Vector particle with electric quadrupole moment in external Coulomb field

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Abstract. We study the problem of vector particles with electric quadrupole moment in presence of an external Coulomb field. Starting with the relativistic Duffin-Kemmer theory, we search for solutions. To this aim, we diagonalize the operators of energy, square of the total angular momentum and its third projection. After separating the variables, we derive a system of 10 radial equations. According to the requirement of diagonalizing the spatial reflection operator, we split the system into two subsystems of 4 and 6 equations respectively, for states with parities $P = (-1)^{j+1}$ and $P = (-1)^j$. Additional interaction terms enter both subsystems. The relativistic radial subsystem of 4 equations reduces to a second order equation which contains two singular points $x = 0$ and $x = \infty$ of ranks 3 and 2, respectively, and four regular points. The local Frobenius solutions near the point $x = 0$ are constructed. It is shown that there exist 8-term recurrence formulas for the involved power series. The condition of transcendency of solutions gives a certain quantization rule for energy levels, which seems to be only partially physically appropriate. The relativistic radial system of 6 equations for states with parity $P = (-1)^j$, turns out to be very complicated. In order to simplify the problem, we perform the transition to the non-relativistic approximation, and consequently derive two associated second order differential equations for two radial functions. We obtain the 4-th order equations for radial functions, and construct four different Frobenius type solutions of these equations. As well, the convergence of the involved power series with 8- and 9-terms recurrence relations, is studied. The transcendency condition gives the formula for energies, which does not depend on the quantum number and on the parameter of quadrupole electric moment, and therefore cannot describe the physical spectrum correctly. The non-relativistic analysis is performed for states with the parity $P = (-1)^j$ as well, but the radial equation for the main function turns out to have more simple structure of singular points. The transcendency condition leads to a formula for energies which only partially correlates with the relativistic one. All the constructed solutions are exact, but they are formal because there exists

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no reliable rule for quantization of energy levels, and the transcendency condition solves this difficulty only partially.

We additionally apply the geometrical method based on the theory of KCC-invariants. The first and the second invariants are calculated, and it is shown that the distinct branches of the solutions converge near the singular points $r = 0, \infty, -\Gamma/2$. This correlates with the expected behavior of solutions for bound states. Within this framework, the explicit Lagrangians related to the geometrical problem are determined. It is shown that the Lagrangians have an arbitrariness degree up to certain terms, which may be considered as a specific gauge freedom.

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1 Introduction. The initial relativistic equation

It is known that in the framework of relativistic wave equation theory, one can propose the so-called *non-minimal equations*, which describe particles with additional electromagnetic characteristics, with spectra of spin and mass states [15]–[18]. Within this approach, in [13]–[11] the problem of spin 1 particles with additional anomalous magnetic and quadrupole electric moments has been investigated. The equations were studied and solved for particles in external homogenous electric and magnetic fields. The equation for a vector particle in the external Coulomb field is rather complicated, even in the case of ordinary particles without additional electromagnetic moments, and has not been completely solved yet. However, at the non-relativistic limit, the equation for ordinary vector particles in the Coulomb field can be solved exactly. In the present work we study the non-relativistic problem of a vector particle with additional quadrupole moment in the external Coulomb field.

The initial equation has the following form (we shall further use the conventional tetrad formalism [16])

$$\left\{ i\beta^c \left[i(e^{\beta}_{(c)}\partial_\beta + \frac{1}{2} j^{ab}\gamma_{abc}(x)) \ - e' A_c \right] + \lambda \frac{e}{M} F_{\alpha\beta}(x) \bar{P} j^{\alpha\beta}(x) - M \right\} \Psi = 0\,; \tag{1.1}$$

related to quadrupole moment free dimensionless parameter λ, $\bar{P}$ is a projective operator separating inside the 10-component wave function its tensor component. We use the notations:

$$\bar{P} = \left| \begin{array}{cc} 0 & 0 \\ 0 & I_6 \end{array} \right| , \quad M = \frac{mc}{\hbar}, \quad e' = \frac{e}{c\hbar}, \quad \Gamma = \lambda \frac{4\alpha}{M}, \quad \alpha = \frac{e^2}{\hbar c}\,. \tag{1.2}$$

With respect to the spherical tetrad [17], equation (1.1) has the form

$$\left[\beta^0 (i\partial_t + \frac{\alpha}{r}) + i \left(\beta^3 \, \partial_r + \frac{1}{r}(\beta^1 j^{31} + \beta^2 j^{32}) \right) + \frac{1}{r}\Sigma_{\theta,\phi} + \frac{\Gamma}{r^2}\bar{P} j^{03} - M \right] \Phi(x) = 0\,,$$

where, depending on the angular variables, the operator $\Sigma_{\theta,\phi}$ is determined by the equality

$$\Sigma_{\theta,\phi} = i\, \beta^1 \partial_\theta + \beta^2 \frac{i\partial_\phi + i j^{12}\cos\theta}{\sin\theta}\,. \tag{1.3}$$

The components of the operator of the total angular momentum are given [17] relative to this basis, by the formulas

$$j_1 = l_1 + \frac{\cos\phi}{\sin\theta} ij^{12}, \quad j_2 = l_2 + \frac{\sin\phi}{\sin\theta} ij^{12}, \quad j_3 = l_3\,, \quad j^{12} = \beta^1\beta^2 - \beta^2\beta^1. \tag{1.4}$$

We shall further use the cyclic basis - the Duffin-Kemmer matrices [17]:

$$\beta^0 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \quad \beta^3 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\beta^1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & 0 & 0 & -i & 0 & +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \quad \beta^2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +i & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +i & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & +i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

The matrix $ij^{12} = i(\beta^1\beta^2 - \beta^2\beta^1)$ has a diagonal structure:

$$ij^{12} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & t_3 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_3 \end{vmatrix}, \quad t_3 = \begin{vmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}.$$

We can express the 10×10 - matrices in terms of the the cyclic basis, by using the similarity transformation

$$\Psi_{cycl} = S\Psi_{Cart}, \quad S = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & U & 0 & 0 \\ 0 & 0 & U & 0 \\ 0 & 0 & 0 & U \end{vmatrix}, \quad U = \begin{vmatrix} -1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ +1/\sqrt{2} & i/\sqrt{2} & 0 \end{vmatrix}, \tag{1.5}$$

$$\beta^c_{cycl} = S\beta^c_{cart}S^{-1}, \quad \bar{P}_{cycl} = S\bar{P}_{cart}S^{-1} \equiv \bar{P}_{Cart}\,.$$

We note that the form of the projective operator $\bar{P}$ does not change. All the forecoming formulas will be expressed in terms of the cyclic basis.

The system of radial equations for the ordinary vector particle in Coulomb field is known [6]. To get a similar system for the particle with quadrupole moment, it suffices to find the explicit form of the additional term in the equation:

$$\frac{\Gamma}{r^2}\bar{P}j^{03} = \frac{\Gamma}{r^2}\bar{P}(\beta^0\beta^3 - \beta^3\beta^0)\,, \tag{1.6}$$

where

$$\beta^0\beta^3 - \beta^3\beta^0 = \begin{vmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +i \\ 0 & 0 & 0 & 0 & +i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \end{vmatrix}, \quad \bar{P}\,j^{03} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +i \\ 0 & 0 & 0 & 0 & +i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \end{vmatrix}.$$

2 Separating the variables in the relativistic equation

The most general form of the 10-component wave function with quantum numbers ϵ, j, m is the following:

$$\Psi(x) = \{\Psi_0(x), \vec{\Psi}(x), \vec{E}(x), \vec{H}(x)\}, \Psi_0(x) = e^{-i\epsilon t} f_0(r) D_0, \vec{\Psi}(x) = e^{-i\epsilon t} \left| \begin{array}{c} f_1(r)D_{-1} \\ f_2(r)D_0 \\ f_3(r)D_{+1} \end{array} \right| ,$$

(2.1)

$$\vec{E}(x) = e^{-i\epsilon t} \left| \begin{array}{c} E_1(r)D_{-1} \\ E_2(r)D_0 \\ E_3(r)D_{+1} \end{array} \right| , \quad \vec{H}(x) = e^{-i\epsilon t} \left| \begin{array}{c} H_1(r)D_{-1} \\ H_2(r)D_0 \\ H_3(r)D_{+1} \end{array} \right| ,$$

where we use the Wigner D-functions [17]: $D_\sigma = D^j_{-m,\sigma}(\phi, \theta, 0)\,, \quad \sigma = 0, -1, +1.$

We start with the known system of radial equations for the ordinary vector particle [6][1]:

$$-(\frac{d}{dr} + \frac{2}{r})\; E_2 - \frac{\nu}{r}(E_1 + E_3) = m f_0\,, +i(\epsilon + \frac{\alpha}{r})E_1 + i(\frac{d}{dr} + \frac{1}{r})H_1 + i\frac{\nu}{r}H_2 = m f_1,$$

$$+i(\epsilon + \frac{\alpha}{r})E_2 - i\frac{\nu}{r}(H_1 - H_3) = m f_2, +i(\epsilon + \frac{\alpha}{r})E_3 - i(\frac{d}{dr} + \frac{1}{r})H_3 - i\frac{\nu}{r}H_2 = m f_3\,;$$

(2.2)

$$-i(\epsilon + \frac{\alpha}{r})\Phi_1 \;+ \frac{\nu}{r}\Phi_0 = m E_1, \quad -i(\epsilon + \frac{\alpha}{r})f_2 - \frac{d}{dr}f_0 = m E_2,$$

$$-i(\epsilon + \frac{\alpha}{r})f_3 + \frac{\nu}{r}f_0 = m E_3, \quad -i(\frac{d}{dr} + \frac{1}{r})f_1 - i\frac{\nu}{r}f_2 = m H_1,$$

$$+i\frac{\nu}{r}(f_1 - f_3) = m H_2, \quad +i(\frac{d}{dr} + \frac{1}{r})f_3 + i\frac{\nu}{r}f_2 = m H_3\,.$$

and accounting for the explicit form of the additional term in the equation[2]:

$$\frac{\Gamma}{r^2}\,\bar{P}j^{03}\,\Psi = \frac{\Gamma}{r^2} \left| \begin{array}{cccccccccc} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&-i&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&+i \\ 0&0&0&0&+i&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-i&0&0&0 \end{array} \right| \left| \begin{array}{c} f_0D_0 \\ f_1D_{-1} \\ f_2D_0 \\ f_3D_{+1} \\ E_1D_{-1} \\ E_2D_0 \\ E_3D_{+1} \\ H_1D_{-1} \\ H_2D_0 \\ H_3D_{+1} \end{array} \right| = \frac{\Gamma}{r^2} \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ -iH_1D_{-1} \\ 0 \\ iH_3D_{+1} \\ iE_1D_{-1} \\ 0 \\ -iE_3D_{+1} \end{array} \right| ,$$

[1]We denote here $\nu = \sqrt{j(j+1)}/\sqrt{2}$.
[2]Here, the multiplier $e^{-i\epsilon t}$ is omitted.

we derive the following system

$$
\begin{aligned}
&-(\frac{d}{dr}+\frac{2}{r})\, E_2 - \frac{\nu}{r}\,(E_1+E_3) = m\, f_0\,,\\
&+i\,(\epsilon+\frac{\alpha}{r})\, E_1 \;\; + i\,(\frac{d}{dr}+\frac{1}{r})\, H_1 + i\frac{\nu}{r}\, H_2 = m f_1\,,\\
&+i\,(\epsilon+\frac{\alpha}{r})\, E_2 - i\frac{\nu}{r}\,(H_1-H_3) = m\, f_2\,,\\
&+i\,(\epsilon+\frac{\alpha}{r})\, E_3 \;\; - i\,(\frac{d}{dr}+\frac{1}{r})\, H_3 - i\frac{\nu}{r}\, H_2 = m\, f_3\,;
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
&-i\,(\epsilon+\frac{\alpha}{r})\, f_1 \;\; + \frac{\nu}{r}\, f_0 - i\frac{\Gamma}{r^2} H_1 = m\, E_1\,,\\
&-i\,(\epsilon+\frac{\alpha}{r})\, f_2 \;\; - \frac{d}{dr}\, f_0 = m\, E_2\,,\; -i\,(\epsilon+\frac{\alpha}{r})\, f_3 \;\; + \frac{\nu}{r}\, f_0 + i\frac{\Gamma}{r^2} H_3 = m\, E_3\,,
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
&-i\,(\frac{d}{dr}+\frac{1}{r})\, f_1 - i\frac{\nu}{r}\, f_2 + i\frac{\Gamma}{r^2} E_1 = m\, H_1\,,\; + i\frac{\nu}{r}\,(f_1-f_3) = m\, H_2\,,\\
&+i\,(\frac{d}{dr}+\frac{1}{r})\, f_3 + i\frac{\nu}{r} f_2 - i\frac{\Gamma}{r^2} E_3 = m\, H_3\,.
\end{aligned} \tag{2.5}
$$

Besides the operators $\vec{j}^{\,2}, j_3$, we diagonalize the operator of spatial reflection $\hat{\Pi}$. In terms of the canonic Cartesian matrix basis β^a, this operator has an ordinary form:

$$
\hat{\Pi} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & +I \end{array} \right| \hat{P}\,, \quad \hat{P}\Psi(\vec{r}) = \Psi(-\vec{r})\,. \tag{2.6}
$$

After transiting this to the spherical tetrad and cyclic representation of the matrix β^a, we obtain

$$
\hat{\Pi}' = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \Pi_3 & 0 & 0 \\ 0 & 0 & \Pi_3 & 0 \\ 0 & 0 & 0 & -\Pi_3 \end{array} \right| \hat{P}\,, \quad \Pi_3 = \left| \begin{array}{ccc} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{array} \right| . \tag{2.7}
$$

The spectral equation $\hat{\Pi}'\Psi = P\Psi$ (accounting for the known property $\hat{P}D_\sigma = (-1)^j D_{-\sigma}$) yields the set of algebraic equations:

$$
\begin{aligned}
&(-1)^j f_0 = P f_0\,,\;\; (-1)^j f_3 = P f_1\,,\;\; (-1)^j f_2 = P f_2\,,\;\; (-1)^j f_1 = P f_3\,,\\
&(-1)^j E_3 = P E_1\,,\;\; (-1)^j E_2 = P E_2\,,\;\; (-1)^j E_1 = P E_3\,,\\
&(-1)^j H_3 = -P H_1\,,\;\; (-1)^j H_2 = -P H_2\,,\;\; (-1)^j H_1 = -P H_3\,.
\end{aligned} \tag{2.8}
$$

This system has two solutions:

$$
P = (-1)^{j+1}, \quad f_0 = 0,\; f_3 = -f_1,\; f_2 = 0, \quad E_3 = -E_1,\; E_2 = 0,\; H_3 = H_1\,; \tag{2.9}
$$

$$P=(-1)^{j}, \quad f_3=+f_1, \quad E_3=+E_1,\; H_3=-H_1,\; H_2=0\,. \tag{2.10}$$

It easy to check that these restrictions are compatible with the above radial equations.

For the parity $P=(-1)^{j+1}$, we get

$$f_0=0,\; f_3=-f_1,\; f_2=0,\; E_3=-E_1\,,_2=0, H_3=H_1,$$
$$0=0, \quad +i(\epsilon+\frac{\alpha}{r})E_1+i(\frac{d}{dr}+\frac{1}{r})H_1+i\frac{\nu}{r}H_2=mf_1,$$
$$0=0, \quad -i(\epsilon+\frac{\alpha}{r})E_1-i(\frac{d}{dr}+\frac{1}{r})H_3-i\frac{\nu}{r}H_2=-mf_1;$$
$$-i(\epsilon+\frac{\alpha}{r})f_1-i\frac{\Gamma}{r^2}H_1=mE_1, \quad 0=0, \quad i(\epsilon+\frac{\alpha}{r})f_1+i\frac{\Gamma}{r^2}H_1=-mE_1,$$
$$-i(\frac{d}{dr}+\frac{1}{r})f_1+i\frac{\Gamma}{r^2}E_1=mH_1, +2\,i\frac{\nu}{r}f_1=mH_2, \quad -i(\frac{d}{dr}+\frac{1}{r})f_1+i\frac{\Gamma}{r^2}E_1=mH_1;$$

so we have only four equations:

$$\begin{aligned} &+i\left(\epsilon+\tfrac{\alpha}{r}\right)E_1+i\left(\tfrac{d}{dr}+\tfrac{1}{r}\right)H_1+i\tfrac{\nu}{r}H_2=mf_1,\\ &-i\,\left(\epsilon+\tfrac{\alpha}{r}\right)\,f_1\;-i\tfrac{\Gamma}{r^2}H_1=mE_1,\\ &-i\,\left(\tfrac{d}{dr}+\tfrac{1}{r}\right)\,f_1+i\tfrac{\Gamma}{r^2}E_1=mH_1,\\ &2i\tfrac{\nu}{r}f_1=mH_2\,. \end{aligned} \tag{2.11}$$

We exclude the variables H_1, H_2 from the first and second equations:

$$\begin{aligned} &+i\left(\epsilon+\frac{\alpha}{r}\right)mE_1+i\left(\frac{d}{dr}+\frac{1}{r}\right)\left[-i\,\left(\frac{d}{dr}+\frac{1}{r}\right)\,f_1+i\frac{\Gamma}{r^2}E_1\right]-\frac{2\nu^2}{r^2}f_1=m^2f_1\,,\\ &-i\,\left(\epsilon+\frac{\alpha}{r}\right)\,mf_1\;-i\frac{\Gamma}{r^2}\left[-i\,\left(\frac{d}{dr}+\frac{1}{r}\right)\,f_1+i\frac{\Gamma}{r^2}E_1\right]=m^2E_1\,. \end{aligned} \tag{2.12}$$

The second equation allows to express E_1 in terms of f_1:

$$-i\,\left(\epsilon+\frac{\alpha}{r}\right)\,mf_1\;-\frac{\Gamma}{r^2}\left(\frac{d}{dr}+\frac{1}{r}\right)\,f_1=(m^2-\frac{\Gamma^2}{r^4})E_1\,; \tag{2.13}$$

and consequently the function E_1 can be excluded. Hence we obtain a 2-nd order equation for the main function f_1:

$$\begin{aligned} &\frac{d^2f_1}{dr^2}+\left[-\frac{2\,r\,m}{mr^2+\Gamma}-\frac{2\,r\,m}{mr^2-\Gamma}+\frac{6}{r}\right]\frac{df_1}{dr}\\ &+\left[\frac{2\,\epsilon\,\alpha}{r}+\frac{-2\,\nu^2+\alpha^2+4}{r^2}+\frac{2\,i\Gamma\epsilon}{mr^3}+\frac{\Gamma\,(\Gamma m+i\alpha)}{mr^4}+\frac{2\,\nu^2\Gamma^2}{m^2r^6}\right.\\ &\left.-m^2+\epsilon^2+\frac{2\,m\,(i\epsilon\,r+i\alpha-1)}{mr^2+\Gamma}-\frac{2\,m\,(i\epsilon\,r+i\alpha+1)}{mr^2-\Gamma}\right]f_1=0\,. \end{aligned} \tag{2.14}$$

This has a singular singular point $r=0$ of rank 3, a singular point $r=\infty$ of rank 2, and four regular points which are determined by the roots of the equation, $(r^2-\Gamma/m)(r^2+\Gamma/m)=0\,.$

Let us consider the states with the other parity:

$$P=(-1)^j,\quad f_3=+f_1\,,\quad E_3=+E_1\,,\ H_3=-H_1\,,\ H_2=0\,; \tag{2.15}$$

under the corresponding restrictions, the radial system takes the form

$$
\begin{gathered}
-(\frac{d}{dr}+\frac{2}{r})\,E_2-2\,\frac{\nu}{r}\,E_1=m\,f_0\,,\quad +i\,(\epsilon+\frac{\alpha}{r})\,E_1\ \ +i\,(\frac{d}{dr}+\frac{1}{r})\,H_1=mf_1\,,\\
+i\,(\epsilon+\frac{\alpha}{r})\,E_2-2\,i\frac{\nu}{r}\,H_1=m\,f_2\,,\quad +i\,(\epsilon+\frac{\alpha}{r})\,E_1\ +i\,(\frac{d}{dr}+\frac{1}{r})\,H_1=m\,f_1\,;\\
-i\,(\epsilon+\frac{\alpha}{r})\,f_1\ \ +\frac{\nu}{r}\,f_0-i\frac{\Gamma}{r^2}H_1=m\,E_1\,,\quad -i\,(\epsilon+\frac{\alpha}{r})\,f_2\ \ -\ \frac{d}{dr}\,f_0=m\,E_2\,,\\
-i\,(\epsilon+\frac{\alpha}{r})\,f_1\ \ +\frac{\nu}{r}\,f_0-i\frac{\Gamma}{r^2}H_1=m\,E_1\,,\quad -i\,(\frac{d}{dr}+\frac{1}{r})\,f_1-i\frac{\nu}{r}\,f_2+i\frac{\Gamma}{r^2}E_1=m\,H_1\,,\\
0=0\,,\quad +i\,(\frac{d}{dr}+\frac{1}{r})\,f_1+i\frac{\nu}{r}f_2-i\frac{\Gamma}{r^2}E_1=-m\,H_1\,;
\end{gathered}
$$

and hence we obtain six different equations:

$$
\begin{gathered}
-\left(\frac{d}{dr}+\frac{2}{r}\right)E_2-2\frac{\nu}{r}E_1=mf_0,\quad +i\left(\epsilon+\frac{\alpha}{r}\right)E_1+i\left(\frac{d}{dr}+\frac{1}{r}\right)H_1=mf_1,\\
+i\left(\epsilon+\frac{\alpha}{r}\right)E_2-2i\frac{\nu}{r}H_1=mf_2,\quad -i\left(\epsilon+\frac{\alpha}{r}\right)f_1+\frac{\nu}{r}f_0-i\frac{\Gamma}{r^2}H_1=mE_1\,,\\
-i\left(\epsilon+\frac{\alpha}{r}\right)f_2-\frac{d}{dr}f_0=mE_2,\quad +i\left(\frac{d}{dr}+\frac{1}{r}\right)f_1+i\frac{\nu}{r}f_2-i\frac{\Gamma}{r^2}E_1=-mH_1\,.
\end{gathered} \tag{2.16}
$$

3 States with parity $P=(-1)^{j+1}$

In eqs. (2.14) we introduce the following new (dimensionless) variables $x=mr,\ E=\frac{\epsilon}{m},\ \gamma=m\Gamma$, in order to obtain a simpler representation

$$
\begin{gathered}
\frac{d^2f_1}{dx^2}+\left[-\frac{2\,x}{x^2+\gamma}-\frac{2x}{x^2-\gamma}+\frac{6}{x}\right]\frac{df_1}{dx}\\
+\left[(E^2-1)+\frac{2E\alpha}{x}+\frac{-2\nu^2+\alpha^2+4}{x^2}+\frac{2i\gamma E}{x^3}+\frac{\gamma(\gamma+i\alpha)}{x^4}+\frac{2\nu^2\gamma^2}{x^6}\right.\\
\left.+\frac{2(iEx+i\alpha-1)}{x^2+\gamma}-\frac{2(iEx+i\alpha+1)}{x^2-\gamma}\right]f_1=0\,.
\end{gathered} \tag{3.1}
$$

Here we have the singular points

$$x=0\,,\mathrm{Rank}=3\,,\quad x=\infty\,,\mathrm{Rank}=2\,,\quad x=-\sqrt{\gamma}\,,\ +\sqrt{\gamma}\,,\ -i\sqrt{\gamma}\,,\ +i\sqrt{\gamma},\mathrm{Rank}=1\,.$$

The equation is suitable to be represented in the symbolic form:

$$
\begin{gathered}
\frac{d^2f_1}{dx^2}+\left[-\frac{2\,x}{x^2+\gamma}-\frac{2x}{x^2-\gamma}+\frac{6}{x}\right]\frac{df_1}{dx}\\
+\left[E^2-1+\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\frac{a_4}{x^4}+\frac{a_6}{x^6}+\frac{L}{x^2+\gamma}+\frac{N}{x^2-\gamma}\right]f_1=0\,.
\end{gathered} \tag{3.2}
$$

We construct its Frobenius type solutions near the point $x = 0$ in the form $f_1(x) = e^{Dx} x^A e^{B/x} e^{C/x^2} F(x)$:

$$\frac{d^2F}{dx^2} + \left[-\frac{2\,x}{x^2+\gamma} - \frac{2\,x}{x^2-\gamma} + \frac{2\,A+6}{x} - \frac{2\,B}{x^2} - \frac{4\,C}{x^3} + 2\,D \right] \frac{dF}{dx}$$
$$+ \left[(E^2 + D^2 - 1) + \frac{2\,AD + 6\,D + a_1}{x} + \frac{A^2 + 5\,A - 2\,BD + a_2}{x^2} \right.$$
$$+ \frac{-2\,AB - 4\,B - 4\,CD + a_3}{x^3} + \frac{-4\,AC + B^2 - 6\,C + a_4}{x^4} + \frac{4\,BC}{x^5} + \frac{4\,C^2 + a_6}{x^6}$$
$$\left. + \frac{-2\,A\gamma - 2\,Bx - 4\,C - 2\,D\gamma x + L\gamma}{\gamma\,(x^2+\gamma)} + \frac{-2\,A\gamma + 2\,Bx + 4\,C - 2\,D\gamma x + N\gamma}{\gamma\,(x^2-\gamma)} \right] F = 0\,.$$

We further impose the restrictions:

(3.3)
$$E^2 + D^2 - 1 = 0 \quad \Rightarrow \quad D = -\sqrt{1-E^2}\,, +\sqrt{1-E^2}\,,$$
$$4\,C^2 + a_6 = 0 \quad \Rightarrow \quad C = \frac{\delta}{2}\sqrt{-a_6} = \delta\frac{1}{2}\sqrt{2\nu^2(-\gamma^2)}\,, \quad \delta = \pm 1\,,$$
$$4\,BC = 0 \quad \Longrightarrow \quad B = 0\,,$$
$$-4\,AC + B^2 - 6\,C + a_4 \quad \Rightarrow \quad A = -\frac{3}{2} + \frac{1}{4}\frac{a_4}{C} = -\frac{3}{2} + \frac{\delta}{2}\frac{\gamma^2 + i\gamma\alpha}{\sqrt{2\nu^2(-\gamma^2)}}\,.$$

On physical grounds, the parameter γ has to be imaginary, so we should make a substitution $i\gamma \Longrightarrow \gamma$. To have regular behavior at $x = 0$, the parameter C should be taken negative. To describe the bound states, we will use following expression for the parameters:

(3.4)
$$A = -\tfrac{3}{2} - \tfrac{1}{2}\,\tfrac{\gamma^2 + i\gamma\alpha}{\sqrt{2\nu^2(-\gamma^2)}}\,,\ B = 0\,,$$
$$C = -\tfrac{1}{2}\sqrt{2\nu^2(-\gamma^2)} < 0\,,\ D = -\sqrt{1-E^2}\,.$$

The equation gets simplified:

$$\frac{d^2F}{dx^2} + \left[-\frac{2\,x}{x^2+\gamma} - \frac{2\,x}{x^2-\gamma} + \frac{2\,A+6}{x} - \frac{2\,B}{x^2} - \frac{4\,C}{x^3} + 2\,D \right] \frac{dF}{dx}$$
$$+ \left[\frac{2\,AD + 6\,D + a_1}{x} + \frac{A^2 + 5\,A - 2\,BD + a_2}{x^2} + \frac{-2\,AB - 4\,B - 4\,CD + a_3}{x^3} \right.$$
$$\left. + \frac{-2\,A\gamma - 2\,Bx - 4\,C - 2\,D\gamma x + L\gamma}{\gamma\,(x^2+\gamma)} + \frac{-2\,A\gamma + 2\,Bx + 4\,C - 2\,D\gamma x + N\gamma}{\gamma\,(x^2-\gamma)} \right] F = 0\,,$$

and we can construct the solutions for $F(x)$ as power series: $F = \sum_{n=0}^{\infty} d_n x^n$, which leads to the recurrence formulas:

$$k = 0\,, \quad 4\,C\,d_1 + (2\,AB + 4\,B + 4\,CD - a_3)\;d_0 = 0\,,$$

$$k = 1\,, \quad 2B\,d_1 + 8\,C\,d_2 + \left(-A^2 - 5\,A + 2\,BD - a_2\right)\,d_0 + (2\,AB + 4\,B + 4\,CD - a_3)\;d_1 = 0\,,$$

$$k=2\,,\quad -2\;(3+A)\;d_1+2B\,2d_2+12\,C\,d_3$$
$$+\,(-2AD-6D-a_1)\,d_0+\big(-A^2-5A+2\,BD-a_2\big)\,d_1+(2AB+4B+4CD-a_3)\,d_2=0\,,$$
$$k=3\qquad -2\gamma^2 d_2-2D\gamma^2 d_1-4\gamma^2\,(3+A)\,d_2+6B\gamma^2 d_3+16C\gamma^2 d_4$$
$$+\,(8C-L\gamma+N\gamma)\,d_0+\gamma^2\,(-2AD-6D-a_1)\,d_1$$
$$+\gamma^2\left(-A^2-5A+2BD-a_2\right)d_2+\gamma^2\,(2AB+4\,B+4CD-a_3)\,d_3=0\,,$$
$$k=4\,,\quad -6\gamma^2 d_3-4Cd_1-4D\gamma^2 d_2-6\gamma^2\,(3+A)\,d_3+8B\gamma^2 d_4+20C\gamma^2 d_5$$
$$+\,(-2AB-4CD+a_3)\,d_0+(8C-L\gamma+N\gamma)\,d_1+\gamma^2\,(-2AD-6\,D-a_1)\,d_2$$
$$+\gamma^2\left(-A^2-5A+2BD-a_2\right)d_3+\gamma^2\,(2AB+4B+4CD-a_3)\,d_4=0\,,$$
$$k=5,\ -12\gamma^2 d_4-2Bd_1-8Cd_2-6D\gamma^2 d_3-8\gamma^2\,(3+A)\;d_4+10B\gamma^2 d_5+24C\gamma^2 d_6$$
$$+\left(A^2+A-2BD+L+N+a_2\right)d_0+(-2AB-4CD+a_3)\,d_1+(8C-L\gamma+N\gamma)\,d_2$$
$$+\gamma^2(-2AD-6D-a_1)d_3+\gamma^2(-A^2-5A+2BD-a_2)d_4+\gamma^2(2AB+4B+4CD-a_3)d_5=0,$$
$$k=6,-20\gamma^2 d_5+(2+2A)\,d_1-4Bd_2-12Cd_3-8D\gamma^2 d_4-10\gamma^2\,(3+A)\,d_5+12B\gamma^2 d_6+28\,C\gamma^2 d_7$$
$$+\,(2\,AD+2\,D+a_1)\;d_0+\left(A^2+A-2\,BD+L+N+a_2\right)\,d_1$$
$$+\,(-2AB-4CD+a_3)\,d_2+(8C-L\gamma+N\gamma)\,d_3+\gamma^2\,(-2AD-6D-a_1)\,d_4$$
$$+\gamma^2\left(-A^2-5\,A+2\,BD-a_2\right)\;d_5+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\;d_6=0\,,$$
$$k=7\,,\quad 2d_2-30\gamma^2\,d_6+2D\,d_1+2\;(2+2\,A)\;d_2-6B\,d_3-16C\,d_4$$
$$-10D\gamma^2\,d_5-12\,\gamma^2\,(3+A)\;d_6+14B\gamma^2\,d_7+32\,C\gamma^2\,d_8$$
$$(2\,AD+2\,D+a_1)\;d_1+\left(A^2+A-2\,BD+L+N+a_2\right)\;d_2$$
$$+\,(-2\,AB-4\,CD+a_3)\;d_3+(8\,C-L\gamma+N\gamma)\;d_4+\gamma^2\,(-2\,AD-6\,D-a_1)\;d_5$$
$$+\gamma^2\left(-A^2-5\,A+2\,BD-a_2\right)\;d_6+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\;d_7=0\,.$$

Thus, we find the following 8-term recurrence relations:

$$\underline{k=6,7,8,9,...}\qquad [2D\,(k-6)+(2\,AD+2\,D+a_1)]\;d_{k-6}$$
$$+\left[(k-5)(k-6)+(2+2\,A)\;(k-5)+\left(A^2+A-2\,BD+L+N+a_2\right)\right]\;d_{k-5}$$
$$+\,[-2B\,(k-4)+(-2\,AB-4\,CD+a_3)]\;d_{k-4}$$
$$+\,[-4C\,(k-3)+(8\,C-L\gamma+N\gamma)]\;d_{k-3}+\left[-2D\gamma^2\,(k-2)+\gamma^2\,(-2\,AD-6\,D-a_1)\right]\;d_{k-2}$$
$$+\left[-\gamma^2\,(k-1)(k-2)-2\,\gamma^2\,(3+A)\;(k-1)+\gamma^2\left(-A^2-5\,A+2\,BD-a_2\right)\right]\;d_{k-1}$$
$$+\left[2B\gamma^2\,k+\gamma^2\,(2\,AB+4\,B+4\,CD-a_3)\right]\;d_k+4\,C\gamma^2\,(k+1)d_{k+1}=0\,.$$

in brief, these relations can be written as

$$P_{k-6}c_{k-6}+P_{k-5}c_{k-5}+P_{k-4}c_{k-4}+P_{k-3}c_{k-3}$$
$$+P_{k-2}c_{k-2}+P_{k-1}c_{k-1}+P_k c_k+P_{k+1}c_{k+1}=0\,. \tag{3.5}$$

By applying the Poincaré-Perrone method, we shall further analyze the convergence radius of the power series. To do this, we divide the last relation by $d_{k-6}k^2$

$$
\begin{aligned}
&[2D\,(k-6) + (2\,AD + 2\,D + a_1)] + \\
&+ \big[(k-5)(k-6) + (2+2\,A)\;(k-5) + \big(A^2 + A - 2\,BD + L + N + a_2\big)\big]\;\frac{d_{k-5}}{d_{k-6}} \\
&+ [-2B\,(k-4) + (-2\,AB - 4\,CD + a_3)]\;\frac{d_{k-4}}{d_{k-5}}\,\frac{d_{k-5}}{d_{k-6}} \\
&+ [-4C\,(k-3) + (8\,C - L\gamma + N\gamma)]\;\frac{d_{k-3}}{d_{k-4}}\,\frac{d_{k-4}}{d_{k-5}}\,\frac{d_{k-5}}{d_{k-6}} \\
&+ \big[-2D\gamma^2\,(k-2) + \gamma^2\,(-2\,AD - 6\,D - a_1)\big]\;\frac{d_{k-2}}{d_{k-3}}\,\frac{d_{k-3}}{d_{k-4}}\,\frac{d_{k-4}}{d_{k-5}}\,\frac{d_{k-5}}{d_{k-6}} \\
&+ \big[-\gamma^2(k-1)(k-2) - 2\gamma^2\,(3+A)\,(k-1) + \gamma^2\,\big(-A^2 - 5A + 2BD - a_2\big)\big] \\
&\times \frac{d_{k-1}}{d_{k-2}}\,\frac{d_{k-2}}{d_{k-3}}\,\frac{d_{k-3}}{d_{k-4}}\,\frac{d_{k-4}}{d_{k-5}}\,\frac{d_{k-5}}{d_{k-6}} \\
&+ \big[2B\gamma^2\,k + \gamma^2\,(2\,AB + 4\,B + 4\,CD - a_3)\big]\;\frac{d_k}{d_{k-1}}\,\frac{d_{k-1}}{d_{k-2}}\,\frac{d_{k-2}}{d_{k-3}}\,\frac{d_{k-3}}{d_{k-4}}\,\frac{d_{k-4}}{d_{k-5}}\,\frac{d_{k-5}}{d_{k-6}} \\
&+4\,C\gamma^2\,(k+1)\;\frac{d_{k+1}}{d_k}\,\frac{d_k}{d_{k-1}}\,\frac{d_{k-1}}{d_{k-2}}\,\frac{d_{k-2}}{d_{k-3}}\,\frac{d_{k-3}}{d_{k-4}}\,\frac{d_{k-4}}{d_{k-5}}\,\frac{d_{k-5}}{d_{k-6}} = 0\,,
\end{aligned}
$$

and tend $k \to \infty$. As a result, we get the algebraic equation for R:

$$R - \gamma^2 R^5 = 0 \qquad \Rightarrow \qquad R = 0\,,\ \pm\frac{1}{\sqrt{\gamma}}\,,\ \pm\frac{1}{\sqrt{-\gamma}}\,.$$

The modulus of the parameter R determines the possible convergence radii.

$$R = \lim_{k\to\infty}\frac{d_{k-5}}{d_{k-6}}, \qquad R_{\text{conv}} = \left|\frac{1}{R}\right| = |\sqrt{\gamma}|\,,\ +\infty\,. \tag{3.6}$$

As a quantization rule we use the restrictions separating the transcendent Frobenius functions (see (3.5))

$$P_{k-6} = 0 \implies 2D\,(k-6) + 2\,AD + 2\,D + a_1 = 0\,, \quad k \ge 6\,, \tag{3.7}$$

where

$$A = -\frac{3}{2} - \frac{1}{2}\,\frac{\gamma^2 + i\gamma\alpha}{\sqrt{2\nu^2(-\gamma^2)}}\,, \quad D = -\sqrt{1-E^2}\,, \quad a_1 = 2E\alpha\,.$$

Taking into account that the parameter γ is imaginary one, we make the change $i\gamma \rightsquigarrow \gamma$, then

$$A = -\frac{3}{2} - \frac{1}{2}\,\frac{-\gamma^2 + \gamma\alpha}{\sqrt{2\nu^2\gamma^2}}\,, \quad D = -\sqrt{1-E^2}\,, \quad a_1 = 2E\alpha\,, \tag{3.8}$$

and the transcendency condition takes the form (let $k-6=n,\ n=0,1,2,...$)

$$-\sqrt{1-E^2}\,n+\left(\frac{3}{2}+\frac{\gamma\alpha-\gamma^2}{2\sqrt{l(l+1)\gamma^2}}\right)\sqrt{1-E^2}-\sqrt{1-E^2}+E\alpha=0\,,$$

or

$$\alpha E=\sqrt{1-E^2}\left(n-1/2-\frac{\gamma\alpha-\gamma^2}{2\sqrt{\gamma^2 l(l+1)}}\right). \tag{3.9}$$

Depending on the sign of parameter γ, there arise two different equations:

$$\gamma>0,\quad \alpha E=\sqrt{1-E^2}\left(n-1/2-\frac{\alpha-\gamma}{2\sqrt{l(l+1)}}\right),$$
$$\gamma<0,\quad \alpha E=\sqrt{1-E^2}\left(n-1/2+\frac{\alpha-\gamma}{2\sqrt{l(l+1)}}\right). \tag{3.10}$$

The structure of these equations is the same, so the energy spectra are similar but non-identical:

$$\alpha E=\sqrt{1-E^2}N \quad\Longrightarrow\quad E=\frac{1}{\sqrt{1+\frac{\alpha^2}{N^2}}}\,;$$
$$\gamma>0\,,\quad N=n-1/2-\frac{\alpha-\gamma}{2\sqrt{l(l+1)}}\,;\quad \gamma<0\,,\quad N=n-1/2+\frac{\alpha-\gamma}{2\sqrt{l(l+1)}}\,. \tag{3.11}$$

4 The case of minimal $j=0$

We shall further consider the case of minimal value of the total momentum $j=0$. We use the following relevant substitution for the corresponding wave function:

$$\Phi_0(x)=e^{-i\epsilon t}f_0(r),\vec{\Phi}(x)=e^{-i\epsilon t}\left|\begin{array}{c}0\\ f_2(r)\\ 0\end{array}\right|,\vec{E}(x)=e^{-i\epsilon t}\left|\begin{array}{c}0\\ E_2(r)\\ 0\end{array}\right|,\vec{H}(x)=e^{-i\epsilon t}\left|\begin{array}{c}0\\ H_2(r)\\ 0\end{array}\right|.$$

The corresponding equations emerge from the general ones, using the restrictions

$$\nu=0\,,\quad f_1=f_3=0\,,\quad E_1=E_3=0\,,\quad H_1=H_3=0\,. \tag{4.1}$$

So, we get

$$-(\frac{d}{dr}+\frac{2}{r})E_2=mf_0\,,\quad +i(\epsilon+\frac{\alpha}{r})E_2=mf_2\,,\quad 0=0\,,\quad 0=0\,,$$
$$-i(\epsilon+\frac{\alpha}{r})f_2-\frac{d}{dr}f_0=mE_2\,,\quad 0=0\,,\quad 0=0\,,\quad 0=mH_2\,,\quad 0=0\,.$$

In the case of minimal $j=0$, the electric quadrupole moment does not manifest itself:

$$-\left(\frac{d}{dr}+\frac{2}{r}\right)E_2=mf_0,\qquad i\left(\epsilon+\frac{\alpha}{r}\right)E_2=mf_2,$$
$$-i\left(\epsilon+\frac{\alpha}{r}\right)f_2-\frac{d}{dr}f_0=mE_2,\qquad H_2=0\,. \tag{4.2}$$

After excluding the variables $f_2,\ E_2$, we obtain a 2-nd order equation for the function $f_0(r)$:

$$\frac{d^2 f_0}{dr^2} + \left(\frac{4}{r} + \frac{-m+\epsilon}{mr-\epsilon\, r-\alpha} + \frac{-m-\epsilon}{mr+\epsilon\, r+\alpha}\right)\frac{df_0}{dr} + \left(\frac{2\,\epsilon\,\alpha}{r} + \frac{\alpha^2}{r^2} - m^2 + \epsilon^2\right) f_0 = 0\,,$$

or in terms of the dimensionless variables $x = mr, E = \epsilon/m$,

$$\frac{d^2 f_0}{dx^2} + \left(\frac{4}{x} + \frac{-1+E}{x-Ex-\alpha} + \frac{-1-E}{x+Ex+\alpha}\right)\frac{df_0}{dx} + \left(\frac{2\,E\alpha}{x} + \frac{\alpha^2}{x^2} + E^2 - 1\right) f_0 = 0\,.$$

This equation has three regular singular points and one singular point of rank 2 at infinity.

Alternatively, by excluding the variables $f_0,\ f_2$, we get the simpler equation for the main function E_2:

$$\frac{d^2 E_2}{dr^2} + \frac{2}{r}\frac{dE_2}{dr} + \left(\epsilon^2 - m^2 + \frac{2\,\epsilon\,\alpha}{r} + \frac{\alpha^2-2}{r^2}\right) E_2 = 0\,.$$

After transforming the last equation in terms of the variable $y = 2\sqrt{m^2-\epsilon^2} r$, we get

$$y\frac{d^2 E_2}{dy^2} + 2\,\frac{dE_2}{dy} + \left(-\frac{1}{4}\,y + \frac{\alpha^2-2}{y} + \frac{\epsilon\,\alpha}{\sqrt{m^2-\epsilon^2}}\right) E_2 = 0\,.$$

Its solutions are constructed in terms of confluent hypergeometric functions, according to the standard procedure:

$$E_2 = y^A e^{By} F(y)\,, \quad y\frac{d^2F}{dy^2} + (2\,A + 2 + 2\,By)\,\frac{dF}{dy}$$
$$+\left[\left(B^2 - \frac{1}{4}\right) y + \frac{A^2 + A + \alpha^2 - 2}{y} + 2\,AB + 2\,B + \frac{\epsilon\,\alpha}{\sqrt{m^2-\epsilon^2+}}\right] F = 0\,.$$

By imposing on the parameters the evident constraints

$$A = -\frac{1}{2} \pm \frac{1}{2}\sqrt{9-4\alpha^2}\,, \quad B = -\frac{1}{2}\,,$$

we simplify the problem, and get the confluent hypergeometric equation

$$y\frac{d^2F}{dy^2} + (2\,A + 2 - y)\,\frac{dF}{dy} + \left(-1 - A + \frac{\epsilon\,\alpha}{\sqrt{m^2-\epsilon^2}}\right), = 0$$

with the parameters

$$a = 1 + A - \frac{\epsilon\,\alpha}{\sqrt{m^2-\epsilon^2}}\,, \quad c = 2A + 2\,.$$

The quantization condition is chosen as usually:

$$a = \frac{1}{2}(1 + \sqrt{9-4\alpha^2}) - \frac{\epsilon\alpha}{\sqrt{m^2-\epsilon^2}} = -n,$$

so we get deriving the formula for energy levels

$$\epsilon = \frac{m}{\sqrt{1+\alpha^2/N^2}}, \quad N = \frac{1}{2}(1 + \sqrt{9-4\alpha^2}) + n,\ n = 0, 1, 2, ... \tag{4.3}$$

5 Nonrelativistic approximation ($P = (-1)^{j+1}, j \geq 1$)

Let us perform the non-relativistic approximation in the radial the system (2.11):

$$P = (-1)^{j+1},\ i\left(\epsilon + \frac{\alpha}{r}\right) E_1 + i\left(\frac{d}{dr} + \frac{1}{r}\right) H_1 + i\frac{\nu}{r} H_2 = m f_1,$$

(5.1)

$$-i\left(\epsilon + \frac{\alpha}{r}\right) f_1 - i\frac{\Gamma}{r^2} H_1 = m E_1, -i\left(\frac{d}{dr} + \frac{1}{r}\right) f_1 + i\frac{\Gamma}{r^2} E_1 = m H_1, 2i\frac{\nu}{r} f_1 = m H_2.$$

By using the third and fourth equations, we exclude non-dynamical variables H_1 and H_2:

$$i(\epsilon + \frac{\alpha}{r})E_1 + \frac{1}{m}(\frac{d}{dr} + \frac{1}{r})\left[(\frac{d}{dr} + \frac{1}{r})f_1 - \frac{\Gamma}{r^2}E_1\right] - \frac{2\nu^2}{mr^2} f_1 = m f_1\,,$$

$$-i(\epsilon + \frac{\alpha}{r})f_1 - \frac{\Gamma}{mr^2}\left[(\frac{d}{dr} + \frac{1}{r})f_1 - \frac{\Gamma}{r^2}E_1\right] = m E_1\,.$$

The big and the small components are introduced by the relations

$$f_1 = (\Psi_1 + \psi_1), \quad iE_1 = (\Psi_1 - \psi_1)\,, \tag{5.2}$$

and then the previous equations take the form[3]:

$$(m + E + \frac{\alpha}{r})(\Psi_1 - \psi_1) + \frac{1}{m}(\frac{d}{dr} + \frac{1}{r})\left[(\frac{d}{dr} + \frac{1}{r})(\Psi_1 + \psi_1) + \frac{i\Gamma}{r^2}(\Psi_1 - \psi_1)\right]$$

$$-\frac{2\nu^2}{mr^2}(\Psi_1 + \psi_1) = m(\Psi_1 + \psi_1)\,,$$

$$(m + E + \frac{\alpha}{r})(\Psi_1 + \psi_1) - \frac{i\Gamma}{mr^2}\left[(\frac{d}{dr} + \frac{1}{r})(\Psi_1 + \psi_1) + \frac{i\Gamma}{r^2}(\Psi_1 - \psi_1)\right] = m(\Psi_1 - \psi_1)\,.$$

By re-grouping the terms and neglecting the small component in comparison with the big one, we derive two equations[4]

$$(E + \frac{\alpha}{r})\Psi_1 + \frac{1}{m}(\frac{d}{dr} + \frac{1}{r})\left[(\frac{d}{dr} + \frac{1}{r})\Psi_1 + \frac{i\Gamma}{r^2}\Psi_1\right] - \frac{j(j+1)}{mr^2}\Psi_1 = 2m\psi_1\,,$$

$$(E + \frac{\alpha}{r})\Psi_1 - \frac{i\Gamma}{mr^2}\left[(\frac{d}{dr} + \frac{1}{r})\Psi_1 + \frac{i\Gamma}{r^2}\Psi_1\right] = -2m\psi_1\,;$$

after summing these, we find a 2-nd order equation for the big component

$$\left\{(\frac{d}{dr} + \frac{1}{r})[(\frac{d}{dr} + \frac{1}{r}) + \frac{i\Gamma}{r^2}] + 2m(E + \frac{\alpha}{r}) - \frac{j(j+1)}{r^2} - \frac{i\Gamma}{r^2}[(\frac{d}{dr} + \frac{1}{r}) + \frac{i\Gamma}{r^2}]\right\}\Psi_1 = 0\,.$$

Then, making the needed change $i\Gamma \rightsquigarrow \Gamma$, we obtain

$$\left\{(\frac{d}{dr} + \frac{1}{r})(\frac{d}{dr} + \frac{1}{r} + \frac{\Gamma}{r^2}) + 2m(E + \frac{\alpha}{r}) - \frac{j(j+1)}{r^2} - \frac{\Gamma}{r^2}(\frac{d}{dr} + \frac{1}{r} + \frac{\Gamma}{r^2})\right\}\Psi_1 = 0\,.$$

---

[3] We also separate the rest energy by the formal change $\epsilon = m + E$, where E stands for the non-relativistic energy

[4] We recall here that $2\nu^2 = j(j+1)$.

The final form of the non-relativistic radial equation is[5]:

$$\left\{ \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2m(E + \frac{\alpha}{r}) - \frac{j(j+1)}{r^2} - \frac{2\Gamma}{r^3} - \frac{\Gamma^2}{r^4} \right\} R(r) = 0\,. \tag{5.3}$$

This equation has two singular points, $r = 0$ and ∞; both of them have rank 2. Therefore, this belongs to the class of double confluent Heun functions. In the vicinity of the point $r = 0$, its Frobenius solutions are constructed in the form

$$R(r) = e^{Cr} r^A e^{\frac{B}{r}} f(r)\,, \qquad \frac{d^2 f}{dr^2} + \left(\frac{2+2\,A}{r} - \frac{2\,B}{r^2} + 2\,C \right) \frac{df}{dr}$$
$$+ \left(\frac{2AC + 2C + 2m\alpha}{r} + \frac{A^2 + A - 2BC - j^2 - j}{r^2} \right.$$
$$\left. + \frac{-2AB - 2\Gamma}{r^3} + \frac{B^2 - \Gamma^2}{r^4} + C^2 + 2mE \right) f = 0.$$

With the following evident restrictions on the parameters[6]:

$$C = -\sqrt{-2\,mE}\,; \qquad B = \Gamma\,, \quad A = -1\,; \qquad B = -\Gamma\,, \quad A = +1 \tag{5.4}$$

the equation becomes simpler. The negative values of the parameter B correspond to the bound states. Depending on the sign of Γ, there exist two different sets of parameters:

$$\begin{array}{l} \Gamma > 0, \quad A = +1, B = -\Gamma\,,\; C = -\sqrt{-2\,mE}\,; \\ \Gamma < 0, \quad A = -1, B = +\Gamma\,,\; C = -\sqrt{-2\,mE}\,. \end{array} \tag{5.5}$$

The equation for $f(r)$ is formally identical:

$$\frac{d^2 f}{dr^2} + \left(2C + \frac{2+2A}{r} - \frac{2B}{r^2} \right) \frac{df}{dr} + \left(\frac{2AC + 2C + 2m\alpha}{r} + \frac{A^2 + A - 2BC - j^2 - j}{r^2} \right) f = 0,$$

or shortly

$$\frac{d^2 f}{dr^2} + \left(a + \frac{a_1}{r} + \frac{a_2}{r^2} \right) \frac{df}{dr} + \left(\frac{b_1}{r} + \frac{b_2}{r^2} \right) f = 0\,.$$

The solutions $f(r)$ are constructed as power series: $f = \sum_{k=0}^{\infty} c_k r^k$, and we obtain the recurrence formulae:

$$k = 0\,, \quad b_2\, c_0 + a_2\, c_1 = 0\,,$$
$$k = 1\,, \quad b_1\, c_0 + (a_1 + b_2)\, c_1 + 2\, a_2\, c_2 = 0\,,$$
$$k = 2\,, \quad (a + b_1)\, c_1 + (2 + 2\, a_1 + b_2)\, c_2 + 3\, a_2\, c_3 = 0\,.$$

Therefore, the general formula for the 3-term recurrence relations is

$$k = 1, 2, 3, 4, ..., \qquad [a\,(k-1) + b_1]\, c_{k-1} + [k(k-1) + a_1\, k + b_2]\, c_k + a_2 (k+1)\, c_{k+1} = 0\,,$$

[5] We denote here $\Psi_1(r) = R(r)$.

[6] we emphasize here that C must be negative.

or shortly $P_{k-1}c_{k-1} + P_k c_k + P_{k+1}c_{k+1} = 0$, where

$$P_{k-1} = a\,(k-1) + b_1\,, \quad P_k = k(k-1) + a_1\,k + b_2\,, \quad P_{k+1} = a_2(k+1)\,.$$

In accordance with the Poincaré-Perrone method, we divide the relation by $k^2 c_{k-1}$ and tend $k \to \infty$:

$$\frac{1}{k^2}\,[a\,(k-1)+b_1] + \frac{1}{k^2}\,[k(k-1)+a_1\,k+b_2]\,\frac{c_k}{c_{k-1}} + \frac{1}{k^2}\,a_2(k+1)\,\frac{c_{k+1}}{c_k}\,\frac{c_k}{c_{k-1}} = 0\,.$$

As a result, we obtain the algebraic equation which determines the possible convergence radius:

$$r = 0 \quad \Rightarrow \quad R_{\mathbf{conv}} = \frac{1}{|r|} = \infty\,. \tag{5.6}$$

We present now the explicit form of the quantities which enter the recurrence relations:

$$P_{k-1} = 2C\,(k-1) + 2\,AC + 2\,C + 2\,m\alpha\,,$$
$$P_k = k(k-1) + (2+2\,A)\,k + A^2 + A - 2\,BC - j^2 - j\,, \quad P_{k+1} = -2B\,(k+1)\,,$$

and consider the transcendency condition for the Heun functions:

$$P_{n-1} = 0 \quad \Longrightarrow \quad C\ = -\,\frac{m\alpha}{[(k-1)+\ A+2\,]}\,, \tag{5.7}$$

where $C = -\sqrt{-2\,mE}$, and

$\Gamma > 0,\ A = +1, B = -\Gamma, C = -\sqrt{-2mE}\,; \quad \Gamma < 0, A = -1, B = +\Gamma, C = -\sqrt{-2mE}\,.$

Depending on the sign of Γ, we obtain different spectra:

$$\Gamma > 0\,,\ R(r) = e^{-\sqrt{-2mE}r} r e^{\frac{-\Gamma}{r}} f(r)\,, \quad E = -\,\frac{m\alpha^2}{2(k+2)^2}\,, \tag{5.8}$$

$$\Gamma < 0\,, R(r) = e^{-\sqrt{-2mE}r} r^{-1} e^{\frac{+\Gamma}{r}} f(r)\,, \quad E = -\,\frac{m\alpha^2}{2k^2}\,. \tag{5.9}$$

The solutions of both types, respectively at $\Gamma > 0$ and $\Gamma < 0$, can describe bound states as they tend to zero at $r \to 0$. However, the formulae for energy do not depend on Γ.

6 Nonrelativistic radial equations (case $j \geq 1$)

We start from equations (2.16):

$$-\left(\frac{d}{dr}+\frac{2}{r}\right)E_2 - 2\frac{\nu}{r}E_1 = mf_0\,, \quad +i\left(\epsilon+\frac{\alpha}{r}\right)E_1 + i\left(\frac{d}{dr}+\frac{1}{r}\right)H_1 = mf_1\,,$$
$$+i\left(\epsilon+\frac{\alpha}{r}\right)E_2 - 2i\frac{\nu}{r}\,H_1 = mf_2\,, \quad -i\left(\epsilon+\frac{\alpha}{r}\right)f_1 + \frac{\nu}{r}f_0 - i\frac{\Gamma}{r^2}H_1 = mE_1\,, \tag{6.1}$$
$$-i\left(\epsilon+\frac{\alpha}{r}\right)f_2 - \frac{d}{dr}f_0 = mE_2\,, \quad +i\,\left(\frac{d}{dr}+\frac{1}{r}\right)f_1 + i\frac{\nu}{r}f_2 - i\frac{\Gamma}{r^2}E_1 = -mH_1\,.$$

Excluding the non-dynamical variables f_0, H_1 (which do not differ in time), from the equations:

$$f_0 = -\frac{1}{m}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)E_2 + 2\frac{\nu}{r}E_1\right], \quad H_1 = -\frac{1}{m}\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)f_1 + i\frac{\nu}{r}f_2 - i\frac{\Gamma}{r^2}E_1\right].$$

we reduce the remaining equations to the form

$$+i\left(\epsilon+\frac{\alpha}{r}\right)E_1 - \frac{i}{m}\left(\frac{d}{dr}+\frac{1}{r}\right)\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)f_1 + i\frac{\nu}{r}f_2 - i\frac{\Gamma}{r^2}E_1\right] = mf_1\,,$$

$$+i\left(\epsilon+\frac{\alpha}{r}\right)E_2 + 2i\frac{\nu}{r}\frac{1}{m}\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)f_1 + i\frac{\nu}{r}f_2 - i\frac{\Gamma}{r^2}E_1\right] = mf_2\,;$$

$$-i\left(\epsilon+\frac{\alpha}{r}\right)f_1 - \frac{\nu}{r}\frac{1}{m}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)E_2 + 2\frac{\nu}{r}E_1\right]$$
$$+i\frac{\Gamma}{r^2}\frac{1}{m}\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)f_1 + i\frac{\nu}{r}f_2 - i\frac{\Gamma}{r^2}E_1\right] = mE_1\,,$$

$$-i\left(\epsilon+\frac{\alpha}{r}\right)f_2 + \frac{1}{m}\frac{d}{dr}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)E_2 + 2\frac{\nu}{r}E_1\right] = mE_2\,.$$

The big and the small components are introduced by the formulae

$$(6.2)\quad f_1 = (\Psi_1+\psi_1)\,, \quad iE_1 = (\Psi_1-\psi_1)\,, \quad f_2 = (\Psi_2+\psi_2)\,, \quad iE_2 = (\Psi_2-\psi_2)\,.$$

Then the previous equations take the form[7]

$$\left(m+E+\frac{\alpha}{r}\right)(\Psi_1-\psi_1) - \frac{i}{m}\left(\frac{d}{dr}+\frac{1}{r}\right)\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)(\Psi_1+\psi_1)\right.$$
$$\left.+i\frac{\nu}{r}(\Psi_2+\psi_2) - \frac{\Gamma}{r^2}(\Psi_1-\psi_1)\right] = m(\Psi_1+\psi_1),$$

$$\left(m+E+\frac{\alpha}{r}\right)(\Psi_1+\psi_1) - \frac{\nu}{r}\frac{1}{m}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)(\Psi_2-\psi_2) + 2\frac{\nu}{r}(\Psi_1-\psi_1)\right]$$
$$-\frac{\Gamma}{r^2}\frac{1}{m}\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)(\Psi_1+\psi_1) + i\frac{\nu}{r}(\Psi_2+\psi_2) - \frac{\Gamma}{r^2}(\Psi_1-\psi_1)\right] = m(\Psi_1-\psi_1),$$

$$\left(m+E+\frac{\alpha}{r}\right)(\Psi_2-\psi_2) + 2i\frac{\nu}{r}\frac{1}{m}\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)(\Psi_1+\psi_1)\right.$$
$$\left.+i\frac{\nu}{r}(\Psi_2+\psi_2) - \frac{\Gamma}{r^2}(\Psi_1-\psi_1)\right] = m(\Psi_2+\psi_2),$$

$$\left(m+E+\frac{\alpha}{r}\right)(\Psi_2+\psi_2) + \frac{1}{m}\frac{d}{dr}\left[\left(\frac{d}{dr}+\frac{2}{r}\right)(\Psi_2-\psi_2) + 2\frac{\nu}{r}(\Psi_1-\psi_1)\right] = m\,(\Psi_2-\psi_2).$$

By regrouping the terms, we obtain

$$\left(E+\frac{\alpha}{r}\right)(\Psi_1-\psi_1) - \frac{i}{m}\left(\frac{d}{dr}+\frac{1}{r}\right)\left[i\left(\frac{d}{dr}+\frac{1}{r}\right)(\Psi_1+\psi_1)\right.$$

[7] We also separate the rest energy by the substitution $\epsilon = m + E$.

$$+i\frac{\nu}{r}(\Psi_2+\psi_2)-\frac{\Gamma}{r^2}(\Psi_1-\psi_1)\Big]=2m\psi_1,$$

$$\Big(E+\frac{\alpha}{r}\Big)(\Psi_1+\psi_1)-\frac{\nu}{r}\frac{1}{m}\Big[\Big(\frac{d}{dr}+\frac{2}{r}\Big)(\Psi_2-\psi_2)+2\frac{\nu}{r}(\Psi_1-\psi_1)\Big]$$

$$-\frac{\Gamma}{r^2}\frac{1}{m}\Big[i\Big(\frac{d}{dr}+\frac{1}{r}\Big)(\Psi_1+\psi_1)+i\frac{\nu}{r}f_2-\frac{\Gamma}{r^2}(\Psi_1-\psi_1)\Big]=-2m\psi_1,$$

$$\Big(E+\frac{\alpha}{r}\Big)(\Psi_2-\psi_2)+2i\frac{\nu}{r}\frac{1}{m}\Big[i\Big(\frac{d}{dr}+\frac{1}{r}\Big)(\Psi_1+\psi_1)+i\frac{\nu}{r}(\Psi_2+\psi_2)-\frac{\Gamma}{r^2}(\Psi_1-\psi_1)\Big]=2m\psi_2,$$

$$\Big(E+\frac{\alpha}{r}\Big)(\Psi_2+\psi_2)+\frac{1}{m}\frac{d}{dr}\Big[\Big(\frac{d}{dr}+\frac{2}{r}\Big)(\Psi_2-\psi_2)+2\frac{\nu}{r}(\Psi_1-\psi_1)\Big]=-2m\psi_2.$$

To derive the needed equations for the big components Ψ_1 and Ψ_2, we sum the equations in each pair and then neglect the small (compared to the big ones) components. This results in

$$2\Big(E+\frac{\alpha}{r}\Big)\Psi_1+\frac{1}{m}\Big(\frac{d}{dr}+\frac{1}{r}\Big)\Big[\Big(\frac{d}{dr}+\frac{1}{r}\Big)\Psi_1+\frac{\nu}{r}\Psi_2+i\frac{\Gamma}{r^2}\Psi_1\Big]$$

$$-\frac{\nu}{r}\frac{1}{m}\Big[\Big(\frac{d}{dr}+\frac{2}{r}\Big)\Psi_2+2\frac{\nu}{r}\Psi_1\Big]-\frac{i\Gamma}{r^2}\frac{1}{m}\Big[\Big(\frac{d}{dr}+\frac{1}{r}\Big)\Psi_1+\frac{\nu}{r}\Psi_2+\frac{i\Gamma}{r^2}\Psi_1\Big]=0\,,$$

$$2\Big(E+\frac{\alpha}{r}\Big)\Psi_2-2\frac{\nu}{r}\frac{1}{m}\Big[\Big(\frac{d}{dr}+\frac{1}{r}\Big)\Psi_1+\frac{\nu}{r}\Psi_2+\frac{i\Gamma}{r^2}\Psi_1\Big]+\frac{1}{m}\frac{d}{dr}\Big[\Big(\frac{d}{dr}+\frac{2}{r}\Big)\Psi_2+2\frac{\nu}{r}\Psi_1\Big]=0.$$

Allowing that Γ is imaginary, we make the change $i\Gamma \rightsquigarrow \Gamma$, and hence produce the system

$$2m\Big(E+\frac{\alpha}{r}\Big)\Psi_1+\Big(\frac{d}{dr}+\frac{1}{r}\Big)\Big[\Big(\frac{d}{dr}+\frac{1}{r}\Big)\Psi_1+\frac{\nu}{r}\Psi_2+\frac{\Gamma}{r^2}\Psi_1\Big]$$

$$-\frac{\nu}{r}\Big[\Big(\frac{d}{dr}+\frac{2}{r}\Big)\Psi_2+2\frac{\nu}{r}\Psi_1\Big]-\frac{\Gamma}{r^2}\Big[\Big(\frac{d}{dr}+\frac{1}{r}\Big)\Psi_1+\frac{\nu}{r}\Psi_2+\frac{\Gamma}{r^2}\Psi_1\Big]=0\,,$$

$$2\,m\Big(E+\frac{\alpha}{r}\Big)\Psi_2-2\,\frac{\nu}{r}\Big[\Big(\frac{d}{dr}+\frac{1}{r}\Big)\Psi_1+\frac{\nu}{r}\Psi_2+\frac{\Gamma}{r^2}\Psi_1\Big]+\frac{d}{dr}\Big[\Big(\frac{d}{dr}+\frac{2}{r}\Big)\Psi_2+2\frac{\nu}{r}\Psi_1\Big]=0\,.$$

We further get[8]:

$$\text{(6.3)}\quad \begin{gathered}\Big(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+2mE+\frac{2m\alpha}{r}-\frac{2\,\nu^2}{r^2}-\frac{2\,\Gamma}{r^3}-\frac{\Gamma^2}{r^4}\Big)\Psi_1-\nu\,\Big(\frac{2}{r^2}+\frac{\Gamma}{r^3}\Big)\Psi_2=0\,,\\ \Big(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+2mE+\frac{2m\alpha}{r}-\frac{2\nu^2+2}{r^2}\Big)\Psi_2-2\,\nu\,\Big(\frac{2}{r^2}+\frac{\Gamma}{r^3}\Big)\Psi_1=0\,.\end{gathered}$$

This system for two functions permits us to construct the fourth order equations for the functions $\Psi_1(r)$ and $\Psi_2(r)$. It suffices to study only one of them, e.g., for the function Ψ_1. We shall use the dimensionless variables:

$$\text{(6.4)}\qquad x=rm\,,\quad \Gamma m=\gamma\,,\quad \epsilon=\frac{E}{m}\,;$$

[8] We recall here that $2\nu^2=j(j+1)$.

as a result we get

$$\frac{d^4\Psi_1}{dx^4}+\left[\frac{10}{x}-\frac{4}{2\,x+\gamma}\right]\frac{d^3\Psi_1}{dx^3}$$

$$+\left[4\,\epsilon+\frac{-24+4\,\alpha\,\gamma}{\gamma x}+\frac{22-4\,\nu^2}{x^2}-\frac{2\,\gamma}{x^3}-\frac{\gamma^2}{x^4}+\frac{48}{(2\,x+\gamma)\,\gamma}+\frac{8}{(2\,x+\gamma)^2}\right]\frac{d^2\Psi_1}{dx^2}$$

$$+\left[\frac{64-16\,\nu^2+20\,\gamma^2\epsilon-8\,\alpha\,\gamma}{\gamma^2 x}+\frac{-24+8\,\nu^2+16\,\alpha\,\gamma}{\gamma x^2}+\frac{8-12\,\nu^2}{x^3}\right.$$

$$\left.+\frac{-128-8\,\gamma^2\epsilon+16\,\alpha\,\gamma+32\,\nu^2}{(2\,x+\gamma)\,\gamma^2}-\frac{32}{(2\,x+\gamma)^2\,\gamma}\right]\frac{d\Psi_1}{dx} \tag{6.5}$$

$$+\left[+4\epsilon^2+\frac{128\nu^2+8\epsilon\alpha\gamma^3+64\alpha\gamma-32\gamma^2\epsilon}{x\gamma^3}+\frac{-24\,\alpha\gamma-48\nu^2+20\gamma^2\epsilon+4\alpha^2\gamma^2-8\epsilon\nu^2\gamma^2}{\gamma^2x^2}\right.$$

$$+\frac{8\,\alpha\,\gamma-4\,\gamma^2\epsilon+16\,\nu^2-8\,\alpha\,\nu^2\gamma}{\gamma x^3}+\frac{-8\,\nu^2-4\,\alpha\,\gamma-2\,\gamma^2\epsilon+4\,\nu^4}{x^4}$$

$$\left.-\frac{2\,\gamma\left(-2+2\nu^2+\alpha\gamma\right)}{x^5}+\frac{2\gamma^2}{x^6}+\frac{-128\alpha\gamma+64\gamma^2\epsilon-256\nu^2}{(2x+\gamma)\,\gamma^3}+\frac{-32\alpha\gamma+16\gamma^2\epsilon-64\nu^2}{(2x+\gamma)^2\,\gamma^2}\right]\Psi_1=0\,.$$

Symbolic structure of the equation (6.5) is written as follows (let $\Psi_1=\Psi$)

$$\frac{d^4}{dx^4}\Psi_1+\left[\frac{10}{x}-\frac{4}{2x+\gamma}\right]\frac{d^3}{dx^3}\Psi+\left[4\epsilon+\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\frac{a_4}{x^4}+\frac{a_5}{2x+\gamma}+\frac{a_6}{(2x+\gamma)^2}\right]\frac{d^2}{dx^2}\Psi$$

$$+\left[\frac{b_1}{x}+\frac{b_2}{x^2}+\frac{b_3}{x^3}+\frac{b_4}{2x+\gamma}+\frac{b_5}{(2x+\gamma)^2}\right]\frac{d}{dx}\Psi$$

$$+\left[4\epsilon^2+\frac{c_1}{x}+\frac{c_2}{x^2}+\frac{c_3}{x^3}+\frac{c_4}{x^4}+\frac{c_5}{x^5}+\frac{c_6}{x^6}+\frac{c_7}{2x+\gamma}+\frac{c_8}{(2x+\gamma)^2}\right]\Psi=0\,.$$

We further search for Frobenius-type solutions of the form

$$\Psi(x)=x^Ae^{Bx}e^{C/x}f(x)\,, \tag{6.6}$$

which yields

$$\frac{d^4f}{dx^4}f+\left[\frac{4A+10}{x}-\frac{4C}{x^2}+4B-\frac{4}{(2x+\gamma)}\right]\frac{d^3f}{dx^3}$$

$$+\left[\frac{-24C+a_1\gamma^2-12A\gamma+30B\gamma^2+12AB\gamma^2}{\gamma^2x}+\frac{6A^2\gamma+a_2\gamma+12C-12BC\gamma+24A\gamma}{\gamma x^2}\right.$$

$$+\frac{a_3-12AC-18C}{x^3}+\frac{6C^2+a_4}{x^4}+\frac{48C+24A\gamma-12B\gamma^2+a_5\gamma^2}{(2x+\gamma)\,\gamma^2}$$

$$\left.+6B^2+4\epsilon+\frac{a_6}{(2x+\gamma)^2}\right]\frac{d^2f}{dx^2}$$

$$+\left[\frac{1}{\gamma^4x}\left(8\epsilon A\gamma^4+2a_1B\gamma^4+24A^2\gamma^2-24A\gamma^2-96C\gamma+96C^2+12AB^2\gamma^4+2a_5\,A\gamma^3\right.\right.$$

$$+4a_5C\gamma^2+2a_6A\gamma^2+8a_6C\gamma-24AB\gamma^3+96AC\gamma+30B^2\gamma^4+b_1\gamma^4-48\,BC\gamma^2\big)$$
$$+\frac{1}{\gamma^3x^2}\big(-8\epsilon C\gamma^3+2a_1A\gamma^3+2a_2B\gamma^3-12A^2\gamma^2+12A\gamma^2+48C\gamma-48C^2$$
$$+48AB\gamma^3+12A^2B\gamma^3-12B^2C\gamma^3-2a_5C\gamma^2-2a_6C\gamma-48AC\gamma+b_2\gamma^3+24BC\gamma^2\big)$$
$$+\frac{1}{\gamma^2x^3}\big(-2a_1C\gamma^2+2a_2A\gamma^2+2a_3B\gamma^2-24C\gamma+24C^2-36BC\gamma^2$$
$$+24AC\gamma+4A^3\gamma^2+18A^2\gamma^2-22A\gamma^2+b_3\gamma^2-24ABC\gamma^2\big)$$
$$+\frac{-2a_2C\gamma+2a_3A\gamma+2a_4B\gamma-12C^2-24AC\gamma+12BC^2\gamma-12A^2C\gamma+36C\gamma}{\gamma x^4}$$
$$+\frac{-2a_3C+2a_4A+12AC^2+6C^2}{x^5}-\frac{2C\left(a_4+2C^2\right)}{x^6}$$
$$+8\epsilon B+4B^3+\frac{2a_6B\gamma^2-4a_6A\gamma-8a_6C+b_5\gamma^2}{(2x+\gamma)^2\gamma^2}$$
$$+\frac{1}{(2\,x+\gamma)\,\gamma^4}\ \big(2\,a_5\,B\gamma^4-48A^2\gamma^2+48A\gamma^2+192C\gamma-192C^2-4a_5A\gamma^3$$
$$-8a_5C\gamma^2-4a_6A\gamma^2-16a_6C\gamma+48AB\gamma^3-192\,AC\gamma-12B^2\gamma^4+b_4\gamma^4+96BC\gamma^2\big)\Big]\ \frac{df}{dx}$$
$$+\Bigg[\frac{1}{\gamma^6x}\big(288AC\gamma^2+384C^2\gamma-96A^2C\gamma^2+96BC^2\gamma^2-32A\gamma^3-16A^3\gamma^3-192AC^2\gamma$$
$$-8a_5C^2\gamma^2-96BC\gamma^3-2a_5A^2\gamma^4+2a_5A\gamma^4+24a_6C\gamma^2+2b_4C\gamma^4+8a_5C\gamma^3+4a_6A\gamma^3$$
$$-24AB\gamma^4-24B^2C\gamma^4-32a_6C^2\gamma+24A^2B\gamma^4-4a_6A^2\gamma^3+4b_5C\gamma^3-8a_5AC\gamma^3+96ABC\gamma^3$$
$$+4a_5BC\gamma^4-24a_6AC\gamma^2+8a_6BC\gamma^3+2a_5AB\gamma^5+2a_6AB\gamma^4-128C^3-12AB^2\gamma^5+48A^2\gamma^3$$
$$-192C\gamma^2+10B^3\gamma^6+b_1B\gamma^6+a_1B^2\gamma^6+4AB^3\gamma^6+b_4A\gamma^5+b_5A\gamma^4+c_1\gamma^6+8\epsilon AB\gamma^6\big)$$
$$+\frac{1}{\gamma^5x^2}\big(-144AC\gamma^2-192C^2\gamma+48A^2C\gamma^2-48BC^2\gamma^2+b_2B\gamma^5+16A\gamma^3+8A^3\gamma^3-4B^3C\gamma^5$$
$$+b_1A\gamma^5+96AC^2\gamma+2a_1AB\gamma^5-8\epsilon BC\gamma^5+4a_5C^2\gamma^2+48BC\gamma^3+a_5A^2\gamma^4-a_5A\gamma^4$$
$$-8a_6C\gamma^2-b_4C\gamma^4-4a_5C\gamma^3-a_6A\gamma^3+12AB\gamma^4+12B^2C\gamma^4+12a_6C^2\gamma-12A^2B\gamma^4$$
$$+a_6\,A^2\gamma^3-b_5C\gamma^3+4a_5AC\gamma^3-48ABC\gamma^3-2a_5BC\gamma^4+8a_6AC\gamma^2-2a_6BC\gamma^3+64C^3$$
$$+24AB^2\gamma^5-24A^2\gamma^3+a_2B^2\gamma^5+4\epsilon A^2\gamma^5+6A^2B^2\gamma^5-4\epsilon A\gamma^5+96C\gamma^2+c_2\gamma^5\big)$$
$$+\frac{1}{\gamma^4x^3}\big(72AC\gamma^2+96C^2\gamma-24A^2C\gamma^2+24BC^2\gamma^2-8A\gamma^3-4A^3\gamma^3-48AC^2\gamma-2a_5C^2\gamma^2$$
$$-24BC\gamma^3+2a_6C\gamma^2+2a_5C\gamma^3-22AB\gamma^4-18B^2C\gamma^4-4a_6C^2\gamma+18A^2B\gamma^4-2a_1BC\gamma^4$$
$$+2a_2AB\gamma^4-8\epsilon AC\gamma^4-12AB^2C\gamma^4-2a_5AC\gamma^3+24ABC\gamma^3-2a_6AC\gamma^2-32C^3+12A^2\gamma^3$$
$$-48C\gamma^2+c_3\gamma^4+4A^3B\gamma^4+b_2A\gamma^4+a_3B^2\gamma^4-a_1A\gamma^4+8\epsilon C\gamma^4+a_1A^2\gamma^4+b_3B\gamma^4-b_1C\gamma^4\big)$$
$$+\frac{1}{\gamma^3x^4}\big(-36\,AC\gamma^2-48C^2\gamma+12A^2C\gamma^2-12BC^2\gamma^2+14A\gamma^3+4A^3\gamma^3+24AC^2\gamma$$

$$+a_5C^2\gamma^2+36BC\gamma^3+a_6C^2\gamma-12A^2BC\gamma^3-2a_1AC\gamma^3+2a_3AB\gamma^3-2a_2BC\gamma^3$$
$$-24ABC\gamma^3+16C^3-19A^2\gamma^3+24C\gamma^2+c_4\gamma^3+A^4\gamma^3-a_2A\gamma^3+2a_1\,C\gamma^3$$
$$+a_4B^2\gamma^3+a_2A^2\gamma^3+4\epsilon C^2\gamma^3-b_2C\gamma^3+6B^2C^2\gamma^3+b_3A\gamma^3\big)$$
$$+\frac{1}{\gamma^2x^5}\left(46AC\gamma^2+a_1C^2\gamma^2+a_3A^2\gamma^2+2a_2C\gamma^2-a_3A\gamma^2-b_3C\gamma^2+24C^2\gamma-8C^3-6A^2C\gamma^2\right.$$
$$\left.+6BC^2\gamma^2-4A^3C\gamma^2-2a_3BC\gamma^2+2a_4AB\gamma^2-12AC^2\gamma-36C\gamma^2+c_5\gamma^2+12ABC^2\gamma^2-2a_2AC\gamma^2\right)$$
$$+\frac{a_2C^2\gamma+a_4A^2\gamma+2a_3C\gamma-a_4A\gamma+4C^3+6A^2C^2\gamma-4BC^3\gamma-2a_3AC\gamma-2a_4BC\gamma-24C^2\gamma+c_6\gamma}{\gamma x^6}$$
$$-\frac{C\left(-a_3C-2a_4+4AC^2+2a_4A-2C^2\right)}{x^7}+\frac{C^2\left(a_4+C^2\right)}{x^8}+B^4+4\,\epsilon^2+4\epsilon B^2$$
$$+\frac{1}{(2x+\gamma)\,\gamma^6}\left(-576AC\gamma^2-768C^2\gamma+192A^2C\gamma^2-192BC^2\gamma^2+64A\gamma^3+32A^3\gamma^3+384AC^2\gamma\right.$$
$$+16a_5C^2\gamma^2+192BC\gamma^3+4a_5A^2\gamma^4-4a_5A\gamma^4-48a_6C\gamma^2-4b_4C\gamma^4-16a_5C\gamma^3-8a_6A\gamma^3$$
$$+48AB\gamma^4+48B^2C\gamma^4+64a_6C^2\gamma-48A^2B\gamma^4+8a_6A^2\gamma^3-8b_5\gamma^3+16a_5AC\gamma^3-192ABC\gamma^3$$
$$-8a_5BC\gamma^4+48a_6AC\gamma^2-16a_6BC\gamma^3-4a_5AB\gamma^5-4a_6AB\gamma^4+256C^3$$
$$\left.+24AB^2\gamma^5-96A^2\gamma^3+384C\gamma^2-4B^3\gamma^6+c_7\gamma^6+a_5B^2\gamma^6+b_4B\gamma^6-2b_4A\gamma^5-2b_5\,A\gamma^4\right)$$
$$+\frac{1}{(2x+\gamma)^2\gamma^4}\left(a_6B^2\gamma^4+b_5B\gamma^4+4a_6A^2\gamma^2+16a_6C^2-2b_5\gamma^3-4a_6A\gamma^2\right.$$
$$\left.-16a_6C\gamma-4b_5C\gamma^2+c_8\gamma^4-4a_6AB\gamma^3+16a_6AC\gamma-8a_6BC\gamma^2\right)\Big]\,f=0\,.$$

We impose restrictions on parameters B and C:

$$B^4+4\,\epsilon^2+4\,\epsilon\,B^2=0\qquad\Longrightarrow\qquad B=-\sqrt{-2\,\epsilon},+\sqrt{-2\,\epsilon}\,,$$

$$\frac{1}{x^8},\quad C^2\left(a_4+C^2\right)=0\;\Rightarrow\;C=0,\;\pm\sqrt{-a_4}\;\Rightarrow\;C_1=0\,,\;C_2=+\gamma\,,\;C_3=-\gamma\,.$$

For $C=C_1=0$, the coefficient of $1/x^7$ vanishes, so we require the coefficient of $1/x^6$ to vanish as well:

$$a_2C^2\gamma+a_4A^2\gamma+2a_3C\gamma-a_4A\gamma+4C^3+6A^2C^2\gamma$$
$$-4BC^3\gamma-2a_3AC\gamma-2a_4BC\gamma-24C^2\gamma+c_6\gamma=0,$$

or

$$a_4A^2\gamma-a_4\,A\gamma+c_6\gamma=0\Longrightarrow-\gamma^3A^2+\gamma^3A+2\gamma^3=0\Longrightarrow A_1=-1,A_2=+2.$$

If $C=C_2=+\gamma$, then we impose the multiplier of $1/x^7$ to be equal to zero:

$$-a_3C-2a_4+4AC^2+2a_4,A-2C^2=0,a_3=-2\gamma\Longrightarrow 2\,\gamma^2+2A\gamma^2=0\Longrightarrow A_3=-1.$$

If $C = C_3 = -\gamma$, then we impose the multiplier of $1/x^7$ to be equal to zero:

$$-a_3 C - 2a_4 + 4AC^2 + 2a_4 A - 2C^2 = 0, a_3 = -2\gamma \Longrightarrow -2\gamma^2 + 2A\gamma^2 = 0 \Longrightarrow A_4 = +1.$$

Thus, there are four types of the solutions (we use below only negative values for B):

$$\begin{array}{l} I, \quad B = -\sqrt{-2\epsilon}, \quad C = 0, \quad A = -1, \quad \Psi = e^{Bx}\frac{1}{x}f_1(x); \\ II, \quad B = -\sqrt{-2\epsilon}, \quad C = 0, \quad A = +2, \quad \Psi = e^{Bx}x^2 f_2(x)\,; \\ III, \quad B = -\sqrt{-2\epsilon}, \quad C = +\gamma, \quad A = -1, \quad \Psi(x) = e^{Bx}\frac{1}{x}e^{+\gamma/x}f_3(x)\,; \\ IV, \quad B = -\sqrt{-2\epsilon}, \quad C_3 = -\gamma, \quad A = +1, \quad \Psi(x) = e^{Bx}x e^{-\gamma/x}f_4(x)\,. \end{array} \tag{6.7}$$

Only three cases may be be suitable for describing the bound states:

$$\text{solutions} \quad II \quad \text{and} \quad III \quad \text{at negative } \gamma\,; \tag{6.8}$$

$$\text{solution} \quad IV \quad \text{at positive } \gamma\,. \tag{6.9}$$

All the constructed solutions are exact, but they are formal, because any reliable rules for quantization of energy levels are not known. The transcendency condition is solving only partially this difficulty.

7 The KCC-geometrical approach

In this section we will study the mathematical tasks which arose above, by applying the special geometric method based on the KCC-invariants [1] – [4]. In this approach, one considers a system of second order differential equations

$$\dot{y}^{\,i}(r) + 2Q^i(r, x, y) = 0\,, \tag{7.1}$$

which corresponds to the the Euler-Lagrange equations of a Lagrangian L. In (7.1), the symbol x^i designates the spatial coordinates, their derivatives in the argument r are $y^i = dx^i/dr = \dot{x}^i$, and the quantities Q_i are determined from a Lagrangian L, as follows

$$Q^i = \frac{1}{4}g^{il}\left(\frac{\partial^2 L}{\partial x^k \partial y^l}y^k - \frac{\partial L}{\partial x^l} + \frac{\partial^2 L}{\partial y^l \partial r}\right), \qquad g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}. \tag{7.2}$$

The first and second invariants, $\varepsilon^i(r, x, y)$ and P^i_j are defined by

$$\begin{array}{l} \varepsilon^i = \frac{\partial Q^i}{\partial y^j}\, y^j - 2Q^i, \\ P^i_j = 2\frac{\partial Q^i}{\partial x^j} + 2Q^s\frac{\partial^2 Q^i}{\partial y^j \partial y^s} - \frac{\partial^2 Q^i}{\partial y^j \partial x^s}y^s - \frac{\partial Q^i}{\partial y^s}\frac{\partial Q^s}{\partial y^j} - \frac{\partial^2 Q^i}{\partial y^j \partial r}. \end{array} \tag{7.3}$$

The second invariant P^i_j relates to the Jacobi stability of the differential system. There is an analogy between the equations of geodesic deviation expressed in terms of the Riemannian curvature and and the equations provided by the second KCC-invariant:

$$\frac{D^2\xi^i}{Ds^2} = R^i_{kjl}\frac{dx^k}{ds}\frac{dx^l}{ds}\xi^j = -K^i_j\xi^j, \qquad \frac{D^2\xi^i}{Dr^2} = P^i_j\xi^j. \tag{7.4}$$

It is known that a pencil of geodesic curves emanating from a given point r_0 converges (or diverges) if the real parts of all eigenvalues of the invariant ${P^i}_j$ are negative (or positive), respectively.

We start from the system of two second-order differential equations for the two radial functions of the spin 1 particle with electric quadrupole moment in the external Coulomb field, namely (6.3). This can be written as:

$$\begin{gathered}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2m\frac{\alpha + Er}{r} - \frac{2\nu^2}{r^2} - \frac{2\Gamma}{r^3} - \frac{\Gamma^2}{r^4}\right)\Psi_1(r) - \nu\frac{2r+\Gamma}{r^3}\Psi_2(r) = 0,\\ \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2m\frac{\alpha + Er}{r} - \frac{2\nu^2}{r^2} - \frac{2}{r^2}\right)\Psi_2(r) - 2\nu\frac{2r+\Gamma}{r^3}\Psi_1(r) = 0.\end{gathered} \tag{7.5}$$

We shall use the following notations $x^i = \Psi_i(r)$, $y^i = (d/dr)\Psi_i(r) = \dot{\Psi}_i(r)$. Then, by comparing the equations (7.5) and (7.1), one finds the quantities Q^i:

$$\begin{gathered}Q^1(r,\Psi_i,\dot{\Psi}_i) = \left(Em + \frac{\alpha m}{r} - \frac{\Gamma^2}{2r^4} - \frac{\Gamma}{r^3} - \frac{\nu^2}{r^2}\right)\Psi_1 - \nu\frac{(\Gamma+2r)}{2r^3}\Psi_2 + \frac{1}{r}\dot{\Psi}_1,\\ Q^2(r,\Psi_i,\dot{\Psi}_i) = \left(Em + \frac{\alpha m}{r} - \frac{\nu^2}{r^2} - \frac{1}{r^2}\right)\Psi_2 - \nu\frac{(\Gamma+2r)}{r^3}\Psi_1 + \frac{1}{r}\dot{\Psi}_2.\end{gathered} \tag{7.6}$$

By direct calculation, according the formula (7.3), the first and second KCC-invariants are explicitly given by:

$$\begin{gathered}\varepsilon^1 = \Psi_1\left(-2Em - \frac{2\alpha m}{r} + \frac{\Gamma^2}{r^4} + \frac{2\Gamma}{r^3} + \frac{2\nu^2}{r^2}\right) + \nu\Psi_2\left(\frac{\Gamma}{r^3} + \frac{2}{r^2}\right) - \frac{\dot{\Psi}_1}{r},\\ \varepsilon^2 = 2\Psi_2\left(-Em - \frac{\alpha m}{r} + \frac{\nu^2+1}{r^2}\right) + 2\nu\Psi_1\left(\frac{\Gamma}{r^3} + \frac{2}{r^2}\right) - \frac{\dot{\Psi}_2}{r};\end{gathered} \tag{7.7}$$

$$P^i_j = \begin{vmatrix} -\frac{\Gamma^2}{r^4} - \frac{2\Gamma}{r^3} - \frac{2\nu^2}{r^2} + 2Em + \frac{2m\alpha}{r} & -\frac{\nu(2r+\Gamma)}{r^3} \\ -\frac{2\nu(2r+\Gamma)}{r^3} & 2Em + \frac{2\alpha m}{r} - \frac{2(\nu^2+1)}{r^2}\end{vmatrix}. \tag{7.8}$$

The eigenvalues Λ_1, Λ_2 of the second invariant are provided by the formulas (7.9)

$$\Lambda_{1,2} = 2Em + \frac{2\alpha m}{r} - \frac{\Gamma^2}{2r^4} - \frac{\Gamma}{r^3} - \frac{2\nu^2+1}{r^2} \pm \frac{\sqrt{(\Gamma^2 - 2r^2 + 2\Gamma r)^2 + 8\nu^2 r^2(\Gamma+2r)^2}}{2r^4}.$$

The typical behavior of eigenvalues for different values of j, is presented in figure 1.

Let us study the behavior of the eigenvalues Λ^i near the singular points $r = 0$, $r = \infty$, $r = -\Gamma/2$. It can be shown that

$$r \to 0,\ \Lambda^1 \to -\frac{2}{r^2} < 0,\ \Lambda^2 \to -\frac{\Gamma^2}{r^4} < 0; \qquad r \to \infty,\ \Lambda^1, \Lambda^2 \to 2Em < 0;$$

$$r \to -\frac{\Gamma}{2},\ \Lambda^1 \to 2Em - \frac{4m\alpha}{\Gamma} - \frac{8\nu^2}{\Gamma^2} - \frac{8}{\Gamma^2} < 0,\ \ \Lambda^2 \to 2Em - \frac{4m\alpha}{\Gamma} - \frac{8\nu^2}{\Gamma^2} < 0.$$

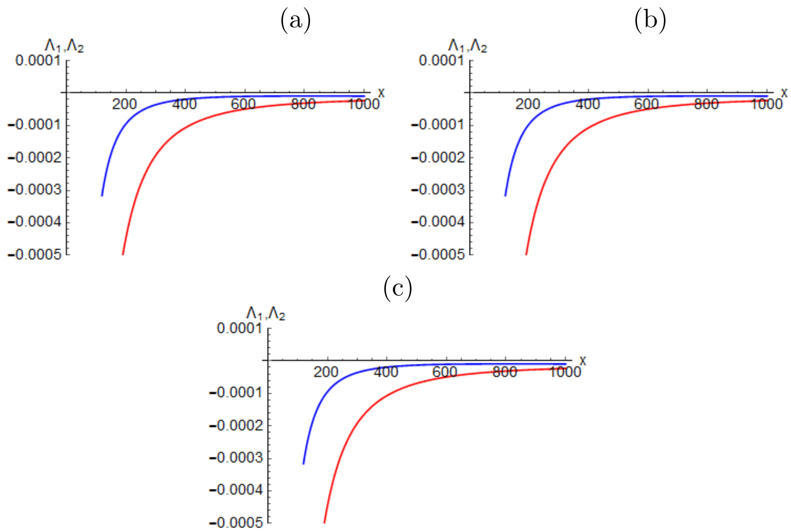

Figure 1: The dependencies of eigenvalues Λ_1 (red) and Λ_1 (blue) on radial coordinate ($x = mr$) at different j: (a) $j = 1$, (b) $j = 2$, (c) $j = 3$. We used following values for the parameters: $\Gamma m = 1$, $E/m = -0.000009$.

Since the real parts of all the eigenvalues of the 2-nd KCC-invariant are negative, the different branches of the solution converge near the singular points $r = 0, \infty, -\Gamma/2$. This correlates with the behavior of solutions near the singular points for bound quantum mechanical states (for discrete spectra).

The third KCC-invariant

$$R^i_{jk} = \frac{1}{3}\left(\frac{\partial P^i_j}{\partial y^k} - \frac{\partial P^i_k}{\partial y^j}\right) \tag{7.10}$$

determines (in Finslerian KCC applications) the torsion of the Berwald connection.

The fourth KCC-invariant is an extension of the Riemann–Christoffel tensor

$$B^i_{jkl} = \frac{\partial R^i_{jk}}{\partial y^l}. \tag{7.11}$$

Finally, the fifth KCC-invariant extends the Duglas tensor.

$$D^i_{jkl} = \frac{\partial^3 Q^i}{\partial y^j \partial y^k \partial y^l}. \tag{7.12}$$

Since the vector field Q^i (7.6) is linear in the coordinates $x^i \equiv \Phi^i$ and $y^i \equiv \dot{\Psi}^i$, the first (7.7) and second invariants (7.8) are functions of the radial coordinate r and do not depend on x^i and y^i, while the third, fourth and fifth invariants identically vanish.

The next step is to construct a Lagrangian function L for the phase space $\dot{\Psi}_i$, Ψ_i, such that the formulas for coefficients Q^i (7.6) hold true, and the dynamics of the system is defined by the equations (7.5). We will search for a function L of the form

$$L = g_{ij}(r)y^i y^j + b_j(r,x)y^j \,. \tag{7.13}$$

Let us assume that the tensor g_{ij} is diagonal, $g_{12} = g_{21} = 0$. In this case, by substituting (7.13) into (7.2), we derive

$$\begin{aligned} Q^1 &= \frac{1}{4g_{11}}\left(2\dot{g}_{11}y^1 + \frac{\partial b_1}{\partial r} + \left(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\right)y^2\right), \\ Q^2 &= \frac{1}{4g_{22}}\left(2\dot{g}_{22}y^2 + \frac{\partial b_2}{\partial r} + \left(\frac{\partial b_2}{\partial x^1} - \frac{\partial b_1}{\partial x^2}\right)y^2\right). \end{aligned} \tag{7.14}$$

By equating the terms from (7.6) with the corresponding terms from (7.14), we obtain the system of equations with respect to $g_{ij}(r)$ and $b_j(r,x)$:

$$\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1} = 0, \qquad \frac{\dot{g}_{11}}{2g_{11}} = \frac{1}{r}, \qquad \frac{\dot{g}_{22}}{2g_{22}} = \frac{1}{r},$$

$$\frac{1}{4g_1}\frac{\partial b_1}{\partial r} = -\frac{x^1\left(\Gamma^2 - 2r^2\left(mr(\alpha+Er) - \nu^2\right) + 2\Gamma r\right)}{2r^4} - \frac{\nu x^1(\Gamma+2r)}{2r^3},$$

$$\frac{1}{4g_2}\frac{\partial b_2}{\partial r} = -\frac{x^2\left(-Emr^2 + \nu^2 - \alpha mr + 1\right)}{r^2} - \frac{\nu x^1(\Gamma+2r)}{r^3}.$$

Its solution is given by the formulas

$$g_{11} = 2C_1 r^2, \qquad g_{22} = C_1 r^2,$$

$$b_1 = B_1(x^1,x^2) - 2C_1\left\{-\frac{2}{3}Emr^3x^1 - \alpha mr^2x^1 - \frac{\Gamma^2 x^1}{r} + \ln r(\Gamma\nu x^2 + 2\Gamma x^1) + 2\nu r(\nu x^1 + x^2)\right\},$$

$$b_2 = B_2(x^1,x^2) - 4C_1\left\{-\frac{1}{3}Emr^3x^2 - \frac{1}{2}\alpha mr^2x^2 + \Gamma\nu x^1\ln r + r\left(2\nu x^1 + \nu^2x^2 + x^2\right)\right\},$$

where C_1 is an arbitrary constant. The two functions $B_1(x^1,x^2)$ and $B_2(x^1,x^2)$ obey the following restriction

$$\frac{\partial B_1(x^1,x^2)}{\partial x^2} - \frac{\partial B_2(x^1,x^2)}{\partial x^1} = 0. \tag{7.15}$$

In accordance with the known theorem, from (7.15) we conclude that this 2-dimensional vector field (B_1, B_2) is the gradient of a scalar function

$$B_1(x^1,x^2) = \frac{\partial}{\partial x^1}\varphi(x^1,x^2), \quad B_2(x^1,x^2) = \frac{\partial}{\partial x^2}\varphi(x^1,x^2), \quad B_i = \operatorname{grad}\varphi\,. \tag{7.16}$$

There exist some freedom in choosing the Lagrangian (the constant C_1 may be taken equal to 1) :

$$\begin{aligned} L = {} & 2r^2(y^1)^2 + r^2(y^2)^2 + 4x^1y^1\left(\frac{2}{3}Emr^3 + \alpha mr^2 + \frac{\Gamma^2}{r} - 2\Gamma\ln r - 2\nu^2 r\right) \\ & + \frac{2}{3}rx^2y^2\left(mr(3\alpha + 2Er) - 6\left(\nu^2+1\right)\right) - 4\nu\left(x^2y^1 + x^1y^2\right)\left(\Gamma\ln r + 2r\right) \\ & + y^1\frac{\partial\varphi}{\partial x^1} + y^2\frac{\partial\varphi}{\partial x^2}, \quad \varphi = \varphi(x^1,x^2). \end{aligned} \tag{7.17}$$

8 Conclusions

In the present work, the quantum-mechanical problem of the spin 1 particle with additional quadrupole moment in the external Coulomb field, has been studied. The separation of variables in the generalized Duffin-Kemmer equation has been performed through the diagonalization of the energy operator, operators of the square and the third projection of the total momentum. The equation system for the ten radial functions has been derived. By the diagonalization of the spatial reflection operator, the system is separated into two subsystems of four and six equations, for the parities $P = (-1)^{j+1}$ and $P = (-1)^{j}$, respectively. The additional terms provided by the electric quadrupole moment are present in both subsystems.

The system of four relativistic radial equations leads to a second-order differential equations for the main function. This equation has two singular singular points of rank 3 and 2, and four regular points with simple indexes. The Frobenius solutions have been constructed as power series, 8-term recurrence relations have been found, and the power convergence has been studied. The transcendence condition for the solutions gives the formula for energy levels, which is suitable from physical point of view.

The relativistic system of six equations turns out to be very complicated. To simplify it, the non-relativistic approximation has been performed. In this case, the radial system is reduced to two linked differential equations of second order for two functions. By using the exclusion method, we get two single-type equations of fourth order for these functions. The Frobenius solutions of these equations has been constructed, and the convergence of the corresponding 8- and 9-term power series has been studied. Among all the solutions we have picked out those solutions which could describe the bound states of the particle.

All the determined Frobenius-type solutions of the 2-d order and 4-th order equations are exact, but no energy quantization rules are unknown at this moment.

We have also used a geometrical KCC-based method to study the problem of spin 1 particle with anomalous magnetic moment in the external Coulomb field. The first and the second invariants were calculated. It was shown that the different branches of the solution converge near the singular points $r = 0, \infty, -\Gamma/2$. This correlates with the behavior of solutions near these points for quantum mechanical bound states. The Lagrangians corresponding to the geometrical problem has been found, and these are demonstrated to have an arbitrariness up to a special term, which may be regarded as a specific gauge freedom.

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