$
$$\tilde{\beta}_3 = 0.5\left(\tilde{h}\Lambda_h k - \left|\tilde{h}\Lambda_h k\right|\right)/\left(h_+\Lambda_h k\right), \qquad \tilde{\beta}_4 = -0.5\left(\tilde{h}\Lambda_h k + \left|\tilde{h}\Lambda_h k\right|\right)/\left(h\Lambda_h k\right),$$
$$g^+ = r^+(\bar{x})\bar{k}(y) \ge 0, \qquad g^- = r^-(\bar{x})\bar{k}(y) \le 0, \qquad r^\pm = 0.5\left(r \pm |r|\right),$$
$$\bar{k}(y) = \left[1/k(y_-) + 1/k(y) + 1/k(y_+)\right]/3, \qquad \bar{\kappa} = \left(1 + \bar{R}\right)^{-1},$$
$$\bar{R} = \frac{h_+ + 2h}{6}g^+ - \frac{2h_+ + h}{6}g^- \ge 0.$$

Now we consider the approximation error of the difference scheme (43)–(44)

$$\psi\left(\bar{x}, \hat{t}\right) = -u_{t(\beta_1\beta_2)} + 0.5\left[\Lambda_h\left(k(u)\widehat{u}\right) + k_{(\beta_1\beta_2)}(u)\Lambda_h\widehat{u} - \widehat{u}_{(\tilde{\beta}_3\tilde{\beta}_4)}\Lambda_h k(u)\right] + \varphi. \tag{45}$$

Taking into account that

$$v_x = v'(\bar{x}) + \frac{h_+ + 2h}{6}v''(\bar{x}) + O\left(\hbar^2\right),$$
$$v_{\bar{x}} = v'(\bar{x}) - \frac{2h_+ + h}{6}v''(\bar{x}) + O\left(\hbar^2\right),$$
$$g^+ + g^- = r(\bar{x})\bar{k}(u), \quad \bar{k}(u) = 1/k(\bar{u}) + O\left(\hbar^2\right), \qquad u = u(\bar{x}),$$

we get

$$g^+ v_x + g^- v_{\bar{x}} = \frac{r(\bar{x})}{k(\bar{u})}v'(\bar{x}) + \bar{R}v''(\bar{x}) + O\left(\hbar^2\right),$$

and hence

$$\Lambda_h v = v''(\bar{x}) + g(\bar{x}, \bar{u})v'(\bar{x}) + O\left(\hbar^2\right) = \Lambda v(\bar{x}) + O\left(\hbar^2\right). \tag{46}$$

It follows from (46), that

$$\Lambda_h\left(k(u)\widehat{u}\right) = \Lambda\left(k(u)u\right)\left(\bar{x}, \hat{t}\right) + O\left(\hbar^2 + \tau\right), \tag{47}$$

$$\Lambda_h\widehat{u} = \Lambda u\left(\bar{x}, \hat{t}\right) + O\left(\hbar^2 + \tau\right), \qquad \Lambda_h k(u) = \Lambda k(u)\left(\bar{x}, \hat{t}\right) + O\left(\hbar^2 + \tau\right). \tag{48}$$

In view of (27), (28), (47), (48) for the approximation error (45) of the difference scheme (43)–(44) we obtain

$$\psi\left(\bar{x}, \hat{t}\right) = O\left(\hbar^2 + \tau\right).$$

We write the difference scheme (43)–(44) in the canonical form (29)–(30) with coefficients

$$A_i^n = -\beta_{2i} + 0.5\tau\left[\left(k_{(\beta_1\beta_2)}\left(y^n\right) + k\left(y_{i-1}^n\right)\right)\left(\bar{\kappa}_i - \hbar_i g^-\right)/\left(\hbar_i h_i\right) - \tilde{\beta}_{4i}\Lambda_{h,i}k\left(y^n\right)\right],$$
$$B_i^n = -\beta_{1i} + 0.5\tau\left[\left(k_{(\beta_1\beta_2)}\left(y^n\right) + k\left(y_{i+1}^n\right)\right)\left(\bar{\kappa}_i + \hbar_i g^+\right)/\left(\hbar_i h_{i+1}\right) - \tilde{\beta}_{3i}\Lambda_{h,i}k\left(y^n\right)\right], \quad (49)$$
$$C_i^n = 1 + A_i^n + B_i^n, \qquad F_i^n = y_{(\beta_1\beta_2)}^n + \tau\varphi_i^{n+1}, \quad \varphi_i^{n+1} = f\left(\bar{x}_i, t_{n+1}\right).$$

It is easy to show [12], that the inequality

$$\tau \geq \frac{\left(1 + 0.5\bar{h}c_0\right)\left\|h_+^2 - h^2\right\|_C}{6k_1},$$
$$\bar{h} = \max_{1\leq i\leq N} h_i, \qquad c_0 = \max_{\substack{0\leq i\leq N\\ 0\leq n\leq N_0}} \frac{|r\left(x_i\right)|}{k\left(u\left(x_i, t_n\right)\right)},$$

guarantees the positivity of the coefficients A_i^n, B_i^n, C_i^n (49). For practical computations as constant c_0 we can use the expression

$$c_0 = \max_{\substack{0\leq i\leq N\\ 0\leq n\leq N_0}} \frac{|r\left(x_i\right)|}{k\left(y\left(x_i, t_n\right)\right)}.$$

Remark 3. Using [9] the results obtained above can be generalized on two-dimensional convection–diffusion equations [22]. Let $\bar{\Omega} = \{x = (x_1, x_2) :\ 0 \leq x_1 \leq l_1,\ 0 \leq x_2 \leq l_2\}$ be a rectangle with boundary Γ. For simplicity we consider the following boundary value problem

$$\frac{\partial^2 u\left(x\right)}{\partial x_1^2} + \frac{\partial^2 u\left(x\right)}{\partial x_2^2} + r_1\left(x\right)\frac{\partial u\left(x\right)}{\partial x_1} + r_2\left(x\right)\frac{\partial u\left(x\right)}{\partial x_2} - q\left(x\right)u\left(x\right) = -f\left(x\right), \quad x \in \Omega, \quad (50)$$
$$u\left(x\right) = \mu\left(x\right), \quad x \in \Gamma, \qquad q\left(x\right) \geq c_1 > 0, \qquad |r_\alpha\left(x\right)| \leq c_2, \quad \alpha = 1, 2. \quad (51)$$

In the domain $\bar{\Omega}$ we introduce arbitrary non-uniform grid

$$\widehat{\bar{\omega}}_h = \left\{x_{i_1 i_2} = \left(x_1^{i_1}, x_2^{i_2}\right),\ x_\alpha^{i_\alpha} = x_\alpha^{i_\alpha - 1} + h_\alpha^{i_\alpha},\ i_\alpha = 1, 2, \ldots, N_\alpha - 1,\ x_\alpha^0 = 0,\ x_\alpha^{N_\alpha} = l_\alpha,\ \alpha = 1, 2\right\},$$

where $\sum_{i_\alpha=1}^{N_\alpha} h_\alpha^{i_\alpha} = l_\alpha$, $\alpha = 1, 2$. Denote by $\widehat{\omega}_h$ the set of the grid internal nodes, and let γ_h be the set of boundary nodes, i.e. $\widehat{\bar{\omega}}_h = \widehat{\omega}_h \cup \gamma_h$. Then the finite difference scheme of second-order of approximation for problem (50)–(51) on non-uniform $\widehat{\bar{\omega}}_h$ has a form

$$\kappa_1 y_{(2)\bar{x}_1\hat{x}_1} + \kappa_2 y_{(1)\bar{x}_2\hat{x}_2} + \bar{r}_1^+ y_{(2)x_1} + \bar{r}_1^- y_{(2)\bar{x}_1} + \bar{r}_2^+ y_{(1)x_2} + \bar{r}_2^- y_{(1)\bar{x}_2} - \bar{q}y_{(\delta_*)} = -\bar{f}, \quad x \in \widehat{\omega}_h, \quad (52)$$
$$y\left(x\right) = \mu\left(x\right), \quad x \in \gamma_h. \quad (53)$$

Here we use the following notations

$$y_{\bar{x}_\alpha\hat{x}_\alpha} = \frac{1}{\hbar_\alpha}\left(y_{x_\alpha} - y_{\bar{x}_\alpha}\right), \quad y_{x_\alpha} = \frac{1}{h_{\alpha+}}\left(y^{(+1_\alpha)} - y\right), \ y_{\bar{x}_\alpha} = \frac{1}{h_\alpha}\left(y - y^{(-1_\alpha)}\right),$$
$$h_\alpha = h_\alpha^{i_\alpha}, \qquad h_{\alpha+} = h_\alpha^{i_\alpha+1}, \qquad \hbar_\alpha = 0.5\left(h_\alpha + h_{\alpha+}\right), \qquad v^{(\pm 1_1)} = v_{i_1\pm 1 i_2}, \qquad v^{(\pm 1_2)} = v_{i_1 i_2 \pm 1},$$
$$\bar{v} = v\left(\bar{x}\right), \quad \bar{x} = \left(\bar{x}_1, \bar{x}_2\right), \qquad \bar{x}_\alpha = x_\alpha + \tilde{h}_\alpha, \quad \tilde{h}_\alpha = \left(h_{\alpha+} - h_\alpha\right)/3, \qquad \delta_\alpha^\pm = 0.5\left(\tilde{h}_\alpha \pm \left|\tilde{h}_\alpha\right|\right),$$
$$\kappa_\alpha = \frac{1}{1 + R_\alpha}, \quad R_\alpha = \bar{r}_\alpha^+\frac{h_{\alpha+} + 2h_\alpha}{6} - \bar{r}_\alpha^-\frac{2h_{\alpha+} + h_\alpha}{6},$$
$$r_\alpha^+ = 0.5\left(r_\alpha + |r_\alpha|\right) \geq 0, \qquad r_\alpha^- = 0.5\left(r_\alpha - |r_\alpha|\right) \leq 0, \qquad y_{(\delta_*)} = y + \sum_{\alpha=1}^{2}\left(\delta_\alpha^- y_{x_\alpha} + \delta_\alpha^+ y_{\bar{x}_\alpha}\right),$$
$$y_{(1)} = y\left(\bar{x}_1, x_2\right) = y + \delta_1^+ y_{x_1} + \delta_1^- y_{\bar{x}_1}, \qquad y_{(2)} = y\left(x_1, \bar{x}_2\right) = y + \delta_2^+ y_{x_2} + \delta_2^- y_{\bar{x}_2}.$$

Approximation error of difference scheme (52)–(53) has the second order of smallness with respect to $|h| = \left(h_1^2 + h_2^2\right)^{1/2}$, $h_\alpha = \max_{i_\alpha} h_\alpha^{i_\alpha}$, $\alpha = 1, 2$, i.e. there exists a constant M_1 independent of h_1, h_2 such that

$$\|\Psi\|_{C(\widehat{\omega}_h)} \leq M_1 |h|^2.$$

For simplicity we consider the case of $h_\alpha^{i_\alpha+1} - h_\alpha^{i_\alpha} > 0$, $\delta_\alpha^- = 0$, $\alpha = 1, 2$. Then difference scheme (52)–(53) takes a simpler form

$$\kappa_1 \left(y + \frac{h_{2+} - h_2}{3} y_{x_2}\right)_{\bar{x}_1 \hat{x}_1} + \kappa_2 \left(y + \frac{h_{1+} - h_1}{3} y_{x_1}\right)_{\bar{x}_2 \hat{x}_2} + \bar{r}_1^+ \left(y + \frac{h_{2+} - h_2}{3} y_{x_2}\right)_{x_1}$$
$$+ \bar{r}_1^- \left(y + \frac{h_{2+} - h_2}{3} y_{x_2}\right)_{\bar{x}_1} + \bar{r}_2^+ \left(y + \frac{h_{1+} - h_1}{3} y_{x_1}\right)_{x_2} + \bar{r}_2^- \left(y + \frac{h_{1+} - h_1}{3} y_{x_1}\right)_{\bar{x}_2}$$
$$- \bar{q} \left(y + \frac{h_{1+} - h_1}{3} y_{\bar{x}_1} + \frac{h_{2+} - h_2}{3} y_{\bar{x}_2}\right) = -\bar{f}, \quad x \in \hat{\omega}_h, \qquad y(x) = \mu(x), \quad x \in \gamma_h. \tag{54}$$

We rewrite difference scheme (54) in the canonical form (3) and check conditions (4)–(5). As a result we obtain that the positivity conditions of all coefficients are satisfied, if

$$\frac{h_{1+} - h_1}{\hbar_1} < \frac{3\kappa_1}{2\kappa_2 + \bar{r}_2^+ h_2 - \bar{r}_2^- h_{2+}} \left(\frac{\hbar_2}{\hbar_1}\right)^2, \tag{55}$$

$$\frac{h_{2+} - h_2}{\hbar_2} < \frac{3\kappa_2}{2\kappa_1 + \bar{r}_1^+ h_1 - \bar{r}_1^- h_{1+}} \left(\frac{\hbar_1}{\hbar_2}\right)^2, \tag{56}$$

i.e. difference scheme (54) is monotone. According to Theorem 1 on the base of the estimate (6) we have

$$\min \left\{ \min_{x \in \gamma_h} \mu(x), \min_{x \in \widehat{\omega}_h} (\bar{f}/\bar{q}) \right\} \le y(x) \le \max \left\{ \max_{x \in \gamma_h} \mu(x), \max_{x \in \widehat{\omega}_h} (\bar{f}/\bar{q}) \right\}, \quad x \in \widehat{\bar{\omega}}_h.$$

Then the difference scheme (54) is stable with respect to the right-hand side and the boundary conditions and the following a priori estimate in the norm C holds

$$\|y\|_{C(\widehat{\bar{\omega}}_h)} \le \max \left\{ \|\mu\|_{C(\gamma_h)}, \|\bar{f}/\bar{q}\|_{C(\widehat{\omega}_h)} \right\}.$$

Restrictions (55)–(56) are due to the fact that already does not exist a point $\bar{x}^* = (\bar{x}_1^*, \bar{x}_2^*)$, such that the relation

$$\Delta u(\bar{x}^*) - u_{\bar{x}_1 \hat{x}_1} - u_{\bar{x}_2 \hat{x}_2} = O(|h|^2),$$

is fulfilled. However, they are considerably weaker than in the case of quasi-uniform grids [23], where the following conditions are required

$$h_{\alpha+} - h_\alpha = O(\hbar_\alpha^2), \quad \alpha = 1, 2.$$

6. Numerical experiments

6.1. The case of non-uniform grid for the quasilinear parabolic equation

We consider the problem (17)–(19) with input data

$$l = 2\pi, \qquad k(u) = u^2,$$
$$f(x,t) = e^t \sin\frac{x}{4} - \frac{1}{8} e^{3t} \cos^2\frac{x}{4} \sin\frac{x}{4} + \frac{1}{16} e^{3t} \sin^3\frac{x}{4},$$
$$u_0(x) = \sin\frac{x}{4}, \qquad \mu_1(t) = 0, \qquad \mu_2(t) = e^t,$$

and the exact solution $u(x,t) = e^t \sin(x/4)$. To verify the efficiency of the new algorithms on non-uniform grids we compare the maximum norm of the error of the method

$$\|z\|_C = \|y - u\|_C = \max_{(x,t) \in \omega} |y(x,t) - u(x,t)|,$$

for difference scheme (20) approximating problem (17)–(19) and the well-known conservative scheme of first order of approximation

$$y_t = (a(y) \widehat{y}_{\bar{x}})_{\widehat{x}} + f(x, \hat{t}), \quad a(y) = 0.5\,[k(y) + k(y_-)]. \tag{57}$$

In Table 1 the starting non-uniform spatial nodes are shown. The numerical results are shown in Table 2. Increasing the number of the nodes of the grid is realized by halving each segment by the law $x_{2i} = (0.375 + r)\, x_{i+1} + (0.625 - r)\, x_i$, where $r \in [0, 0.25)$—is random variable of the normal distribution.

The computational experiment illustrates the higher accuracy of the new scheme on coarse space grids. For the scheme (20) the accuracy of order $O(h^2 + \tau)$ is reached on the coarse grids, in the scheme (57) only for sufficiently small h, τ.

Table 1
Starting non-uniform spatial nodes ($N = 10$).

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
0	0.2	1.6	2.1	2.8	3.2	4.8	5	5.4	5.9	2π

Table 2
Numerical results on non-uniform spatial grids for Eqs. (17)–(19) (FDS—finite difference scheme).

$N, h_{\max}, \tau = 0.025$	$10, 1.6, \tau$		$20, 0.916, \tau/4$		$40, 0.484, \tau/16$	
FDS	(20)	(57)	(20)	(57)	(20)	(57)
$\|z\|_C$	0.02	0.083	0.00465	0.051	0.00146	0.0279

Table 3
Numerical results on non-uniform spatial grids for Eqs. (35)–(36).

$N, h_{\max}$	$\|z\|_C$
10, 1.6	0.468
20, 0.93	0.3
40, 0.495	0.0835
80, 0.271	0.022
160, 0.149	0.0058

6.2. *The case of the non-uniform grid for the stationary linear convection–diffusion equation*

We consider the problem (35)–(36) with input data

$$k(x) = e^x, \qquad f(x) = \left(4e^x + x^2\right)\cos 2x + 2\left(e^x + \sin x\right)\sin 2x,$$
$$l = 2\pi, \qquad r(x) = \sin x, \qquad d(x) = x^2, \qquad \mu_1 = \mu_2 = 1,$$

and the exact solution $u(x) = \cos 2x$. We use starting space grids given in Table 1. In Table 3 we show the error of the method in maximum norm

$$\|z\|_C = \|y - u\|_C = \max_{(x,t)\in\omega} |y(x) - u(x)|,$$

for the difference scheme (39).

The numerical experiments illustrate the higher accuracy of the new scheme and the order of accuracy $O\left(h^2\right)$ is reached on coarse grids.

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