Rashba spin-orbit interaction in quantum ring with confining potential of finite depth

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We present the exact wave functions and energy levels for an electron in a two-dimensional quantum ring with confining potential of finite depth in the presence of the Rashba spin-orbit interaction.

Keywords: Quantum ring; Rashba spin-orbit interaction; Exact wave functions

1. Introduction

Experimental and theoretical investigations of the spin-orbit coupling in semiconductor nanostructures have attached considerable attention.

The Rashba spin-orbit interaction in law-dimensional semiconductor structures has been the object of many investigations in recent years. The Rashba interaction is of the form [1, 2]

$$V_R = \beta_R(\sigma_x p_y - \sigma_y p_x) \tag{1}$$

with standard Pauli spin-matrices σ_x and σ_y. The Rashba interaction can be strong in semiconductor heterostructures and its strength can be controlled by an external electric field.

Recently, advanced growth techniques have made it possible to fabricate high quality semiconductor rings. The circular quantum rings (nanorings) in semiconductors can be described as effectively two-dimensional systems in confining potential $V_c(\rho)$ ($\rho = \sqrt{x^2 + y^2}$). The parabolic and infinite hard wall confining potentials are the most commonly used approximations for the quantum rings [3]. As it was noticed in [4], these models do not allow us to study the tunneling effects and do not permit the existence of unbounded states. In paper [4], a more realistic model which corresponds to a quantum ring with a potential well of finite depth $V = constant$ was proposed:

$$V_c(\rho) = \begin{cases} V, & 0 < \rho < \rho_i, \\ 0, & \rho_i < \rho < \rho_o, \\ V, & \rho_o < \rho < \infty, \end{cases} \tag{2}$$

where ρ_i and ρ_o are the inner and outer radii of the ring, respectively. However, this potential was not used in the presence of the Rashba spin-orbit interaction. For example, in paper [5], the Rashba interaction was considered for the infinite hard wall. In this model $V = \infty$.

In the present paper, we examine a model which corresponds to the Rashba interaction (1) and the confining potential (2). We shall present the exact coordinate wave functions and energy levels for this model.

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2. Analytical solutions of the Schrödinger equation

The single-electron wave functions satisfy the Schrödinger equation

$$\left(\frac{\mathbf{p}^2}{2\mu}+V_c(\rho)+V_R\right)\Psi=E\Psi, \tag{3}$$

where μ is the effective electron mass.

The Schrödinger equation is considered in the cylindrical coordinates $x=\rho\cos\varphi, y=\rho\sin\varphi$. Further it is convenient to employ dimensionless quantities

$$\begin{aligned} e &= \frac{2\mu}{\hbar^2}\rho_o^2E, \quad v=\frac{2\mu}{\hbar^2}\rho_o^2V, \quad \beta=\frac{2\mu}{\hbar}\rho_o\beta_R, \\ r &= \frac{\rho}{\rho_o}, \quad r_i=\frac{\rho_i}{\rho_o}. \end{aligned} \tag{4}$$

Note that $1-r_i=(\rho_o-\rho_i)/\rho_o$ is a relative width of a ring.

As it was shown in [6, 7], Eq. (3) permits the separation of variables

$$\begin{aligned} \Psi_m(r,\varphi) &= u(r)e^{im\varphi}\begin{pmatrix}1\\0\end{pmatrix}+w(r)e^{i(m+1)\varphi}\begin{pmatrix}0\\1\end{pmatrix}, \\ m &= 0,\pm1,\pm2,\ldots \end{aligned} \tag{5}$$

due to conservation of the total angular momentum $L_z+\frac{\hbar}{2}\sigma_z$.

In the examined model, we look for the radial wave functions $u(r)$ and $w(r)$ regular at the origin $r=0$ and decreasing at infinity $r\to\infty$.

We consider three regions $0<r<r_i$ (region 1), $r_i<r<1$ (region 2) and $1<r<\infty$ (region 3) separately.

In the regions 1 and 3 ($v>0$) we have the following radial equations

$$\begin{aligned} r^2\frac{d^2u}{dr^2} &+ r\frac{du}{dr}-(k_o^+k_o^-r^2+m^2)u=-i(k_o^+-k_o^-)r^2\left(\frac{dw}{dr}+\frac{m+1}{r}w\right), \\ r^2\frac{d^2w}{dr^2} &+ r\frac{dw}{dr}-(k_o^+k_o^-r^2+(m+1)^2)w=i(k_o^+-k_0^-)r^2\left(\frac{du}{dr}-\frac{m}{r}u\right), \end{aligned} \tag{6}$$

where

$$k_o^\pm(e,v,\beta)=\sqrt{v-e-\frac{\beta^2}{4}}\pm i\frac{\beta}{2}. \tag{7}$$

Note that in the case $\beta=0$ ($k_o^+=k_o^-$) Eqs. (6) are the modified Bessel equations. In the regions 1 and 2 we must select the different particular solutions of Eqs. (6) in order to reproduce the correct behavior of the wave functions at $r\to0$ and at $r\to\infty$.

In the region 1 using the known properties [8]

$$\begin{aligned} \left(\frac{d}{dr}+\frac{n}{r}\right)I_n(kr) &= kI_{n-1}(kr), \\ \left(\frac{d}{dr}-\frac{n}{r}\right)I_n(kr) &= kI_{n+1}(kr) \end{aligned} \tag{8}$$

of the modified Bessel functions $I_n(kr)$ it is easily to get the exact solutions

$$\begin{aligned} u_1(r) &= c_1 f_1(m,r) + d_1 g_1(m,r), \\ w_1(r) &= -c_1 g_1(m+1,r) + d_1 f_1(m+1,r) \end{aligned} \tag{9}$$

of system (6) where

$$\begin{aligned} f_1(m,r) &= \frac{1}{2}\left(I_m(k_o^- r) + I_m(k_o^+ r)\right), \\ g_1(m,r) &= \frac{i}{2}\left(I_m(k_o^- r) - I_m(k_o^+ r)\right) \end{aligned} \tag{10}$$

are the real linear combinations of the modified Bessel functions with complex arguments. Here c_1 and d_1 are arbitrary coefficients. The radial wave functions $u_1(r)$ and $w_1(r)$ have the desirable behavior at the origin.

In the region 3 using the known properties [8]

$$\begin{aligned} \left(\frac{d}{dr} + \frac{n}{r}\right) K_n(kr) &= -kK_{n-1}(kr), \\ \left(\frac{d}{dr} - \frac{n}{r}\right) K_n(kr) &= -kK_{n+1}(kr) \end{aligned} \tag{11}$$

of the modified Bessel functions $K_n(kr)$ it is simply to derive the exact solutions

$$\begin{aligned} u_3(r) &= c_3 f_3(m,r) + d_3 g_3(m,r), \\ w_3(r) &= c_3 g_3(m+1,r) - d_3 f_3(m+1,r) \end{aligned} \tag{12}$$

of system (6) where

$$\begin{aligned} f_3(m,r) &= \frac{1}{2}\left(K_m(k_o^- r) + K_m(k_o^+ r)\right), \\ g_3(m,r) &= \frac{i}{2}\left(K_m(k_o^- r) - K_m(k_o^+ r)\right). \end{aligned} \tag{13}$$

Here c_3 and d_3 are arbitrary coefficients. At large values of r the functions $f_3(m,r)$ and $g_3(m,r)$ behave as

$$\begin{aligned} f_3(m,r) &\sim \sqrt{\frac{\pi}{2}}\frac{e^{-r\sqrt{v-e-\beta^2/4}}}{(v-e)^{1/4}\sqrt{r}}\cos\left(\frac{\beta r+\gamma}{2}\right), \\ g_3(m,r) &\sim -\sqrt{\frac{\pi}{2}}\frac{e^{-r\sqrt{v-e-\beta^2/4}}}{(v-e)^{1/4}\sqrt{r}}\sin\left(\frac{\beta r+\gamma}{2}\right), \end{aligned} \tag{14}$$

where γ is defined as follows

$$\cos(\gamma) = \frac{\sqrt{v-e-\beta^2/4}}{\sqrt{v-e}}, \quad \sin(\gamma) = \frac{\beta}{2\sqrt{v-e}}. \tag{15}$$

Thus, the radial wave functions $u_3(r)$ and $w_3(r)$ have the appropriate behavior at infinity. It should be noted that the radial wave functions have the infinite number of zeros for the finite value of energy.

In the region 2 ($v = 0$) the radial equations may be written in the suitable form

$$\begin{aligned} r^2\frac{d^2u}{dr^2} &+ r\frac{du}{dr} + (k_i^+k_i^-r^2 - m^2)u = (k_i^+ - k_i^-)r^2\left(\frac{dw}{dr} + \frac{m+1}{r}w\right), \\ r^2\frac{d^2w}{dr^2} &+ r\frac{dw}{dr} + (k_i^+k_i^-r^2 - (m+1)^2)w = -(k_i^+ - k_i^-)r^2\left(\frac{du}{dr} - \frac{m}{r}u\right), \end{aligned} \tag{16}$$

where

$$k_i^{\pm}(e,\beta) = \sqrt{e + \frac{\beta^2}{4}} \pm \frac{\beta}{2}. \tag{17}$$

Now in the case $\beta = 0$ ($k_i^+ = k_i^-$) Eqs. (16) are the Bessel equations. Therefore we use the known properties [8]

$$\begin{aligned} \left(\frac{d}{dr} + \frac{n}{r}\right) J_n(kr) &= kJ_{n-1}(kr), \\ \left(\frac{d}{dr} - \frac{n}{r}\right) J_n(kr) &= -kJ_{n+1}(kr), \end{aligned} \tag{18}$$

$$\begin{aligned} \left(\frac{d}{dr} + \frac{n}{r}\right) Y_n(kr) &= kY_{n-1}(kr), \\ \left(\frac{d}{dr} - \frac{n}{r}\right) Y_n(kr) &= -kY_{n+1}(kr) \end{aligned} \tag{19}$$

of the Bessel functions in order to obtain the exact solutions

$$\begin{aligned} u_2(r) &= c_{21}f_{21}(m,r) + d_{21}g_{21}(m,r) + c_{22}f_{22}(m,r) + d_{22}g_{22}(m,r), \\ w_2(r) &= c_{21}g_{21}(m+1,r) + d_{21}f_{21}(m+1,r) + c_{22}g_{22}(m+1,r) + d_{22}f_{22}(m+1,r) \end{aligned} \tag{20}$$

of system (16) where

$$\begin{aligned} f_{21}(m,r) &= \frac{1}{2}\left(J_m(k_i^-r) + J_m(k_i^+r)\right), \\ g_{21}(m,r) &= \frac{1}{2}\left(J_m(k_i^-r) - J_m(k_i^+r)\right), \end{aligned} \tag{21}$$

$$\begin{aligned} f_{22}(m,r) &= \frac{1}{2}\left(Y_m(k_i^-r) + Y_m(k_i^+r)\right), \\ g_{22}(m,r) &= \frac{1}{2}\left(Y_m(k_i^-r) - Y_m(k_i^+r)\right). \end{aligned} \tag{22}$$

Here c_{21}, d_{21}, c_{22} and d_{22} are arbitrary coefficients.

The continuity conditions

$$\begin{aligned} u_1(r_i) &= u_2(r_i), \quad u_1'(r_i) = u_2'(r_i), \\ w_1(r_i) &= w_2(r_i), \quad w_1'(r_i) = w_2'(r_i), \\ u_2(1) &= u_3(1), \quad u_2'(1) = u_3'(1), \\ w_2(1) &= w_3(1), \quad w_2'(1) = w_3'(1) \end{aligned} \tag{23}$$

for the radial wave functions and their derivatives at the boundary points $r = r_i$ and $r = 1$ lead to the algebraic equations

$$M(m,e,v,\beta)\mathbf{X} = 0 \tag{24}$$

for eight coefficients where

$$\mathbf{X} = (c_1, d_1, c_{21}, d_{21}, c_{22}, d_{22}, c_3, d_3), \tag{25}$$

and $M(m,e,v,\beta)$ is 8×8 matrix

$$M(m,e,v,\beta) = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}, \tag{26}$$

$$M_1 = \begin{pmatrix} f_1(m,r_i) & g_1(m,r_i) & -f_{21}(m,r_i) & -g_{21}(m,r_i) \\ f_1'(m,r_i) & g_1'(m,r_i) & -f_{21}'(m,r_i) & -g_{21}'(m,r_i) \\ -g_1(m+1,r_i) & f_1(m+1,r_i) & -g_{21}(m+1,r_i) & -f_{21}(m+1,r_i) \\ -g_1'(m+1,r_i) & f_1'(m+1,r_i) & -g_{21}'(m+1,r_i) & -f_{21}'(m+1,r_i) \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -f_{22}(m,r_i) & -g_{22}(m,r_i) & 0 & 0 \\ -f_{22}'(m,r_i) & -g_{22}'(m,r_i) & 0 & 0 \\ -g_{22}(m+1,r_i) & -f_{22}(m+1,r_i) & 0 & 0 \\ -g_{22}'(m+1,r_i) & -f_{22}'(m+1,r_i) & 0 & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & f_{21}(m,1) & g_{21}(m,1) \\ 0 & 0 & f_{21}'(m,1) & g_{21}'(m,1) \\ 0 & 0 & g_{21}(m+1,1) & f_{21}(m+1,1) \\ 0 & 0 & g_{21}'(m+1,1) & f_{21}'(m+1,1) \end{pmatrix},$$

$$M_4 = \begin{pmatrix} f_{22}(m,1) & g_{22}(m,1) & -f_3(m,1) & -g_3(m,1) \\ f_{22}'(m,1) & g_{22}'(m,1) & -f_3'(m,1) & -g_3'(m,1) \\ g_{22}(m+1,1) & f_{22}(m+1,1) & -g_3(m+1,1) & f_3(m+1,1) \\ g_{22}'(m+1,1) & f_{22}'(m+1,1) & -g_3'(m+1,1) & f_3'(m+1,1) \end{pmatrix}.$$

Hence, the exact equation for energy spectrum is

$$\det M(m,e,v,\beta) = 0. \tag{27}$$

It should be stressed that in the explored model the number of admissible energy levels is finite for the fixed angular momentum. Note that Eq. (27) is invariant under two replacements $m \to -(m+1)$ and $\beta \to -\beta$.

If the energy values are found from Eq. (27) then it is simply to get the values of coefficients c_1, d_1, c_{21}, d_{21}, c_{22}, d_{22}, c_3 and d_3 from Eq. (24) and the following normalization condition

$$\int_0^\infty \left(u^2(r) + w^2(r)\right) r dr = 1.$$

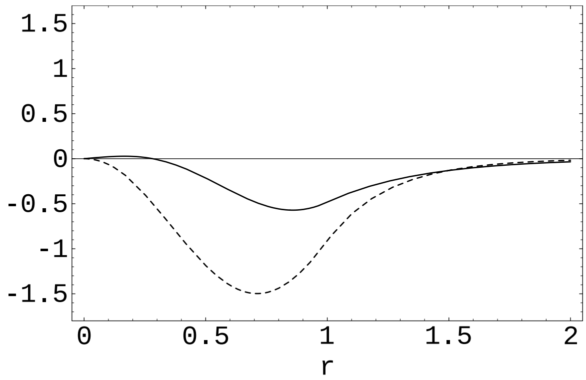

FIG. 1. Radial wave functions for $m = 1, v = 25, \beta = 1$,$r_i = 0.2$. Solid line for $u(r)$, dashed line for $w(r)$.

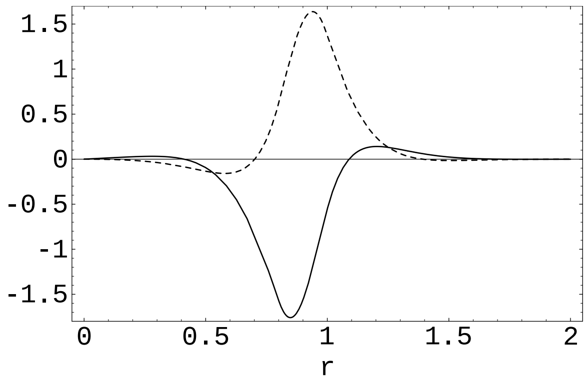

FIG. 2. Radial wave functions for $m = 1, v = 100, \beta = 10$,$r_i = 0.8$. Solid line for $u(r)$, dashed line for $w(r)$.

3. Numerical and graphic illustrations

Now we present some numerical and graphic illustrations in addition to the analytic results.

Figures 1 and 2 demonstrate the continuous radial wave functions for $m = 1, e = 17.88591, \beta = 1, r_i = 0.2$ in the case $v = 25$ and for $m = 1, e = 21.4541, \beta = 10, r_i = 0.8$ in the case $v = 100$, respectively. Solid lines correspond to the functions $u(r)$ and dashed lines correspond to the functions $w(r)$.

Tables 1 and 2 show the dependence of energy e on the ring width parameter r_i and the Rashba parameter β for two values of angular momentum number m and two values of the well depth v. We see that the number of energy levels increases if the well depth v increases.

4. Conclusion

In our opinion the examined exactly solvable model with the realistic potential well of finite depth is physically adequate in order to describe the behavior of electron in a semiconductor finite width quantum ring with account of the Rashba spin-orbit interaction. Further we intend to generalize our consideration by including the magnetic field effects on the orbital motion of electron in a quantum ring.

Table 1. Energy levels for $v = 25$

r_i	β	e		
		$m = 0$		
0.2	0	5.58	10.10	22.69
0.2	1	5.10	10.07	22.24
0.2	5	−2.42	3.61	16.27
0.5	0	10.62	13.09	
0.5	1	10.15	13.05	
0.5	5	2.97	7.39	
0.8	0	19.90	21.64	
0.8	1	19.45	21.59	
0.8	5	12.82	15.27	
		$m = 1$		
0.2	0	10.10	17.47	
0.2	1	9.18	17.86	
0.2	5	−2.29	9.59	
0.5	0	13.09	18.76	
0.5	1	12.18	19.14	
0.5	5	1.37	13.05	
0.8	0	21.64		
0.8	1	20.78		
0.8	5	11.25	18.18	

Table 2. Energy levels for $v = 100$

r_i	β	e					
		$m = 0$					
0.2	0	8.72	12.77	36.62	43.42	79.52	88.94
0.2	2	6.97	12.47	34.95	43.08	77.90	88.50
0.2	10	−17.01	−14.43	8.23	20.60	53.15	64.08
0.5	0	19.23	21.22	71.61	74.30		
0.5	2	17.54	20.91	69.98	73.91		
0.5	10	−7.93	−2.62	45.67	50.05		
0.8	0	54.30	55.61				
0.8	2	52.66	55.24				
0.8	10	26.57	32.94				
		$m = 1$					
0.2	0	12.77	21.74	43.42	57.91	88.94	
0.2	2	9.56	22.71	40.43	58.84	86.19	
0.2	10	−21.43	−12.83	9.39	36.34	62.39	
0.5	0	21.22	27.03	74.30	81.32		
0.5	2	18.15	28.05	71.39	82.15		
0.5	10	−12.64	2.34	45.78	59.81		
0.8	0	55.61	59.49				
0.8	2	52.69	60.38				
0.8	10	21.45	40.12				

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