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ЗАДАЧИ ЛИНЕЙНОГО ПОЛУОПРЕДЕЛЕННОГО ПРОГРАММИРОВАНИЯ: РЕГУЛЯРИЗАЦИЯ И ДВОЙСТВЕННЫЕ ФОРМУЛИРОВКИ В СТРОГОЙ ФОРМЕ

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Регуляризация задачи оптимизации состоит в ее сведении к эквивалентной задаче, удовлетворяющей условиям регулярности, которые гарантируют выполнение соотношений двойственности в строгой форме. В настоящей статье для линейных задач полуопределенного программирования предлагается процедура регуляризации, основанная на понятии неподвижных индексов и их свойствах. Эта процедура описана в виде алгоритма, который за конечное число шагов преобразует любую задачу линейного полубесконечного программирования в эквивалентную задачу, удовлетворяющую условию Слейтера. В результате использования свойств неподвижных индексов и предложенной процедуры регуляризации получены новые двойственные задачи полубесконечного программирования в явной и неявной формах. Доказано, что для этих двойственных задач и исходной задачи соотношения двойственности выполняются в строгой форме.

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LINEAR SEMIDEFINITE PROGRAMMING PROBLEMS: REGULARISATION AND STRONG DUAL FORMULATIONS

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Regularisation consists in reducing a given optimisation problem to an equivalent form where certain regularity conditions, which guarantee the strong duality, are fulfilled. In this paper, for linear problems of semidefinite programming (SDP), we propose a regularisation procedure which is based on the concept of an immobile index set and its properties. This procedure is described in the form of a finite algorithm which converts any linear semidefinite problem to a form that satisfies the Slater condition. Using the properties of the immobile indices and the described regularisation procedure, we obtained new dual SDP problems in implicit and explicit forms. It is proven that for the constructed dual problems and the original problem the strong duality property holds true.

Keywords: linear semidefinite programming; strong duality; normalised immobile index set; regularisation; constraint qualifications.

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Introduction

In this paper, we consider semidefinite programming (SDP) problems which are particular case of semi-infinite programming (SIP) and conic optimisation. The interest to SDP problems stems from their numerous applications [1; 2].

In conic optimisation, optimality conditions and duality results are usually formulated under some regularity conditions, so-called constraint qualifications (CQ) (see e. g. [2–5]). Such conditions should guarantee the fulfillment of the Karush – Kuhn – Tucker (KKT) type optimality conditions and the *strong duality* property consisting in the fact that the optimal values of the objective function in the primal problem and its Lagrangian dual problem are equal and the objective function in the dual problem attains its maximum.

If a given SDP problem does not satisfy CQs then, in general, the optimality conditions in the KKT form degenerate (i. e. are satisfied for all feasible solutions and, hence, are not helpful for the search for optimal solutions) and the strong duality relations are violated. This makes it difficult to solve such problems numerically. To avoid these obstacles, regularisation procedures are applied to the original irregular problems. The purpose of such procedures is to obtain equivalent formulations satisfying some CQs.

One of the most commonly used CQ is the Slater condition that consists in non-emptiness of the interior of the feasible set. For SDP problems that do not satisfy this CQ, several regularisation procedures based on a so-called *facial reduction approach* (FRA) were proposed in [6–9]. Being very general (they are designed for a wider class of problems than SDP), these procedures are not detailed.

In [6; 10], the authors used the FRA to construct an explicit dual SDP problem which satisfies the strong duality property without any CQ.

For the study of SIP problems not satisfying CQ, we proposed (see [11; 12]) other approach which is based on the concept of *immobile indices*, i. e. indices of the constraints that are active for all feasible solutions. In this paper, we will apply this approach to SDP problems.

The main aim of this paper is to show that the concept of immobile indices is an efficient tool that can be used in SDP:

1) to design and justify a new regularisation procedure;

2) formulate explicitly dual SDP problems satisfying the strong duality relations.

Linear semidefinite programming problem: problem statement and basic definitions

Given an integer $p > 1$, denote by $\mathbb{R}^p_+$ the set of all p vectors with non-negative components, by $\mathcal{S}^p$ and $\mathcal{S}^p_+$ the space of real symmetric $p \times p$ matrices and the cone of symmetric positive semidefinite $p \times p$ matrices, respectively. The space $\mathcal{S}^p$ is considered here as a vector space with the trace inner product $A \bullet B := trace(AB)$.

Consider a linear SDP problem in the form

$$\min_x c^\top x, \text{ s.t. } \mathcal{A}(x) \in \mathcal{S}^p_+, \tag{1}$$

where $x = (x_1, \ldots, x_n)^\top$ is the vector of decision variables and the constraints matrix function $\mathcal{A}(x)$ is defined as

$$\mathcal{A}(x) := \sum_{j=1}^{n} A_j x_j + A_0, \tag{2}$$

matrices $A_j \in \mathcal{S}^p$, $j = 0, 1, \ldots, n$, and vector $c \in \mathbb{R}^n$ are given. Here and below, for a given vector or matrix ν, we denote by $\nu^\top$ its transpose.

It is well known that the SDP problem in the form (1) is equivalent to the following convex SIP problem:

$$\min_x c^\top x, \text{ s.t. } t^\top \mathcal{A}(x) t \geq 0 \;\; \forall t \in T, \tag{3}$$

with a p – dimensional compact index set

$$T := \{t \in \mathbb{R}^p_+ : \|t\| = 1\}. \tag{4}$$

Denote by X the feasible set of the equivalent problems (1) and (3):

$$X := \{x \in \mathbb{R}^n : \mathcal{A}(x) \in S^p_+\} = \{x \in \mathbb{R}^n : t^\top \mathcal{A}(x) t \geq 0 \;\; \forall t \in T\}.$$

Evidently, the set X is convex.

Remark 1. In what follows, we will suppose that $X \neq \varnothing$. Then there exists a feasible solution $y \in X$ and, without lost of generality, we can consider that $A_0 \in S^p_+$. In fact, having substituted x by a new variable $z := x - y$, we can replace the original problem (1) by the following one in terms of z:

$$\min_z c^\top z, \text{ s.t. } \bar{\mathcal{A}}(z) \in S^p_+,$$

with $\bar{\mathcal{A}}(z) := \sum_{i=1}^{n} A_i z_i + \bar{A}_0$, $\bar{A}_0 = \mathcal{A}(y) \in S^p_+$.

According to the commonly used definition, the constraints of the SDP problem (1) satisfy the Slater condition if $\exists\, \bar{x} \in \mathbb{R}^n$ such that

$$\mathcal{A}(\bar{x}) \in \text{int } S^p_+ = \{D \in \mathcal{S}(p) : t^\top D t > 0 \;\forall t \in \mathbb{R}^p,\; t \neq 0\}. \tag{5}$$

Here int$\mathcal{B}$ stays for the interior of a set $\mathcal{B}$.

Following [11; 12], let's define the set of immobile indices R_{im} and the set of normalised immobile indices T_{im} in problem (1):

$$R_{im} := \{t \in \mathbb{R}^p : t^\top \mathcal{A}(x) t = 0 \;\; \forall x \in X\},$$

$$T_{im} := \{t \in T : t^\top \mathcal{A}(x) t = 0 \;\; \forall x \in X\} = \{t \in R_{im} : \|t\| = 1\}.$$

The following proposition is a consequence of the well know fact: if $A \in \mathcal{S}_+(p)$, then $t^\top A t = 0$ if and only if $At = 0$.

Proposition 1. If $t \in R_{im}$ then $\mathcal{A}(x) t = 0$ for all $x \in X$.

The lemma below can be proved following the scheme of the proof of lemma 1 and proposition 1 in [13].

Lemma 1. *Given the linear SDP problem (1), the following statements are true:*

1) the Slater condition (5) is satisfied if and only if $R_{im} = \{\mathbf{0}\}$;

2) the set of immobile indices R_{im} either contains a single zero vector or is a subspace in $\mathbb{R}^p$.

Here in what follows, **0** denotes a null vector of a given finite dimensional vector space.

It follows from lemma 1 that if $T_{im} = \varnothing$, then the constraints of problem (1) satisfy the Slater condition. This case is well studied in literature. In this paper, we will consider a general case, where we do not suppose that $T_{im} = \varnothing$ and concentrate our main attention on the case $T_{im} \neq \varnothing$.

Let us consider any finite non-empty set of indices

$$V = \{\tau(i) \in T_{im},\ i \in I\},\ 1 \leq |I| < \infty. \tag{6}$$

Lemma 2. *Given any set V in the form (6), the feasible set X of problem (1) coincides with the set*

$$X(V) := \{x \in \mathbb{R}^n : \mathcal{A}(x)\tau(i) = 0,\ i \in I,\ t^\top \mathcal{A}(x)t \geq 0\ \forall t \in T(V)\},$$

where $T(V) := \{t \in T : t^\top \tau(i) = 0,\ i \in I\}$.

P r o o f. It follows from proposition 1 that $X \subset X(V)$. Let us show that $X(V) \subset X$. Suppose that $\bar{x} \in X(V)$ whereform

$$\mathcal{A}(\bar{x})\tau(i) = 0,\ i \in I, \tag{7}$$

$$t^\top \mathcal{A}(\bar{x})t \geq 0\ \ \forall t \in T(V). \tag{8}$$

It is evident that any vector $t \in \mathbb{R}^p$ can be presented in the form

$$t = l + \mu,\ \text{where}\ l \in \operatorname{span}\{\tau(i),\ i \in I\},\ \mu \in \left(\operatorname{span}\{\tau(i),\ i \in I\}\right)^\perp.$$

Here for $S \subset \mathbb{R}^p$, we denote by $S^\perp$ the orthogonal complement to S in $\mathbb{R}^p$. Then, it follows from (7) that $\mathcal{A}(\bar{x})l = 0$ and hence

$$t^\top \mathcal{A}(\bar{x})t = \mu^\top \mathcal{A}(\bar{x})\mu. \tag{9}$$

If $\mu = 0$, then it follows from (9) that $t^\top \mathcal{A}(\bar{x})t = 0$.

Suppose that $\mu \neq 0$. Then, by construction, for $\bar{\mu} = \mu/\|\mu\|$ it holds $\bar{\mu} \in T(V)$. Taking into account this inclusion and relations (8), (9) we conclude that $t^\top \mathcal{A}(\bar{x})t \geq 0$. Thus we have shown that, for any $\bar{x} \in X(V)$ and any $t \in \mathbb{R}^p$, the inequality $t^\top \mathcal{A}(\bar{x})t \geq 0$ holds true. This implies that $\bar{x} \in X$, and hence $X(V) \subset X$. The lemma is proved.

Notice that, by construction, we have $\operatorname{span}\{V\} \cap T(V) = \varnothing$.

A regularisation procedure based on immobile indices

In this section, we will describe a regularisation procedure which constructs an SDP problem that is equivalent to the original problem (1) and satisfies the Slater condition. At the beginning of the procedure we suppose that the matrix function $\mathcal{A}(x),\ x \in \mathbb{R}^n$, defined in (2) is known. Remind that $A_0 \in S_+^p$.

Let us describe and justify the regularisation procedure in steps.

Step #0. Given the SDP problem in the form (1), consider the following SIP problem:

$$\text{SIP}_0\text{: } \min_{(x,\ \mu)} \mu,\ \text{s. t. } t^\top \mathcal{A}(x)t + \mu \geq 0,\ t \in T,$$

with the index set T defined in (4).

If there exists a feasible solution $(\bar{x}, \bar{\mu})$ of this problem with $\bar{\mu} < 0$, then set $m_* := 0$ and go to the *final step*. Otherwise the vector ($x = \mathbf{0}$, $\mu = 0$) is an optimal solution of the problem SIP_0.

It should be noticed that the problem SIP_0 is regular since the index set T is a compact set, and the constraints satisfy the Slater condition. Hence, (see e. g. [14]), it follows from the optimality conditions for the vector ($x = \mathbf{0}$, $\mu = 0$) in problem SIP_0 that there exist indices and numbers $\tau(i) \in T,\ \gamma(i) > 0,\ i \in I_1,\ |I_1| \leq n + 1$, such that

$$\sum_{i \in I_1} \gamma(i)(\tau(i))^\top A_j \tau(i) = 0,\ \ j = 0, 1, \ldots, n;\ \ \sum_{i \in I_1} \gamma(i) = 1.$$

It follows from the relations above that $I_1 \neq \varnothing$ and $\tau(i) \in T_{im} \subset T,\ i \in I_1$.

Go to the next step with the data $\gamma(i) > 0,\ \tau(i) \in T_{im},\ i \in I_1$, and $I_0 = \varnothing$.

Step #*m*, *m* ≥ 1. By the beginning of the iteration, we have numbers, indices and vectors

$$\gamma(i) > 0,\ \tau(i) \in T_{im},\ i \in I_m,\ \lambda^{m-1}(i) \in \mathbb{R}^p,\ i \in I_{m-1}, \tag{10}$$

which satisfy the relations

$$\sum_{i \in I_m} \gamma(i)\big(\tau(i)\big)^\top A_j \tau(i) + \sum_{i \in I_{m-1}} \big(\lambda^{m-1}(i)\big)^\top A_j \tau(i) = 0,\ j = 0, 1, \ldots, n. \tag{11}$$

Notice that if $m = 1$, then $I_{m-1} = I_0 = \varnothing$. Hence $\sum_{i \in I_{m-1}} \ldots = 0$ and we do not need vectors $\lambda^{m-1}(i)$, $i \in I_{m-1}$.

Consider a SIP problem:

$$\text{SIP}_m:\ \min_{(x,\,\mu)} \mu,\ \text{s. t. } \mathcal{A}(x)\tau(i) = 0,\ i \in I_m,\ t^\top \mathcal{A}(x) t + \mu \ge 0,\ t \in T_m,$$

with the index set $T_m := \{t \in T : t^\top \tau(i) = 0,\ i \in I_m\}$.

By construction, in this problem the index set T_m is a compact set, there is a finite number of linear equality constraints, and there exists a feasible solution ($x^* = \mathbf{0}$, $\mu^* = 1$) such that $t^\top \mathcal{A}(x^*) t + \mu^* > 0$ for all $t \in T_m$. Hence, problem SIP_m is regular.

Notice that it follows from lemma 2 that for $\mu = 0$, the set of feasible solutions in the problem SIP_m coincides with the set of feasible solutions X in the original problem (1).

If there exists a feasible solution $(\bar{x}, \bar{\mu})$ of SIP_m with $\bar{\mu} < 0$, then set $m_* := m$ and go to the *final step.*

Otherwise the vector ($x = \mathbf{0}$, $\mu = 0$) is an optimal solution of the problem SIP_m. Since this problem is regular, it follows from the optimality of ($x = \mathbf{0}$, $\mu = 0$) (see [14]) that there exist indices, numbers and vectors

$$\tau(i) \in T_m,\ \gamma(i) > 0,\ i \in \Delta I_m,\ |\Delta I_m| \le n+1,\ \lambda(i) \in \mathbb{R}^p,\ i \in I_m,$$

such that

$$\sum_{i \in \Delta I_m} \gamma(i)\big(\tau(i)\big)^\top A_j \tau(i) + \sum_{i \in I_m} \big(\lambda(i)\big)^\top A_j \tau(i) = 0,\ j = 0, 1, \ldots, n;\ \sum_{i \in \Delta I_m} \gamma(i) = 1. \tag{12}$$

It follows from the relations above that $\Delta I_m \ne \varnothing$, $\tau(i) \in T_{im} \cap T_m$, $i \in \Delta I_m$, and

$$\text{rank}\big(\tau(i),\ i \in I_{m+1}\big) \ge \text{rank}\big(\tau(i),\ i \in I_m\big) + 1,\ I_{m+1} := I_m \cup \Delta I_m. \tag{13}$$

From (11) and (12), we have

$$\sum_{i \in I_{m+1}} \gamma(i)\big(\tau(i)\big)^\top A_j \tau(i) + \sum_{i \in I_m} \big(\lambda^m(i)\big)^\top A_j \tau(i) = 0,\ j = 0, 1, \ldots, n, \tag{14}$$

where $\lambda^m(i) = \lambda(i)$, $i \in I_m \backslash I_{m-1}$; $\lambda^m(i) = \lambda(i) + \lambda^{m-1}(i)$, $i \in I_{m-1}$.

Go to the next iteration $\#(m+1)$ with the new data

$$\gamma(i) > 0,\ \tau(i) \in T_{im},\ i \in I_{m+1};\ \lambda^m(i) \in \mathbb{R}^p,\ i \in I_m, \tag{15}$$

which satisfies (13) and (14).

Final step. It follows from (13) that the algorithm consists of a finite number m_* of steps and $m_* \le p$. Hence, for some $0 \le m_* \le p$, the problem SIP_{m_*} has a feasible solution $(\bar{x}, \bar{\mu})$ with $\bar{\mu} < 0$.

If $m_* = 0$, then the constraints of the original SDP problem (1) satisfy the Slater condition. Hence, this problem is regular.

Suppose that $m_* > 0$. By the beginning of the final step, the immobile indices $\tau(i)$, $i \in I_{m_*}$, have been constructed. Consider a problem

$$\min_{x \in \mathbb{R}^n} c^\top x,\ \text{s. t. } \mathcal{A}(x)\tau(i) = 0,\ i \in I_{m_*},\ t^\top \mathcal{A}(x) t \ge 0,\ t \in T_{m_*}, \tag{16}$$

where $T_{m_*} = \{t \in T : t^\top \tau(i) = 0,\ i \in I_{m_*}\}$. Since $\tau(i) \in T_{im}$, $i \in I_{m_*}$, it follows from lemma 2 that the problem above is equivalent to problem (1) and can be considered as its regularisation since, by construction,

$$\mathcal{A}(\bar{x})\tau(i) = 0,\ i \in I_{m_*},\ t^\top \mathcal{A}(\bar{x}) t \ge -\bar{\mu} > 0,\ t \in T_{m_*},$$

and the index set T_{m_*} is a compact set. The regularisation procedure is described.

The described in this section regularisation procedure has the form of the algorithm whose iterations are described in all details and, therefore, it is more constructive than other regularisation procedures proposed for SDP problems in [6; 8]. It should be noted also that unlike the procedures from [6; 8] which are based on the FRA, the presented here procedure is based on the properties of the set of immobile indices that proves once again the important role that the immobile indices play in the study of optimisation problems.

Dual SDP formulations based on the immobile indices

In this section, for the linear SDP problem (1), we formulate several types of dual problems and discuss their properties.

The standard Lagrangian dual problem (SLD). For the SDP problem (1), the SLD has the form (see [10])

$$\text{SLD: } \max -U \bullet A_0, \text{ s. t. } U \bullet A_j = c_j,\ j = 1, 2, \dots, n;\ U \in \mathcal{S}_+^p.$$

Given a pair of mutually dual optimisation problems, the deference between the optimal values of the primal and dual objective functions is called the *duality gap*.

For the pair of dual problems (1) and SLD, the following results are known:

• *weak duality* (the duality gap is not negative);

• *strong duality* (if the constraints of problem (1) satisfy the Slater condition, then the duality gap vanishes);

• if the Slater condition is not satisfied, then the duality gap may be positive.

Let us now formulate for problem (1), two new dual problems (one in an implicit and another in an explicit form), for which the duality gap vanishes without any additional assumptions.

An implicit dual problem (IDP). Suppose that for problem (1), the set of immobile indices R_{im} is known. By lemma 1, the set R_{im} either consists of a unique zero element: $R_{im} = \{\mathbf{0}\}$, or it is a subspace in $\mathbb{R}^p$. Denote by $\{\xi(i),\ i \in I_b\}$, where $0 \le |I_b| \le p$, any basis of the subspace R_{im}. In the case $R_{im} = \{\mathbf{0}\}$, we consider that $I_b := \varnothing$.

Define a matrix $V_b := (\xi(i),\ i \in I_b) \in \mathbb{R}^{p \times p_*},\ p_* := |I_b|$, and consider a problem

$$\text{IDP: } \max -U \bullet A_0, \text{ s. t. } (U + \Lambda V_b) \bullet A_j = c_j,\ j = 1, \dots, n;\ U \in \mathcal{S}_+^p,\ \Lambda \in \mathbb{R}^{p \times p_*}.$$

Note that in the case $R_{im} = \{\mathbf{0}\}$, the problem IDP coincides with the problem SLD.

Based on lemma 2, we can prove the following theorem.

Theorem 1. *The problem IDP is dual to problem (1) and the strong duality relations hold true.*

To formulate the dual problem IDP, we have to know a basis of the subspace R_{im}. Hence this dual formulation can be considered as an implicit one.

A dual problem in the explicit form (EDP). Evidently, we are interested to have a dual SDP problem in an explicit form, where only data of the original primal problem (1) is used.

For a given finite integer $k_* \ge 0$, let us consider the following problem:

$$\max -(U + W_{k_*}) \bullet A_0,$$

$$\text{s. t. } (U + W_{m_*}) \bullet A_j = c_j,\ j = 1, 2, \dots, n;\ U \in \mathcal{S}_+^p,\ W_0 = \mathbb{O}_p;$$

$$\text{EDP}(k_*):\ (U_m + W_{m-1}) \bullet A_j = 0,\ j = 0, 1, \dots, n, \tag{17}$$

$$\begin{pmatrix} U_m & W_m \\ W_m^\top & D_m \end{pmatrix} \in \mathcal{S}_+^{2p},\ m = 1, \dots, k_*, \tag{18}$$

where $U_m \in S^p,\ D_m \in S^p,\ W_m \in \mathbb{R}^{p \times p},\ m = 1, \dots, k_*$, and $\mathbb{O}_p$ stays for the $p \times p$ null matrix.

Notice that in the case $k_* = 0$, the index set $\{1, \dots, k_*\}$ is supposed to be empty and then constraints (17) and (18) are missing in the problem $\text{EDP}(k_*)$. Hence, the problem $\text{EDP}(0)$ coincides with the problem SLD.

Lemma 3 [weak duality]. *Let $x \in X$ be a feasible solution of the primal linear SDP problem (1) and $U_m,\ W_m,\ D_m,\ m = 1, \dots, k_*$; U be a feasible solution of the problem $\text{EDP}(k_*)$. Then the following inequality holds*

$$c^\top x \ge -(U + W_{k_*}) \bullet A_0. \tag{19}$$

P r o o f. Given $m = 1, \ldots, k_*$, it follows from the condition (18) that there exists a matrix B_m in the form

$$B_m = \begin{pmatrix} V_m \\ L_m \end{pmatrix} \in \mathbb{R}^{2p \times k(m)} \text{ such that } \begin{pmatrix} U_m & W_m \\ W_m^\top & D_m \end{pmatrix} = B_m B_m^\top = \begin{pmatrix} V_m \\ L_m \end{pmatrix} \begin{pmatrix} V_m^\top & L_m^\top \end{pmatrix}.$$

The matrix B_m above is composed by the blocks containing matrices

$$V_m = \left(\tau^m(i),\ i \in I_m\right) \text{ and } L_m = \left(\lambda^m(i),\ i \in I_m\right), \tag{20}$$

where $\tau^m(i) \in \mathbb{R}^p$, $\lambda^m(i) \in \mathbb{R}^p$, $i \in I_m$, $k(m) := |I_m|$.

Hence, for $m = 1, \ldots, k_*$, the matrices U_m, W_m, D_m in $\mathrm{EDP}(k_*)$ admit representation

$$U_m = V_m V_m^\top,\ W_m = V_m L_m^\top,\ D_m = L_m L_m^\top. \tag{21}$$

Let us prove, first that for all $m = 1, \ldots, k_*$, it holds

$$\mathcal{A}(x)\tau^m(i) = 0,\ i \in I_m,\ \forall x \in X. \tag{22}$$

Consider the following constraints of the problem $\mathrm{EDP}(k_*)$:

$$U_1 \bullet A_j = 0,\ j = 0, 1, \ldots, n.$$

Due to (20) and (21), these constraints can be rewritten in the form

$$\sum_{i \in I_1} \left(\tau^1(i)\right)^\top A_j \tau^1(i) = 0,\ j = 0, 1, \ldots, n.$$

It follows from these equalities that for any $x \in \mathbb{R}^n$, we have

$$\sum_{i \in I_1} \left(\tau^1(i)\right)^\top \mathcal{A}(x) \tau^1(i) = 0. \tag{23}$$

Taking into account that the inequalities

$$t^\top \mathcal{A}(x) t \geq 0\ \forall t \in \mathbb{R}^p,\ \forall x \in X, \tag{24}$$

should be fulfilled, equality (23) implies $\left(\tau^1(i)\right)^\top \mathcal{A}(x)\tau^1(i) = 0$, $i \in I_1$, $\forall x \in X$.

Thus, one can conclude that $\tau^1(i) \in T_{im}$, $i \in I_1$, and, consequently (see proposition 1), it holds $\mathcal{A}(x)\tau^1(i) = 0$, $i \in I_1$, $\forall x \in X$. Hence, equalities (22) are valid for $m = 1$.

Suppose that for some $m \geq 1$, equalities (22) are proved. Due to (20) and (21), the constraints $\left(U_{m+1} + W_m\right) \bullet A_j = 0$, $j = 0, 1, \ldots, n$, of the problem $\mathrm{EDP}(k_*)$ can be rewritten in the form

$$\sum_{i \in I_{m+1}} \left(\tau^{m+1}(i)\right)^\top A_j \tau^{m+1}(i) + \sum_{i \in I_m} \left(\lambda^m(i)\right)^\top A_j \tau^m(i) = 0,\ j = 0, 1, \ldots, n.$$

It follows from the latter equalities that for any $x \in \mathbb{R}^n$, we have

$$\sum_{i \in I_{m+1}} \left(\tau^{m+1}(i)\right)^\top \mathcal{A}(x) \tau^{m+1}(i) + \sum_{i \in I_m} \left(\lambda^m(i)\right)^\top \mathcal{A}(x) \tau^m(i) = 0. \tag{25}$$

By the hypothesis above, equalities (22) are satisfied. Then, taking into account inequalities (24), we conclude from (25) that

$$\left(\tau^{m+1}(i)\right)^\top \mathcal{A}(x) \tau^{m+1}(i) = 0,\ i \in I_{m+1},\ i \in I_m,\ \forall x \in X.$$

Hence, $\tau^{m+1}(i) \in T_{im}$, $i \in I_{m+1}$, and, according to proposition 1, it holds

$$\mathcal{A}(x)\tau^{m+1}(i) = 0,\ i \in I_{m+1},\ \forall x \in X.$$

Replace m by $m + 1$ and repeat the considerations for all $1 < m < k_*$.

Let $m = k_*$. In this case, relations (22) have the form

$$\mathcal{A}(x)\tau^{k_*}(i) = 0,\ i \in I_{k_*},\ \forall x \in X, \tag{26}$$

and for $U = \sum_{i \in I} \tau(i)(\tau(i))^\top$, $\tau(i) \in \mathbb{R}^p$, $i \in I$, the constraints $(U + W_{k_*}) \bullet A_j = c_j$, $j = 1, \ldots, n$, of the problem EDP(k_*) can be presented in the form

$$\sum_{i \in I} (\tau(i))^\top A_j \tau(i) + \sum_{i \in I_{k_*}} \left(\lambda^{k_*}(i)\right)^\top A_j \tau^{k_*}(i) = c_j,\ j = 1, \ldots, n.$$

Then, it follows from the equalities above and relations (24), (26) that

$$\sum_{j=1}^{n} c_j x_j = \sum_{i \in I} (\tau(i))^\top \mathcal{A}(x) \tau(i) - (U + W_{k_*}) \bullet A_0 \geq -(U + W_{k_*}) \bullet A_0.$$

The lemma is proved.

Theorem 2 [strong duality]. *Suppose that the linear SDP problem (1) admits an optimal solution x^0. Then there exists a finite integer k_*, $0 \leq k_* \leq \min\{p, n\}$, such that the problem EDP(k_*) is dual to problem (1) and the strong duality relations are satisfied, i. e. the dual problem has an optimal solution*

$$\left(U_m^0,\ W_m^0,\ D_m^0,\ m = 1, \ldots, k_*;\ U^0\right) \tag{27}$$

and the following (the strong duality) equality holds

$$c^\top x^0 = -\left(U^0 + W_{k_*}^0\right) \bullet A_0. \tag{28}$$

P r o o f. To prove the theorem, we will construct the number k_* and matrices (27) using the regularisation procedure described in section 3. This procedure consists of the steps numbered from 0 to m_*, where $m_* \leq p$. At the beginning of each step #m, we have indices $\tau(i)$, $i \in I_m$. Denote

$$r(m) := \operatorname{rank}\begin{pmatrix} A_j \tau(i),\ j = 1, \ldots, n \\ i \in I_m \end{pmatrix} \leq n \text{ for } m \geq 1,\ r(0) = 0.$$

By construction, $r(m+1) \geq r(m)$, $m = 0, \ldots, m_* - 1$.

Let $m(k)$, $k = 0, 1, \ldots, k_*$, be the numbers of the steps on which the following relations are satisfied:

$$m(k) = m(k-1) + s(k), \text{ where } s(k) \geq 1,\ k = 0, 1, \ldots, k_*;$$

$$m(-1) = -1,\ m(k_*) = m_* \leq p,$$

$$r(m(k-1)+1) = r(m(k-1)+2) = \ldots = r(m(k-1) + s(k)),\ k = 0, 1, \ldots, k_*; \tag{29}$$

$$r(m(k+1)) > r(m(k)),\ k = 0, 1, \ldots, k_* - 1.$$

These relations and the conditions $r(m) \leq n$, $m(k_*) = m_* \leq p$, imply $k_* \leq \min\{p, n\}$.

Let us group together the steps of the procedure described in section 3, into iterations that have the numbers $k = 0, 1, \ldots, k_*$, as follows: the iteration #k consists of the steps with the numbers $m(k-1)+1$, $m(k-1) + 2, \ldots, m(k-1) + s(k) =: m(k)$.

For $0 \leq k \leq k_* - 1$, let us consider iteration #k and its endmost step, i. e. the step with the number $m = m(k)$. This step starts having the initial data (10) and ends having new data (15) that satisfies the relations (14). It can be shown, taking into account the equalities (29), that relations (14) with $m = m(k)$ can be rewritten in the form

$$\sum_{i \in I_{m(k)+1}} \gamma(i)(\tau(i))^\top A_j \tau(i) + \sum_{i \in I_{m(k-1)+1}} \left(\hat{\lambda}^k(i)\right)^\top A_j \tau(i) = 0,\ j = 0, 1, \ldots, n, \tag{30}$$

where $\hat{\lambda}^k(i) \in \mathbb{R}^p$, $i \in I_{m(k-1)+1}$, are some vectors which can differ from the vectors $\lambda^m(i)$, $i \in I_{m-1}$, in (14) with $m = m(k)$. Denote $\tilde{I}_{k+1} := I_{m(k)+1}$, $\tilde{I}_k := I_{m(k-1)+1}$ and set

$$V_{k+1} := \left(\sqrt{\gamma(i)}\tau(i),\ i \in \tilde{I}_{k+1}\right),\ \Lambda_k := \left(\hat{\lambda}^k(i)/\sqrt{\gamma(i)},\ i \in \tilde{I}_k\right),$$

$$W_k^0 := \Lambda_k V_k^\top,\ U_{k+1}^0 := V_{k+1} V_{k+1}^\top,\ D_k^0 := \Lambda_k \Lambda_k^\top. \tag{31}$$

By construction,

$$\begin{pmatrix} V_k \\ \Lambda_k \end{pmatrix} \begin{pmatrix} V_k^\top \Lambda_k^\top \end{pmatrix} = \begin{pmatrix} U_k^0 & W_k^0 \\ W_k^{0\top} & D_k^0 \end{pmatrix} \in S_+^{2p},$$

where V_k was constructed at the previous iteration $\#(k-1)$. Relations (30) take the form

$$U_{k+1}^0 \bullet A_j + W_k^0 \bullet A_j = 0,\ j = 0, 1, \dots, n. \tag{32}$$

Notice that for $k = 0$, the set $\tilde{I}_k = I_{m(k-1)+1} = I_0$ is empty, by construction. Hence at the step with the number $\#m(0)$ of the iteration #0, we construct only a matrix U_1^0 by the rule

$$U_1^0 := V_1 V_1^\top, \text{ where } V_1 := \left(\sqrt{\gamma(i)}\tau(i),\ i \in \tilde{I}_1\right). \tag{33}$$

Then relations (30) with $k = 0$ can be rewritten in the form

$$U_1^0 \bullet A_j = 0,\ j = 0, 1, \dots, n;\ U_1^0 \in S_+^p.$$

Consider the (last) iteration $\#k_*$. According to the regularisation procedure, this iteration ends on the step $\#m(k_*) = m_*$ after which we pass to the final step. Notice that, by construction, $0 \le k_* \le m_*$.

Let us consider the final step of the regularisation procedure.

If $m_* = 0$ and, consequently $k_* = 0$, then the constraints of the original SDP problem (1) satisfy the Slater condition. In this case, according to the well-known optimality conditions (see e. g. [1; 2]), if x^0 is an optimal solution of problem (1), then there exists a matrix $U^0 \in \mathcal{S}_+^p$ such that

$$U^0 \bullet A_j = c_j,\ j = 1, 2, \dots, n;\ U^0 \bullet \mathcal{A}\left(x^0\right) = 0.$$

It follows from the relations above that U^0 is a feasible solution of the dual problem $\mathrm{EDP}(0)$ and equality (28) holds.

Suppose that $m_* > 0$. By the beginning of the final step, the immobile indices and numbers $\tau(i),\ \gamma(i) > 0$, $i \in I_{m_*}$, are found and it was shown at the final step that problem (16) is regular and equivalent to the original problem (1).

Let x^0 be an optimal solution of the problem (1). Then vector x^0 is optimal in problem (16) as well. Hence, taking into account the regularity of this problem, we concluder that there exist indices, numbers and vectors $\tau(i) \in T_{m_*},\ \gamma(i) > 0,\ i \in I;\ \lambda^{m_*}(i) \in \mathbb{R}^p,\ i \in I_{m_*}$, such that

$$\begin{gathered} \sum_{i \in I} \gamma(i)\left(\tau(i)\right)^\top A_j \tau(i) + \sum_{i \in I_{m_*}} \left(\lambda^{m_*}(i)\right)^\top A_j \tau(i) = c_j,\ j = 1, \dots, n, \\ \left(\tau(i)\right)^\top \mathcal{A}\left(x^0\right)\tau(i) = 0,\ i \in I;\ \left(\lambda^{m_*}(i)\right)^\top \mathcal{A}\left(x^0\right)\tau(i) = 0,\ i \in I_{m_*}. \end{gathered} \tag{34}$$

Suppose that $k_* = 0$. Then, from equalities (29) with $k = 0$, it follows $A_j\tau(i) = \mathbf{0},\ j = 1, \dots, n,\ i \in I_{m_*}$, and, consequently, relations (34) take the form

$$\sum_{i \in I} \gamma(i)\left(\tau(i)\right)^\top A_j \tau(i) = c_j,\ j = 1, \dots, n,\ \left(\tau(i)\right)^\top \mathcal{A}\left(x^0\right)\tau(i) = 0,\ i \in I.$$

Let us set $V^0 := \left(\tau(i)\sqrt{\gamma(i)},\ i \in I\right),\ U^0 := V^0\left(V^0\right)^\top \in S_+^p$. Then it follows from the above relations that $U^0 \bullet A_j = c_j$, $j = 1, \dots, n$, and $-U^0 \bullet A_0 = c^\top x^0$. These equalities and inequality (19) imply that the matrix U^0 is an optimal solution of the problem $\mathrm{EDP}(0)$ and the strong duality relation holds.

Now suppose that $k_* > 0$. Notice that in this case we have found the matrices

$$U_k^0,\ W_k^0,\ D_k^0,\ k = 1, \dots, k_* - 1;\ W_0 = \mathbb{O}_p,\ U_{k_*}^0 = V_{k_*}^0\left(V_{k_*}^0\right)^\top,$$

where the matrix $V_{k_*}^0$ was constructed using the index set $\tilde{I}_{k_*} := I_{m(k_*-1)+1}$. Taking into account equalities (29) with $k = k_*$, it is easy to show that the optimality relations (34) can be rewritten in the form

$$\sum_{i \in I} \gamma(i)\left(\tau(i)\right)^{\top} A_j \tau(i) + \sum_{i \in \tilde{I}_{k_*}} \left(\hat{\lambda}^{k_*}(i)\right)^{\top} A_j \tau(i) = c_j, \; j = 1, \ldots, n, \tag{35}$$

$$\left(\tau(i)\right)^{\top} \mathcal{A}\left(x^0\right)\tau(i) = 0, \, i \in I; \; \left(\hat{\lambda}^{k_*}(i)\right)^{\top} \mathcal{A}\left(x^0\right)\tau(i) = 0, \; i \in \tilde{I}_{k_*} := I_{m(k_* - 1) + 1}. \tag{36}$$

Let us set

$$V^0 := \left(\tau(i)\sqrt{\gamma(i)}, \, i \in I\right), \; L^0_{k_*} := \left(\frac{\hat{\lambda}^{k_*}(i)}{\sqrt{\gamma(i)}}, \, i \in \tilde{I}_{k_*}\right),$$

$$U^0 := V^0\left(V^0\right)^{\top}, \; W^0_{k_*} := L^0_{k_*}\left(V^0_{m_*}\right)^{\top}, \; D^0_{m_*} := L^0_{m_*}\left(L^0_{m_*}\right)^{\top}. \tag{37}$$

Then, equalities (35) take the form

$$\left(U^0 + W^0_{k_*}\right) \bullet A_j = c_j, \; j = 1, \ldots, n, \tag{38}$$

and relations (36) imply

$$\left(U^0 + W^0_{k_*}\right) \bullet \mathcal{A}\left(x^0\right) = 0. \tag{39}$$

It follows from (32) and (38) that the constructed set of matrices (27) is a feasible solution of the problem $\text{EDP}(k_*)$, where $0 < k_* \le \min\{n, p\}$. Taking into account (38) and (39), we obtain

$$\sum_{j=1}^{n} c_j x_j^0 = \left(U^0 + W^0_{k_*}\right) \bullet \mathcal{A}\left(x^0\right) - \left(U^0 + W^0_{k_*}\right) \bullet A_0 = -\left(U^0 + W^0_{k_*}\right) \bullet A_0.$$

It follows from these relations and inequality (19) that the constructed set of matrices (27) is an optimal solution of the problem $\text{EDP}(k_*)$, and the strong duality relation holds. The theorem is proved.

The dual problem $\text{EDP}(k_*)$ is explicit and similar to the dual problem, which was obtained on the base of the FRA in [10]. There is only one difference in the formulations of these two problems, namely, in the dual problem in [10], all matrices D_m are equal to the identity matrix. Notice that the dual problem $\text{EDP}(k_*)$ is more general than the dual one in [10] since it is evident that any feasible solution of the dual problem from [10] is feasible for $\text{EDP}(k_*)$ as well.

In this paper, the dual problem $\text{EDP}(k_*)$ was obtained, using the concept of immobile indices and, as result, we have relations (31), (33), and (37), which provide us with additional information about matrices U_m, W_m, D_m, $m = 1, \ldots, k_*$; U that form a feasible solution of this dual problem and illustrate how these matrices can be expressed via immobile indices. This information can be used for constructing new and possibly less complex forms of dual SDP problems.

Conclusion

The main contribution of the paper consists of derivation of new theoretical results and algorithmic procedure for regularisation of the linear SDP problems, which permits to obtain new explicit dual formulations satisfying the strong duality property. This regularisation is based on the properties of immobile indices and is more constructive than the previously suggested in [6–8] procedures based on the FRA.

The results of the paper justify the fact that the concept of immobile indices plays an important role in the study of optimisation problems not satisfying regularity conditions and can be used for other classes of optimisation problems.

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