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APPLICATION OF GEOMETRICAL METHODS TO STUDY THE SPIN 1 PARTICLE WITH ELECTRIC QUADRUPOLE MOMENT IN THE COULOMB FIELD

The quantum-mechanical problem for a spin 1 particles with electric quadrupole moment in the presence of external Coulomb field was studied in [1]. The system of two second-order differential equations for two radial functions of spin 1 particle with electric quardupole moment was obtained in non-relativistic approximation from a relativistic Duffing-Kemmer like equation by diagonalization of the operators of particle energy and total angular momentum and of the spatial inversion operator. In this study we apply Kosambi–Cartan–Chern geometrical approach (KCC-theory) to investigate this system. KCC-theory was developed in detail in many mathematical books and papers [2]. At that, a system of differential equations of second order is considered:

$$\dot{y}^i(r) + 2Q^i(r, x, y) = 0. \tag{1}$$

In (1), the symbol x^i designates coordinates, their derivatives in argument r are $y^i = dx^i/dr = \dot{x}^i$, and quantities Q_i are determined through some Lagrangian L as

$$Q^i = \frac{1}{4} g^{il}\left(\frac{\partial^2 L}{\partial x^k \partial y^l} y^k - \frac{\partial L}{\partial x^l} + \frac{\partial^2 L}{\partial y^l \partial r}\right), \qquad g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}. \tag{2}$$

The first and second invariants, $\varepsilon^i(r, x, y)$ and P^i_j are introduced by the definitions

$$\varepsilon^i = \frac{\partial Q^i}{\partial y^j} y^j - 2Q^i, \qquad P^i_j = 2\frac{\partial Q^i}{\partial x^j} + 2Q^s \frac{\partial^2 Q^i}{\partial y^j \partial y^s} - \frac{\partial^2 Q^i}{\partial y^j \partial x^s} y^s - \frac{\partial Q^i}{\partial y^s}\frac{\partial Q^s}{\partial y^j} - \frac{\partial^2 Q^i}{\partial y^j \partial r}. \tag{3}$$

We start from the system of two second-order differential equations for two radial functions of spin 1 particle with electric quardupole moment in the external Coulomb field:

$$\begin{aligned}
&\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2m\frac{\alpha + Er}{r} - \frac{2\nu^2}{r^2} - \frac{2\Gamma}{r^3} - \frac{\Gamma^2}{r^4}\right)\Psi_1(r) - \nu\frac{2r+\Gamma}{r^3}\Psi_2(r) = 0,\\
&\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2m\frac{\alpha + Er}{r} - \frac{2\nu^2}{r^2} - \frac{2}{r^2}\right)\Psi_2(r) - 2\nu\frac{2r+\Gamma}{r^3}\Psi_1(r) = 0.
\end{aligned} \tag{4}$$

We will apply the notations $x^i = \Psi_i(r),\ y^i = (d/dr)\Psi_i(r) = \dot{\Psi}_i(r)$. Comparing equations (4) and (1), one finds the quantities Q^i:

$$\begin{aligned}
Q^1(r, \Psi_i, \dot{\Psi}_i) &= \left(Em + \frac{\alpha m}{r} - \frac{\Gamma^2}{2r^4} - \frac{\Gamma}{r^3} - \frac{\nu^2}{r^2}\right)\Psi_1 - \nu\frac{(\Gamma+2r)}{2r^3}\Psi_2 + \frac{1}{r}\dot{\Psi}_1,\\
Q^2(r, \Psi_i, \dot{\Psi}_i) &= \left(Em + \frac{\alpha m}{r} - \frac{\nu^2}{r^2} - \frac{1}{r^2}\right)\Psi_2 - \nu\frac{(\Gamma+2r)}{r^3}\Psi_1 + \frac{1}{r}\dot{\Psi}_2.
\end{aligned} \tag{5}$$

Direct calculations according the formula (3) give the first and second KCC-invariants:

$$\begin{aligned}
\varepsilon^1 &= \Psi_1\left(-2Em - \frac{2\alpha m}{r} + \frac{\Gamma^2}{r^4} + \frac{2\Gamma}{r^3} + \frac{2\nu^2}{r^2}\right) + \nu\Psi_2\left(\frac{\Gamma}{r^3} + \frac{2}{r^2}\right) - \frac{\dot{\Psi}_1}{r},\\
\varepsilon^2 &= 2\Psi_2\left(-Em - \frac{\alpha m}{r} + \frac{\nu^2+1}{r^2}\right) + 2\nu\Psi_1\left(\frac{\Gamma}{r^3} + \frac{2}{r^2}\right) - \frac{\dot{\Psi}_2}{r};
\end{aligned} \tag{6}$$

$$
P_j^i = \begin{vmatrix} -\dfrac{\Gamma^2}{r^4} - \dfrac{2\Gamma}{r^3} - \dfrac{2\nu^2}{r^2} + 2Em + \dfrac{2m\alpha}{r} & -\dfrac{\nu(2r+\Gamma)}{r^3} \\ -\dfrac{2\nu(2r+\Gamma)}{r^3} & 2Em + \dfrac{2\alpha m}{r} - \dfrac{2\left(\nu^2+1\right)}{r^2} \end{vmatrix}. \tag{7}
$$

The eigenvalues Λ_1, Λ_2 of the second invariant are given by the formulas

$$
\Lambda_{1,2} = 2Em + \frac{2\alpha m}{r} - \frac{\Gamma^2}{2r^4} - \frac{\Gamma}{r^3} - \frac{2\nu^2+1}{r^2} \pm \frac{\sqrt{\left(\Gamma^2 - 2r^2 + 2\Gamma r\right)^2 + 8\nu^2 r^2 (\Gamma + 2r)^2}}{2r^4}. \tag{8}
$$

In Fig. 1 the dependencies of eigenvalues Λ_1, Λ_2 at different j are shown.

(a) (b) (c)

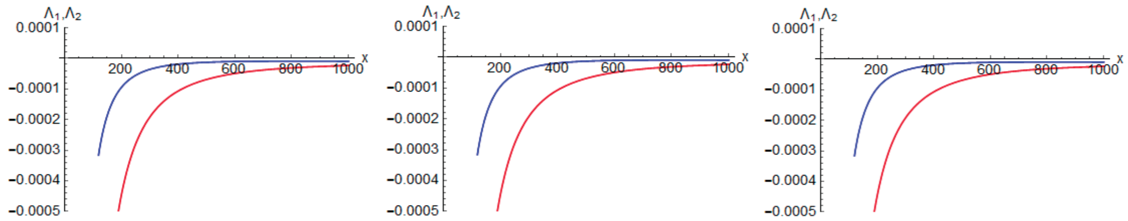

Figure 1. The dependencies of eigenvalues Λ_1 (red) and Λ_1 (blue) on radial coordinate ($x = mr$) at different j: (a) $j=1$, (b) $j=2$, (c) $j=3$. We used the following parameters: $\Gamma m = 1$, $E/m = -0.000009$.

Let study the behavior of the eigenvalues Λ^i near the singular points $r=0$, $r=\infty$. It was found out that

$$
r \to 0, \Lambda^1 \to -\frac{2}{r^2} < 0, \Lambda^2 \to -\frac{\Gamma^2}{r^4} < 0; \qquad r \to \infty, \Lambda^1, \Lambda^2 \to 2Em < 0. \tag{9}
$$

Since the real parts of all eigenvalues of the 2-nd KCC-invariant are negative, the different branches of the solution converges near the singular points $r = 0, \infty$. This correlates with behavior of solutions near the points $r = 0, \infty$ for quantum mechanical states (discrete spectra).

We constructed a Lagrangian function L for the phase space $\dot{\Psi}_i, \Psi_i$, defined by (5). The function has been found in the form

$$
\begin{gathered}
L = 2r^2(y^1)^2 + r^2(y^2)^2 + 4x^1y^1\left(\frac{2}{3}Emr^3 + \alpha mr^2 + \frac{\Gamma^2}{r} - 2\Gamma\ln r - 2\nu^2 r\right) + \\
+\frac{2}{3}rx^2y^2\left(mr(3\alpha + 2Er) - 6\left(\nu^2+1\right)\right) - 4\nu\left(x^2y^1 + x^1y^2\right)(\Gamma\ln r + 2r) + \\
+y^1\frac{\partial\varphi}{\partial x^1} + y^2\frac{\partial\varphi}{\partial x^2}, \quad \varphi = \varphi(x^1, x^2),
\end{gathered} \tag{10}
$$

where φ is some arbitrary scalar function. So, there exist some freedom in choosing the Lagrangian.

Therefore, we apply the geometrical KCC-based method to study the quantum-mechanical problem of spin 1 particle with electric quadrupole moment in the external Coulomb field. The first and the second invariants were calculated. It has been shown that the different branches of the solution converges near the singular points $r = \infty$, $r = 0$. The Lagrangian corresponding to the geometrical problem has been found. It has been shown that the Lagrangian posesses the arbitrariness up to some special term, which may be considered as specific gauge freedom.

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