УДК 512.542

О НЕКОТОРЫХ СВОЙСТВАХ РЕШЕТКИ ТОТАЛЬНО σ-ЛОКАЛЬНЫХ ФОРМАЦИЙ КОНЕЧНЫХ ГРУПП

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Все рассматриваемые в статье группы являются конечными. Пусть $\sigma = \{\sigma_i | i \in I\}$ – некоторое разбиение множества всех простых чисел $\mathbb{P}$. Если n – целое число, G – группа и $\mathfrak{F}$ – класс групп, то $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \varnothing\}$, $\sigma(G) = \sigma(|G|)$ и $\sigma(\mathfrak{F}) = \cup_{G \in \mathfrak{F}} \sigma(G)$. Функция f вида $f : \sigma \to$ {формации групп} называется формационной σ-функцией. Для всякой формационной σ-функции f класс $LF_\sigma(f)$ определяется следующим образом:

$$LF_\sigma(f) = (G | G = 1 \text{ или } G \neq 1 \text{ и } G/O_{\sigma_i', \sigma_i}(G) \in f(\sigma_i) \text{ для всех } \sigma_i \in \sigma(G)).$$

Если для некоторой формационной σ-функции f имеет место $\mathfrak{F} = LF_\sigma(f)$, то класс $\mathfrak{F}$ называют σ-локальным, а формационную σ-функцию f – σ-локальным определением $\mathfrak{F}$. Всякую формацию считают 0-кратно σ-локальной.

Образец цитирования:
Сафонова ИН, Сафонов ВГ. О некоторых свойствах решетки тотально σ-локальных формаций конечных групп. *Журнал Белорусского государственного университета. Математика. Информатика.* 2020;3:6–16 (на англ.).
https://doi.org/10.33581/2520-6508-2020-3-6-16

For citation:
Safonova IN, Safonov VG. On some properties of the lattice of totally σ-local formations of finite groups. *Journal of the Belarusian State University. Mathematics and Informatics.* 2020; 3:6–16.
https://doi.org/10.33581/2520-6508-2020-3-6-16

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При $n > 0$ формацию $\mathfrak{F}$ называют n-кратно σ-локальной, если $\mathfrak{F} = (1)$ – класс всех единичных групп или $\mathfrak{F} = LF_\sigma(f)$, где $f(\sigma_i)$ является $(n-1)$-кратно σ-локальной формацией для всех $\sigma_i \in \sigma(\mathfrak{F})$. Формацию называют тотально σ-локальной, если она n-кратно σ-локальна для всякого целого неотрицательного числа n. Цель данной работы – изучение свойств решетки тотально σ-локальных формаций. В частности, мы доказываем, что решетка всех тотально σ-локальных формаций является алгебраической и дистрибутивной.

Ключевые слова: конечная группа; формационная σ-функция; формация конечных групп; тотально σ-локальная формация; решетка формаций.

ON SOME PROPERTIES OF THE LATTICE OF TOTALLY σ-LOCAL FORMATIONS OF FINITE GROUPS

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Throughout this paper, all groups are finite. Let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of the set of all primes $\mathbb{P}$. If n is an integer, G is a group, and $\mathfrak{F}$ is a class of groups, then $\sigma(n) = \{\sigma_i \mid \sigma_i \cap \pi(n) \neq \varnothing\}$, $\sigma(G) = \sigma(|G|)$, and $\sigma(\mathfrak{F}) = \cup_{G \in \mathfrak{F}} \sigma(G)$. A function f of the form $f : \sigma \to \{\text{formations of groups}\}$ is called a formation σ-function. For any formation σ-function f the class $LF_\sigma(f)$ is defined as follows:

$$LF_\sigma(f) = \left(G \text{ is a group} \mid G = 1 \text{ or } G \neq 1 \text{ and } G/O_{\sigma_i', \sigma_i}(G) \in f(\sigma_i) \text{ for all } \sigma_i \in \sigma(G)\right).$$

If for some formation σ-function f we have $\mathfrak{F} = LF_\sigma(f)$, then the class $\mathfrak{F}$ is called σ-local and f is called a σ-local definition of $\mathfrak{F}$. Every formaton is called 0-multiply σ-local. For $n > 0$, a formation $\mathfrak{F}$ is called n-multiply σ-local provided either $\mathfrak{F} = (1)$ is the class of all identity groups or $\mathfrak{F} = LF_\sigma(f)$, where $f(\sigma_i)$ is $(n-1)$-multiply σ-local for all $\sigma_i \in \sigma(\mathfrak{F})$. A formation is called totally σ-local if it is n-multiply σ-local for all non-negative integer n. The aim of this paper is to study properties of the lattice of totally σ-local formations. In particular, we prove that the lattice of all totally σ-local formations is algebraic and distributive.

Keywords: finite group; formation σ-function; formation of finite groups; totally σ-local formation; lattice of formations.

Introduction

All groups under consideration are finite. The notations and definitions we use are borrowed from [1–3]. The basic properties and various applications of σ-local formations can be found in the articles [4–10].

A. Skiba presented [4] the concept of generalised locality or σ-locality of formations as a tool for studying the σ-properties of groups, i. e. properties depending on some partition σ of the set of all primes. In [4], using σ-local formations, A. Skiba studied (weakly) S_t^σ-closed and (weakly) M_t^σ-closed classes of finite groups. Some general properties of σ-local formations as well as their applications for studying Σ_t^σ-closed classes of meta-σ-nilpotent groups [5] and (weakly) Γ_t^σ-closed classes of finite groups [6], were obtained Ch. Zhang and A. Skiba. Applications of the theory of σ-local formations were obtained by A. Skiba [7] for a lattice characterization of σ-soluble $P\sigma T$-groups, and also for constructing new sublattices of the lattice of all subgroups of the group generated by formation Fitting sets [10].

In [8; 9] Ch. Zhang, V. Safonov and A. Skiba described some general properties and examples of n-multiply σ-local formations and also consider one application of such formations in the theory of finite factorisable groups. In particular, in their paper [9] it was proved that the lattice of all n-multiply σ-local formations of finite groups is algebraic and modular.

A. Tsarev [11] proved that every law of the lattice of all formations is fulfilled in the lattice of all n-multiply σ-local formations of finite groups and that the lattice of all n-multiply σ-local formations of finite groups is modular but is not distributive for any non-negative integer n.

At the same time, the question on the algebraicity, modularity or distributivity of the lattice of all totally σ-local formations was an open problem. Note that the question on the distributivety or modularty of the lattice of all totally σ-local formations of finite groups was discussed by A. Tsarev in [11, question 3.2].

In this paper we will prove that the set l_∞^σ of all totally σ-local formations of finite groups is a complete algebraic and distributive lattice. In the work, we study also some general properties of totally σ-local formations of finite groups.

We also note that the concept of generalised locality of formations was developed in papers [12; 13], where the main properties and some examples of Baer-σ-local formations were considered.

Definitions and notations

The basic definitions, notations and general properties of σ-local formations were discussed in the papers [4–10]. Recall some of the basic concepts of the theory of σ-local formations.

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$. If n is an integer, G is a group, and $\mathfrak{F}$ is a class of groups, then $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \varnothing\}$, $\sigma(G) = \sigma(|G|)$, and $\sigma(\mathfrak{F}) = \cup_{G \in \mathfrak{F}} \sigma(G)$.

A group G is called [14]: σ-primary if G is a σ_i-group for some i; σ-nilpotent if every chief factor H/K of G is σ-central in G, that is, the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary; σ-soluble if $G = 1$ or $G \neq 1$ and every chief factor of G is σ-primary.

We write $\mathfrak{S}_\sigma$ to denote the class of all σ-soluble groups and $\mathfrak{N}_\sigma$ to denote the class of all σ-nilpotent groups. A class of groups $\mathfrak{F}$ is called a formation if: (1) $G/N \in \mathfrak{F}$ whenever $G \in \mathfrak{F}$, and (2) $G/N \cap R \in \mathfrak{F}$ whenever $G/N \in \mathfrak{F}$ and $G/R \in \mathfrak{F}$.

Any function f of the form $f : \sigma \to \{\text{formations of groups}\}$ is called a formation σ-function. For any formation σ-function f the class $LF_\sigma(f)$ is defined as follows:

$$LF_\sigma(f) = (G \text{ is a group} | G = 1 \text{ or } G \neq 1 \text{ and } G/O_{\sigma_i', \sigma_i}(G) \in f(\sigma_i) \text{ for all } \sigma_i \in \sigma(G)).$$

If for some formation σ-function f we have $\mathfrak{F} = LF_\sigma(f)$, then the class $\mathfrak{F}$ is called σ-local and f is called a σ-local definition of $\mathfrak{F}$. We write $F_{\{\sigma_i\}}(G)$ instead of $O_{\sigma_i', \sigma_i}(G) = G_{\mathfrak{G}_{\sigma_i'}\mathfrak{G}_{\sigma_i}}$.

Every formation is called 0-multiply σ-local. For $n > 0$, the formation $\mathfrak{F}$ is called n-multiply σ-local provided either $\mathfrak{F} = (1)$ is the class of all identity groups or $\mathfrak{F} = LF_\sigma(f)$, where $f(\sigma_i)$ is $(n-1)$-multiply σ-local for all $\sigma_i \in \sigma(\mathfrak{F})$. A formation is called totally σ-local if it is n-multiply σ-local for all non-negative integer n.

The symbol l_∞^σ denotes the set of all totally σ-local formations. Formations from l_∞^σ are called l_∞^σ-formations.

For any collection of groups $\mathfrak{X}$, $l_\infty^\sigma \text{ form } (\mathfrak{X})$ denotes the totally σ-local formation generated by $\mathfrak{X}$, i. e. $l_\infty^\sigma \text{ form } (\mathfrak{X})$ is the intersection of all totally σ-local formations containing the collection of groups $\mathfrak{X}$. If $\mathfrak{X} = \{G\}$ for some group G, then $\mathfrak{F} = l_\infty^\sigma \text{ form } (G)$ is called a one-generated totally σ-local formation. For any two classes of groups $\mathfrak{M}$ and $\mathfrak{H}$ we put $\mathfrak{M} \vee_\infty^\sigma \mathfrak{H} = l_\infty^\sigma \text{ form } (\mathfrak{M} \cup \mathfrak{H})$.

If f is a formation σ-function, then the symbol $\mathrm{Supp}(f)$ denotes the support of f, that is, the set of all σ_i such that $f(\sigma_i) \neq \varnothing$. A formation σ-function f is called l_∞^σ-valued if $f(\sigma_i)$ is a totally σ-local formation for every $\sigma_i \in \mathrm{Supp}(f)$; integrated if $f(\sigma_i) \subseteq LF_\sigma(f)$ for all i.

If m and h are l_∞^σ-valued formation σ-functions, then $m \vee_\infty^\sigma h$ is a formation σ-function such that $(m \vee_\infty^\sigma h)(\sigma_i) = m(\sigma_i) \vee_\infty^\sigma h(\sigma_i)$ for all i; we use also $m \cap h$ to denote the formation σ-function such that $(m \cap h)(\sigma_i) = m(\sigma_i) \cap h(\sigma_i)$ for all i.

Every sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ from σ is called a σ-sequence. For any σ-sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ and for any collection of groups $\mathfrak{X}$, the class of groups $\mathfrak{X}(\alpha_1\alpha_2 \ldots \alpha_n)$ is defined recursively in the following way: (1) $\mathfrak{X}(\alpha_1) = (G/F_{\{\alpha_1\}}(G) | G \in \mathfrak{X})$; (2) $\mathfrak{X}(\alpha_1\alpha_2 \ldots \alpha_n) = (G/F_{\{\alpha_n\}}(G) | G \in \mathfrak{X}(\alpha_1\alpha_2 \ldots \alpha_{n-1}))$.

For any l_∞^σ-formation $\mathfrak{F}$, we set $\mathfrak{F}_\infty^\sigma(\sigma_i) = l_\infty^\sigma \text{ form } (\mathfrak{F}(\sigma_i))$, if $\sigma_i \in \sigma(\mathfrak{F})$, and $\mathfrak{F}_\infty^\sigma(\sigma_i) = \varnothing$, if $\sigma_i \notin \sigma(\mathfrak{X})$.

If $\mathfrak{F} \in l_\infty^\sigma$, then the symbol $\mathfrak{F}_\infty^\sigma$ denotes the smallest l_∞^σ-valued definition of $\mathfrak{F}$, i. e. $\mathfrak{F}_\infty^\sigma = \cap_{j \in J} f_j$, where $\{f_j | j \in J\}$ is the set of all l_∞^σ-valued definitions of $\mathfrak{F}$.

We say that a σ-sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ is suitable for $\mathfrak{F}$ (or $\mathfrak{F}$-suitable), if $\alpha_1 \in \sigma(\mathfrak{F})$ and for any $j \in \{2, \ldots, n\}$ we have $\alpha_j \in \sigma(\mathfrak{F}(\alpha_1\alpha_2 \ldots \alpha_{j-1}))$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an $\mathfrak{F}$-suitable σ-sequence. Then the l_∞^σ-valued σ-function $\mathfrak{F}_\infty^\sigma \alpha_1\alpha_2 \ldots \alpha_n$ is defined recursively as follows: (1) $\mathfrak{F}_\infty^\sigma \alpha_1 = \left(\mathfrak{F}_\infty^\sigma(\alpha_1)\right)_\infty^\sigma$; (2) $\mathfrak{F}_\infty^\sigma \alpha_1 \ldots \alpha_n = \left(\mathfrak{F}_\infty^\sigma \alpha_1 \ldots \alpha_{n-1}(\alpha_n)\right)_\infty^\sigma$.

For any group G and a non-empty formation $\mathfrak{F}$ by $G^{\mathfrak{F}}$ denote the $\mathfrak{F}$-residual of G, i. e. the intersection of all subgroups N of G such that $G/N \in \mathfrak{F}$. If $\mathfrak{F}$ and $\mathfrak{H}$ are formations, then $\mathfrak{F}\mathfrak{H} = \left\{G \mid G^{\mathfrak{H}} \in \mathfrak{F}\right\}$ is called the Gaschütz product of formations $\mathfrak{F}$ and $\mathfrak{H}$.

Auxiliary results

We need some well-known results, which we present in the form of the following lemmas.

Lemma 1 [9]. *If the class of groups $\mathfrak{F}_j$ is an n-multiply σ-local formation for all $j \in J$, then the class $\cap_{j \in J} \mathfrak{F}_j$ is also n-multiply σ-local formation.*

Recall that if f is a formation σ-function, then the symbol $\mathrm{Supp}(f)$ denotes the support of f, that is, the set of all σ_i such that $f(\sigma_i) \neq \varnothing$.

Lemma 2 [5; 9]. *Let f and h be formation σ-functions and let $\Pi = \mathrm{Supp}(f)$. Suppose that $\mathfrak{F} = LF_\sigma(f) = LF_\sigma(h)$.*

(1) $\Pi = \sigma(\mathfrak{F})$.

(2) $\mathfrak{F} = \left(\cap_{\sigma_i \in \Pi} \mathfrak{G}_{\sigma_i'} \mathfrak{G}_{\sigma_i} f(\sigma_i)\right) \cap \mathfrak{G}_\Pi$. *Hence $\mathfrak{F}$ is a saturated formation.*

(3) If every group in $\mathfrak{F}$ is σ-soluble, then $\mathfrak{F} = \left(\cap_{\sigma_i \in \Pi} \mathfrak{S}_{\sigma_i'} \mathfrak{S}_{\sigma_i} f(\sigma_i)\right) \cap \mathfrak{S}_\Pi$.

(4) If $\sigma_i \in \Pi$, then $\mathfrak{G}_{\sigma_i}\left(f(\sigma_i) \cap \mathfrak{F}\right) = \mathfrak{G}_{\sigma_i}\left(h(\sigma_i) \cap \mathfrak{F}\right) \subseteq \mathfrak{F}$.

(5) $\mathfrak{F} = LF_\sigma(F)$, where F is the unique formation σ-function such that $F(\sigma_i) = \mathfrak{G}_{\sigma_i} F(\sigma_i) \subseteq \mathfrak{F}$ for all $\sigma_i \in \Pi$ and $F(\sigma_i) = \varnothing$ for all $\sigma_i \in \Pi'$. Moreover, $F(\sigma_i) = \mathfrak{G}_{\sigma_i}\left(f(\sigma_i)\right) \cap \mathfrak{F}$ for all i.

Lemma 3 [9]. *If $\mathfrak{F}$ is a non-empty formation and $f(\sigma_i) = \mathfrak{F}$ for all i, then $LF_\sigma(f) = \mathfrak{N}_\sigma \mathfrak{F}$.*

Lemma 4 [9]. *If $\mathfrak{F} = \cap_{j \in J} \mathfrak{F}_j$ and $\mathfrak{F}_j = LF_\sigma(f_j)$ for all $j \in J$, then $\mathfrak{F} = LF_\sigma(f)$, where $f(\sigma_i) = \cap_{j \in J} f_j(\sigma_i)$ for all $\sigma_i \in \sigma(\mathfrak{F}) = \cap_{j \in J} \sigma(\mathfrak{F}_j)$ and $f(\sigma_i) = \varnothing$ for all $\sigma_i \in \sigma \backslash \sigma(\mathfrak{F})$. Moreover, if f_j is integrated for all $j \in J$, then f is also integrated.*

Lemma 5 [2, p. 41]. *Let A be a monolithic group and let $\mathrm{Soc}(A)$ be a non-abelian group. Let $\mathfrak{M}$ be some homomorph. If $A \in l_n$* form *$\mathfrak{M}$, then $A \in \mathfrak{M}$.*

Lemma 6 [2, p. 152]. *Let G be a group such that $O_p(G) = 1$, let $N_1 \times \ldots \times N_k = \mathrm{Soc}(G)$, where N_i is a minimal normal subgroup of G (k ≥ 2). Let M_i denote a maximal normal subgroup of G, which contains $N_1 \times \ldots \times N_{i-1} \times N_{i+1} \times \ldots \times N_k$ and does not contain N_i, $i \in \{1, \ldots, k\}$. Then*

(a) the group G/M_i is a monolithic and $\mathrm{Soc}(G/M_i) = N_i M_i / N_i$ for any $i \in \{1, \ldots, k\}$;

(b) $N_i M_i / N_i$ is G-isomorphic to N_i;

(c) $O_p(G/M_i) = 1$;

(d) $M_1 \cap \ldots \cap M_k = 1$.

The main results

Let $\mathfrak{X}$ be some collection of groups, $\sigma_i \in \sigma(\mathfrak{X})$, then the class of groups $\mathfrak{X}(\sigma_i)$ is defined as follows: $\mathfrak{X}(\sigma_i) = \left(G/F_{\{\sigma_i\}}(G) \mid G \in \mathfrak{X}\right)$.

Lemma 7. *Let $\mathfrak{F} = l_\infty^\sigma$* form *$(\mathfrak{X}) = LF_\sigma(f)$ be the totally σ-local formation generated by $\mathfrak{X}$, where f is an l_∞^σ-valued definition of $\mathfrak{F}$, and let $\Pi = \sigma(\mathfrak{X})$. Let h be the formation σ-function such that $h(\sigma_i) = l_\infty^\sigma$* form *$\left(\mathfrak{X}(\sigma_i)\right)$ for all $\sigma_i \in \Pi$ and $h(\sigma_i) = \varnothing$ for all $\sigma_i \in \Pi'$. Then*

(1) $\Pi = \sigma(\mathfrak{F})$;

(2) h is an l_∞^σ-valued definition of $\mathfrak{F}$;

(3) $h(\sigma_i) \subseteq f(\sigma_i) \cap \mathfrak{F}$ and for all i.

P r o o f. Since $\mathfrak{X} \subseteq \mathfrak{F}$ we have $\Pi \subseteq \sigma(\mathfrak{F})$. In view of [9, remark 2.4 (ii)], the class of all Π-groups $\mathfrak{G}_\Pi$ is a totally σ-local formation. Hence $\mathfrak{F} \subseteq \mathfrak{G}_\Pi$. Therefore, $\sigma(\mathfrak{F}) \subseteq \Pi$ and statement (1) holds.

Let $\mathfrak{H} = LF_\sigma(h)$. Then it is clear that $\mathfrak{X} \subseteq \mathfrak{H}$. On the other hand, since h is an l_∞^σ-valued, which implies that $\mathfrak{H}$ is a totally σ-local formation. Therefore, $\mathfrak{F} \subseteq \mathfrak{H}$.

Since $\mathfrak{X}(\sigma_i) \subseteq f(\sigma_i)$ and the formation $f(\sigma_i)$ is totally σ-local we have $h(\sigma_i) \subseteq f(\sigma_i) \cap \mathfrak{F}$ for all $\sigma_i \in \sigma$. Therefore, $\mathfrak{H} \subseteq \mathfrak{F}$. Hence $\mathfrak{F} = \mathfrak{H}$ and so statements (2) and (3) hold. The lemma is proved.

Lemma 8. *Let $\mathfrak{F}_j = LF_\sigma(f_j)$ be a totally σ-local formation, where f_i is the smallest l_∞^σ-valued definition of $\mathfrak{F}_j$, $j \in J$. Then $\vee_\infty^\sigma(f_j \mid j \in J)$ is the smallest l_∞^σ-valued definition of $\mathfrak{F} = \vee_\infty^\sigma(\mathfrak{F}_j \mid j \in J)$.*

P r o o f. Let l be the smallest l_∞^σ-valued definition of $\mathfrak{F}$, $f = \vee_\infty^\sigma(f_j \mid j \in J)$, and $\Pi = \sigma(\cup_{j \in J} \mathfrak{F}_j) = \cup_{j \in J} \sigma(\mathfrak{F}_j)$. Then $\sigma(\mathfrak{F}) = \Pi$ by lemma 7 (1). Now we show that $l(\sigma_i) = f(\sigma_i)$ for all $\sigma_i \in \sigma$.

Let $\sigma_i \in \sigma \backslash \Pi$. Then for any $j \in J$ we have $f_j(\sigma_i) = \varnothing$ by lemma 7. Hence $f(\sigma_i) = \varnothing$. Similary, in view of lemma 7, $l(\sigma_i) = \varnothing$. Therefore, $l(\sigma_i) = f(\sigma_i)$.

Now suppose that $\sigma_i \in \Pi$. Then there exists $j_t \in J$ such that $\sigma_i \in \sigma(\mathfrak{F}_{j_t})$. From lemma 7 it follows that $f_{j_t}(\sigma_i) \neq \varnothing$ and

$$l(\sigma_i) = l_\infty^\sigma \text{ form}\left(G/F_{\{\sigma_i\}}(G) \middle| G \in \cup_{j \in J} \mathfrak{F}_j\right) = l_\infty^\sigma \text{ form}\left(\cup_{j \in J} l_\infty^\sigma \text{ form}\left(G/F_{\{\sigma_i\}}(G) \middle| G \in \mathfrak{F}_j\right)\right) =$$

$$= l_\infty^\sigma \text{ form}\left(\cup_{j \in J} f_j(\sigma_i) \middle| j \in J\right) = \left(\vee_\infty^\sigma(f_j \mid j \in J)\right)(\sigma_i) = f(\sigma_i).$$

Therefore, $l(\sigma_i) = f(\sigma_i)$ for all $\sigma_i \in \Pi$. Thus, $l = f$. The lemma is proved.

Lemma 9. *Let $\mathfrak{H}_j = LF_\sigma(h_j)$, where h_j is integrated l_∞^σ-valued definition of $\mathfrak{H}_j$, $j = 1, 2$. Then $\mathfrak{H} = \mathfrak{H}_1 \vee_\infty^\sigma \mathfrak{H}_2 = LF_\sigma(h)$, where $h = h_1 \vee_\infty^\sigma h_2$ is integrated.*

P r o o f. Let l be the smallest l_∞^σ-valued definition of $\mathfrak{H}$ and let H be the canonical σ-local definition of $\mathfrak{H}$. Let l_j be the smallest l_∞^σ-valued definition of $\mathfrak{H}_j$ and let H_j be the canonical σ-local definition of $\mathfrak{H}_j$, $j = 1, 2$. In view of lemmas 2 (5) and 7 we have $l_j(\sigma_i) \subseteq h_j(\sigma_i) \subseteq H_j(\sigma_i)$ for all σ_i. Besides, lemmas 2 (5) and 7 imply also that

$$l(\sigma_i) = l_\infty^\sigma \text{ form}\left((\mathfrak{H}_1 \cup \mathfrak{H}_2)(\sigma_i)\right) = l_\infty^\sigma \text{ form}\left(\mathfrak{H}_1(\sigma_i) \cup \mathfrak{H}_2(\sigma_i)\right) = l_\infty^\sigma \text{ form}\left(l_1(\sigma_i) \cup l_2(\sigma_i)\right) \subseteq$$

$$\subseteq l_\infty^\sigma \text{ form}\left(h_1(\sigma_i) \cup h_2(\sigma_i)\right) = h(\sigma_i) \subseteq \mathfrak{G}_{\sigma_i} l(\sigma_i) = H(\sigma_i).$$

Hence $l(\sigma_i) \subseteq h(\sigma_i) \subseteq H(\sigma_i)$ for all σ_i. Therefore, $\mathfrak{H} = LF_\sigma(h)$. The lemma is proved.

Lemma 10. *Let $\mathfrak{F}$ be a non-empty formation. Then the formation $\mathfrak{S}_\sigma \mathfrak{F}$ is totally σ-local.*

P r o o f. Let $\mathfrak{M} = \mathfrak{S}_\sigma \mathfrak{F}$. By lemma 3 the formation $\mathfrak{H} = \mathfrak{N}_\sigma \mathfrak{M}$ is σ-local and $\mathfrak{H} = LF_\sigma(h)$, where $h(\sigma_i) = \mathfrak{M}$ for all $\sigma_i \in \sigma$. Since the Gaschütz product of formations is associative,

$$\mathfrak{H} = \mathfrak{N}_\sigma \mathfrak{M} = \mathfrak{N}_\sigma(\mathfrak{S}_\sigma \mathfrak{F}) = (\mathfrak{N}_\sigma \mathfrak{S}_\sigma)\mathfrak{F} = \mathfrak{S}_\sigma \mathfrak{F} = \mathfrak{M}.$$

Therefore $\mathfrak{M}$ is a σ-local formation. On the other hand, $h(\sigma_i) = \mathfrak{M}$ for all $\sigma_i \in \sigma$. Hence $\mathfrak{M} = \mathfrak{H} = LF_\sigma(h)$ is 2-multyply σ-local. Therefore, the formation $\mathfrak{M}$ is n-multiply σ-local for any positive integer n. Consequently $\mathfrak{M}$ is totally σ-local. The lemma is proved.

Lemma 11. *If $\mathfrak{F} = LF_\sigma(f)$ and $G/O_\sigma(G) \in f(\sigma_i) \cap \mathfrak{F}$ for some $\sigma_i \in \sigma(G)$, then $G \in \mathfrak{F}$.*

P r o o f. Since $G/O_{\sigma_i}(G) \in f(\sigma_i) \cap \mathfrak{F}$, we have $f(\sigma_i) \neq \varnothing$. But then $\sigma_i \in \sigma(\mathfrak{F})$ by lemma 2 (1). Moreover, $G/O_{\sigma_i}(G) \in f(\sigma_i) \cap \mathfrak{F}$ imply also that $G^{f(\sigma_i) \cap \mathfrak{F}} \subseteq O_{\sigma_i}(G) \in \mathfrak{G}_{\sigma_i}$. Hence $G \in \mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F})$. In view of lemma 2 (4) we have $G \in \mathfrak{G}_{\sigma_i}(f(\sigma_i) \cap \mathfrak{F}) \subseteq \mathfrak{F}$. The lemma is proved.

Lemma 12. *Let $\mathfrak{F}_j = LF_\sigma(f_j)$, where f_j is an integrated l_∞^σ-valued definition of $\mathfrak{F}_j$, $j \in J$. If $\mathfrak{F} = \cap(\mathfrak{F}_j \mid j \in J)$, then $f = \cap(f_j \mid j \in J)$ is an integrated l_∞^σ-valued definition of $\mathfrak{F}$.*

P r o o f. First note that $\mathfrak{F}_j$ is an n-multiply σ-local formation for all positive integer n. Then in view of lemma 1, $\cap_{j \in J} \mathfrak{F}_j$ is an n-multiply σ-local formation for all positive integer n. Therefore $\cap_{j \in J} \mathfrak{F}_j$ is totally σ-local. Besides, since $f = \cap(f_j \mid j \in J)$ and f_j is an l_∞^σ-valued definition of $\mathfrak{F}_j$, we see that f is an l_∞^σ-valued σ-function.

Let $\mathfrak{H} = LF_\sigma(f)$. Now we show that $\mathfrak{H} = \mathfrak{F}$. First assume that $G \in \mathfrak{H}$ and let $\sigma_i \in \sigma(G)$. Then $G/F_{\{\sigma_i\}}(G) \in f(\sigma_i) = \cap_{j \in J} f_j(\sigma_i)$. Hence $G/F_{\{\sigma_i\}}(G) \in f_j(\sigma_i)$ for all $j \in J$. Therefore for all $\sigma_i \in \sigma(G)$ we obtain $G/F_{\{\sigma_i\}}(G) \in f_j(\sigma_i)$. But then $G \in \mathfrak{F}_j$ for all $j \in J$ and consequently, $G \in \cap_{j \in J} \mathfrak{F}_j = \mathfrak{F}$. Thus $\mathfrak{H} \subseteq \mathfrak{F}$.

Now suppose that $G \in \mathfrak{F} = \cap_{j \in J} \mathfrak{F}_j$ and let $\sigma_i \in \sigma(G)$. Then $G \in \mathfrak{F}_j$ for all $j \in J$. Therefore, $G/F_{\{\sigma_i\}}(G) \in f_j(\sigma_i)$ for all $j \in J$. Hence $G/F_{\{\sigma_i\}}(G) \in \cap_{j \in J} f_j(\sigma_i) = f(\sigma_i)$ for all $\sigma_i \in \sigma(G)$. Consequently, $G \in \mathfrak{H}$ and $\mathfrak{F} \subseteq \mathfrak{H}$.

Finally, since f_j is integrated for all $j \in J$, we have $f(\sigma_i) = \cap_{j \in J} f_j(\sigma_i) \subseteq f_j(\sigma_i) \subseteq \mathfrak{F}_j$ for all $\sigma_i \in \sigma$. Hence $f(\sigma_i) \subseteq \cap_{j \in J} \mathfrak{F}_j = \mathfrak{F}$ and f is an integrated l_∞^σ-valued definition of $\mathfrak{F}$. The lemma is proved.

Algebraicity of the lattice of all totally σ-local formations

Theorem 1. *The set l_∞^σ of all totally σ-local formations is a complete algebraic lattice in which, for any set $\{\mathfrak{F}_j \mid j \in J\} \subseteq l_\infty^\sigma$, the intersection $\cap_{j \in J} \mathfrak{F}_j$ is the greatest lower bound and $l_\infty^\sigma \operatorname{form}(\cup_{j \in J} \mathfrak{F}_j)$ is the smallest upper bound of $\{\mathfrak{F}_j \mid j \in J\}$ in l_∞^σ.*

P r o o f. It is clear that the set l_∞^σ is partially ordered with respect to set inclusion. Since by [9, remark 2.4 (ii)], the formation of all groups $\mathfrak{G}$ is totally σ-local we have $\mathfrak{G}$ is the largest element in l_∞^σ. It follows from lemma 12 that $\cap_{j \in J} \mathfrak{F}_j \in l_\infty^\sigma$. Therefore, $\cap_{j \in J} \mathfrak{F}_j$ is the greatest lower bound of $\{\mathfrak{F}_j \mid j \in J\}$ in l_∞^σ, which implies that $l_\infty^\sigma \operatorname{form}(\cup_{j \in J} \mathfrak{F}_j)$ is the smallest upper bound of $\{\mathfrak{F}_j \mid j \in J\}$ in l_∞^σ.

Now we show that for every group A the one-generated totally σ-local formation $l_\infty^\sigma \operatorname{form}(A)$ is a compact element in l_∞^σ. Let A is counterexample minimal order and

$$\mathfrak{F} = l_\infty^\sigma \operatorname{form}(A) \subseteq l_\infty^\sigma \operatorname{form}(\cup_{j \in J} \mathfrak{F}_j),$$

where $\mathfrak{F}_j$ is a totally σ-local formation, $j \in J$. If A is a σ_i-group for some i, then $\mathfrak{F} = \mathfrak{G}_{\sigma_i}$. Since $\mathfrak{F} \subseteq l_\infty^\sigma \operatorname{form}(\cup_{j \in J} \mathfrak{F}_j)$, we have $\sigma(\mathfrak{F}) \subseteq \sigma(\cup_{j \in J} \mathfrak{F}_j) = \cup_{j \in J} \sigma(\mathfrak{F}_j)$ by lemma 7. Therefore, there is j_s such that $\sigma_i \in \sigma(\mathfrak{F}_{j_s})$. But then $\mathfrak{F} = \mathfrak{G}_{\sigma_i} \subseteq \mathfrak{F}_{j_s}$ by lemma 2. This contradiction shows that A is not σ-primary.

Now we show that A is monolithic. Suppose that it is false and let N_1, N_2 be minimal normal subgroups of A, where $N_1 \neq N_2$. Let $\mathfrak{L} = l_\infty^\sigma \operatorname{form}(A/N_1)$, $\mathfrak{M} = l_\infty^\sigma \operatorname{form}(A/N_2)$. It is clear that $\mathfrak{F} = \mathfrak{L} \vee_\infty^\sigma \mathfrak{M}$. By inductive hypothesis for groups A/N_1 and A/N_2 our statement is true. Then since

$$\mathfrak{L} = l_\infty^\sigma \operatorname{form}(A/N_1) \subseteq \mathfrak{F} \subseteq l_\infty^\sigma \operatorname{form}(\cup_{j \in J} \mathfrak{F}_j),\ \ \mathfrak{M} = l_\infty^\sigma \operatorname{form}(A/N_2) \subseteq \mathfrak{F} \subseteq l_\infty^\sigma \operatorname{form}(\cup_{j \in J} \mathfrak{F}_j),$$

there are $j_1, \ldots, j_k$ and $s_1, \ldots, s_n$ such that

$$\mathfrak{L} \subseteq l_\infty^\sigma \operatorname{form}(\mathfrak{F}_{j_1} \cup \ldots \cup \mathfrak{F}_{j_k}) \text{ and } \mathfrak{M} \subseteq l_\infty^\sigma \operatorname{form}(\mathfrak{F}_{s_1} \cup \ldots \cup \mathfrak{F}_{s_n}).$$

But then we have

$$\mathfrak{F} = \mathfrak{L} \vee_\infty^\sigma \mathfrak{M} \subseteq l_\infty^\sigma \operatorname{form}(\mathfrak{F}_{j_1} \cup \ldots \cup \mathfrak{F}_{j_k} \cup \mathfrak{F}_{s_1} \cup \ldots \cup \mathfrak{F}_{s_n}),$$

a contradiction. Hence A is a monolithic group.

Let $P = \operatorname{Soc}(A)$. Assume that P is not a σ-primary group. Since $A \in l_\infty^\sigma \operatorname{form}(\cup_{j \in J} \mathfrak{F}_j)$ we have $A \in \mathfrak{G}_\sigma \operatorname{form}(\cup_{j \in J} \mathfrak{F}_j)$ by lemma 10. But P is not a σ-primary group, therefore, $A \in \operatorname{form}(\cup_{j \in J} \mathfrak{F}_j)$. Using lemma 5, we obtain $A \in \cup_{j \in J} \mathfrak{F}_j$. Hence there is $j_m \in J$ such that $A \in \mathfrak{F}_{j_m}$. This contradiction shows that P is a σ_i-group for some i. Therefore, $F_{\{\sigma_i\}}(A) = O_{\sigma_i}(A)$.

Let f_i, f, h are smallest l_∞^σ-valued definitions of formations $\mathfrak{F}_j$, $\mathfrak{F}$ and $\mathfrak{H} = l_\infty^\sigma \operatorname{form}(\cup_{j \in J} \mathfrak{F}_j)$, respectively. In view of lemma 8 we have $h = \vee_\infty^\sigma (f_j \mid j \in J)$. Since $O_{\sigma_i}(A) = F_{\{\sigma_i\}}(A)$ and $A \in \mathfrak{H}$, we have

$$A/O_{\sigma_i}(A) = A/F_{\{\sigma_i\}}(A) \in h(\sigma_i) = \vee_\infty^\sigma (f_j(\sigma_i) \mid j \in J).$$

Since $|A/O_{\sigma_i}(A)| < |A|$, by our inductive hypothesis there are $j_1, \ldots, j_t \in J$ such that

$$l_\infty^\sigma \operatorname{form}\left(A/O_{\sigma_i}(A)\right) \subseteq f_{j_1}(\sigma_i) \vee_\infty^\sigma \ldots \vee_\infty^\sigma f_{j_t}(\sigma_i).$$

By lemma 8, $l = f_{j_1} \vee_\infty^\sigma \ldots \vee_\infty^\sigma f_{j_t}$ is the smallest l_∞^σ-valued definition of $\mathfrak{L} = \mathfrak{F}_{j_1} \vee_\infty^\sigma \ldots \vee_\infty^\sigma \mathfrak{F}_{j_t}$. But then $A/O_{\sigma_i}(A) \in l(\sigma_i) = f_{j_1}(\sigma_i) \vee_\infty^\sigma \ldots \vee_\infty^\sigma f_{j_t}(\sigma_i)$. Since l is integrated, $A \in \mathfrak{L}$ by lemma 11.

Thus, $\mathfrak{F} = l_\infty^\sigma \operatorname{form}(A) \subseteq \mathfrak{L} = \mathfrak{F}_{j_1} \vee_\infty^\sigma \ldots \vee_\infty^\sigma \mathfrak{F}_{j_t}$. This contradiction shows that every one-generated totally σ-local formation $l_\infty^\sigma \operatorname{form}(A)$ is a compact element in l_∞^σ.

It is clear that for any totally σ-local formation $\mathfrak{F}$ we have $\mathfrak{F} = l_\infty^\sigma \operatorname{form}\left(\cup_{t \in T} \mathfrak{F}_t\right)$, where $\left\{\mathfrak{F}_t \mid t \in T\right\}$ is the set of all one-generated totally σ-local formations contained in $\mathfrak{F}$. Hence the lattice l_∞^σ is algebraic. The theorem is proved.

In the classical case, when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \ldots\}$ we get from theorem 1 the following known results.

Corollary 1 [15]. *The lattice l_∞ of all totally local formations is algebraic.*

Corollary 2 [2, p. 180]. *The lattice of all soluble totally local formations is algebraic.*

Distributivety of the lattice of all totally σ-local formations

Recall that if $\mathfrak{F}$ is a totally σ-local formation, then $\mathfrak{F}_\infty^\sigma$ denotes the smallest l_∞^σ-valued definition of $\mathfrak{F}$. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is an $\mathfrak{F}$-suitable σ-sequence, then the l_∞^σ-valued σ-function $\mathfrak{F}_\infty^\sigma \alpha_1\alpha_2 \ldots \alpha_n$ is defined recursively as follows: (1) $\mathfrak{F}_\infty^\sigma \alpha_1 = \left(\mathfrak{F}_\infty^\sigma(\alpha_1)\right)_\infty^\sigma$; (2) $\mathfrak{F}_\infty^\sigma \alpha_1 \ldots \alpha_n = \left(\mathfrak{F}_\infty^\sigma \alpha_1 \ldots \alpha_{n-1}(\alpha_n)\right)_\infty^\sigma$.

Let $\mathfrak{F}$, $\mathfrak{M}$, and $\mathfrak{X}$ be totally σ-local formations. Let $\alpha_1, \ldots, \alpha_n$ be some suitable σ-sequence for $\mathfrak{F}$, $\mathfrak{M}$, and $\mathfrak{X}$. Then by $\widetilde{\mathfrak{L}}_\infty^\sigma$, $\widetilde{\mathfrak{H}}_\infty^\sigma$, $\widetilde{\mathfrak{L}}_\infty^\sigma\alpha_1$, $\widetilde{\mathfrak{H}}_\infty^\sigma\alpha_1$, …, $\widetilde{\mathfrak{L}}_\infty^\sigma\alpha_1 \ldots \alpha_n$, $\widetilde{\mathfrak{H}}_\infty^\sigma\alpha_1 \ldots \alpha_n$ we denotes formation σ-functions such that

$$\widetilde{\mathfrak{L}}_\infty^\sigma = \left(\mathfrak{X}_\infty^\sigma \cap \mathfrak{F}_\infty^\sigma\right) \vee_\infty^\sigma \left(\mathfrak{M}_\infty^\sigma \cap \mathfrak{F}_\infty^\sigma\right),\ \widetilde{\mathfrak{H}}_\infty^\sigma = \left(\mathfrak{X}_\infty^\sigma \vee_\infty^\sigma \mathfrak{M}_\infty^\sigma\right) \cap \mathfrak{F}_\infty^\sigma,$$

$$\widetilde{\mathfrak{L}}_\infty^\sigma\alpha_1 = \left(\mathfrak{X}_\infty^\sigma\alpha_1 \cap \mathfrak{F}_\infty^\sigma\alpha_1\right) \vee_\infty^\sigma \left(\mathfrak{M}_\infty^\sigma\alpha_1 \cap \mathfrak{F}_\infty^\sigma\alpha_1\right),\ \widetilde{\mathfrak{H}}_\infty^\sigma\alpha_1 = \left(\mathfrak{X}_\infty^\sigma\alpha_1 \vee_\infty^\sigma \mathfrak{M}_\infty^\sigma\alpha_1\right) \cap \mathfrak{F}_\infty^\sigma\alpha_1,$$

$$\ldots,$$

$$\widetilde{\mathfrak{L}}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_n = \left(\mathfrak{X}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_n \cap \mathfrak{F}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_n\right) \vee_\infty^\sigma \left(\mathfrak{M}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_n \cap \mathfrak{F}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_n\right),$$

$$\widetilde{\mathfrak{H}}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_n = \left(\mathfrak{X}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_n \vee_\infty^\sigma \mathfrak{M}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_n\right) \cap \mathfrak{F}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_n.$$

Lemma 13. *Let $\mathfrak{L} = (\mathfrak{X} \cap \mathfrak{F}) \vee_\infty^\sigma (\mathfrak{M} \cap \mathfrak{F})$, $\mathfrak{H} = \left(\mathfrak{X} \vee_\infty^\sigma \mathfrak{M}\right) \cap \mathfrak{F}$, where $\mathfrak{M}$, $\mathfrak{X}$, and $\mathfrak{F}$ are totally* σ*-local formations. Then*

(1) $\sigma(\mathfrak{L}) = \sigma(\mathfrak{H})$*;*

(2) if $\alpha_1, \ldots, \alpha_n$ *is a suitable* σ*-sequence for* $\mathfrak{X}$, $\mathfrak{M}$, *and* $\mathfrak{F}$, *then the formation* σ*-functions*

$$\widetilde{\mathfrak{L}}_\infty^\sigma,\ \widetilde{\mathfrak{H}}_\infty^\sigma,\ \widetilde{\mathfrak{L}}_\infty^\sigma\alpha_1,\ \widetilde{\mathfrak{H}}_\infty^\sigma\alpha_1,\ \ldots,\ \widetilde{\mathfrak{L}}_\infty^\sigma\alpha_1\ldots\alpha_n,\ \widetilde{\mathfrak{H}}_\infty^\sigma\alpha_1\ldots\alpha_n$$

are integrated l_∞^σ-valued definitions of the formations

$$\mathfrak{L},\ \mathfrak{H},\ \widetilde{\mathfrak{L}}_\infty^\sigma(\alpha_1),\ \widetilde{\mathfrak{H}}_\infty^\sigma(\alpha_1),\ \ldots,\ \widetilde{\mathfrak{L}}_\infty^\sigma\alpha_1\ldots\alpha_{n-1}(\alpha_n),\ \widetilde{\mathfrak{H}}_\infty^\sigma\alpha_1\ldots\alpha_{n-1}(\alpha_n),$$

respectively.

P r o o f. Let $\mathfrak{H}_1 = \mathfrak{X} \vee_\infty^\sigma \mathfrak{M}$, $\mathfrak{L}_1 = \mathfrak{X} \cap \mathfrak{F}$, $\mathfrak{L}_2 = \mathfrak{M} \cap \mathfrak{F}$, $h_1 = \mathfrak{X}_\infty^\sigma \vee_\infty^\sigma \mathfrak{M}_\infty^\sigma$, $l_1 = \mathfrak{X}_\infty^\sigma \cap \mathfrak{F}_\infty^\sigma$, and $l_2 = \mathfrak{M}_\infty^\sigma \cap \mathfrak{F}_\infty^\sigma$. By lemmas 8 and 12, it follows that $\mathfrak{H}_1 = LF_\sigma(h_1)$, $\mathfrak{L}_1 = LF_\sigma(l_1)$, $\mathfrak{L}_2 = LF_\sigma(l_2)$, and h_1, l_1, l_2 are integrated l_∞^σ-valued σ-functions.

(1) Since the inclusion $\mathfrak{L} \subseteq \mathfrak{H}$ is obvious, we obtain $\sigma(\mathfrak{L}) \subseteq \sigma(\mathfrak{H})$. Let $\sigma_i \in \sigma(\mathfrak{H}) \setminus \sigma(\mathfrak{L})$. Since $\sigma_i \in \sigma(\mathfrak{H}_1)$, we have $h_1(\sigma_i) \neq \varnothing$ by lemma 7. If $\sigma_i \notin \sigma(\mathfrak{X}) \cup \sigma(\mathfrak{M})$, then it follows from lemma 7 that $\mathfrak{X}_\infty^\sigma(\sigma_i) = \varnothing$ and $\mathfrak{M}_\infty^\sigma(\sigma_i) = \varnothing$. Consequently, $h_1(\sigma_i) = \varnothing$, a contradiction. Hence $\sigma_i \in \sigma(\mathfrak{X}) \cup \sigma(\mathfrak{M})$. But in this case $\sigma_i \in \sigma(\mathfrak{X} \cap \mathfrak{F}) \cup \sigma(\mathfrak{M} \cap \mathfrak{F}) = \sigma(\mathfrak{L})$. Thus, $\sigma(\mathfrak{L}) = \sigma(\mathfrak{H})$.

(2) Since $\mathfrak{L} = \mathfrak{L}_1 \vee_\infty^\sigma \mathfrak{L}_2$ and $\mathfrak{H} = \mathfrak{H}_1 \cap \mathfrak{F}$, it follows from lemmas 9 and 12, that

$$\widetilde{\mathfrak{L}}_\infty^\sigma = l_1 \vee_\infty^\sigma l_2 = \left(\mathfrak{X}_\infty^\sigma \cap \mathfrak{F}_\infty^\sigma\right) \vee_\infty^\sigma \left(\mathfrak{M}_\infty^\sigma \cap \mathfrak{F}_\infty^\sigma\right),\ \widetilde{\mathfrak{H}}_\infty^\sigma = h_1 \cap \mathfrak{F}_\infty^\sigma = \left(\mathfrak{X}_\infty^\sigma \vee_\infty^\sigma \mathfrak{M}_\infty^\sigma\right) \cap \mathfrak{F}_\infty^\sigma$$

are integrated l_∞^σ-valued definitions of the formations $\mathfrak{L}$ and $\mathfrak{H}$, respectively.

Let $\alpha_1, \ldots, \alpha_n$ be a suitable sequence for $\mathfrak{X}$, $\mathfrak{M}$, and $\mathfrak{F}$. Since by the definition $\mathfrak{X}_\infty^\sigma\alpha_1\ldots\alpha_j$, $\mathfrak{M}_\infty^\sigma\alpha_1\ldots\alpha_j$, $\mathfrak{F}_\infty^\sigma\alpha_1\ldots\alpha_j$ are smallest l_∞^σ-valued definitions of the formations

$$\mathfrak{X}_\infty^\sigma\alpha_1\ldots\alpha_{j-1}(\alpha_j),\ \mathfrak{M}_\infty^\sigma\alpha_1\ldots\alpha_{j-1}(\alpha_j),\text{ and }\mathfrak{F}_\infty^\sigma\alpha_1\ldots\alpha_{j-1}(\alpha_j)$$

$\left(j=\overline{1, n}\right)$, respectively, it follows from lemmas 9 and 12 that

$$\widetilde{\mathfrak{L}}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_j=\left(\mathfrak{X}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_j\cap\mathfrak{F}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_j\right)\vee_\infty^\sigma\left(\mathfrak{M}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_j\cap\mathfrak{F}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_j\right),$$

$$\widetilde{\mathfrak{H}}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_j=\left(\mathfrak{X}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_j\vee_\infty^\sigma\mathfrak{M}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_j\right)\cap\mathfrak{F}_\infty^\sigma\alpha_1\alpha_2\ldots\alpha_j$$

are integrated l_∞^σ-valued definitions of the formations

$$\widetilde{\mathfrak{L}}_\infty^\sigma\alpha_1\ldots\alpha_{j-1}(\alpha_j)\text{ and }\widetilde{\mathfrak{H}}_\infty^\sigma\alpha_1\ldots\alpha_{j-1}(\alpha_j),$$

respectively. The lemma is proved.

Lemma 14. *Let $\mathfrak{M}$, $\mathfrak{X}$, and $\mathfrak{F}$ be totally σ-local formations. Let G be a monolithic group and* $\mathrm{Soc}(G)$ *is not σ-primary. If* $G\in\mathfrak{F}\cap\left(\mathfrak{X}\vee_\infty^\sigma\mathfrak{M}\right)$, *then* $G\in(\mathfrak{X}\cap\mathfrak{F})\vee_\infty^\sigma(\mathfrak{M}\cap\mathfrak{F})$.

P r o o f. Let $G\in\mathfrak{F}\cap\left(\mathfrak{X}\vee_\infty^\sigma\mathfrak{M}\right)$. It follows from lemma 10 that $\mathfrak{S}_\sigma\operatorname{form}(\mathfrak{X}\cup\mathfrak{M})$ is a totally σ-local formation. Therefore, $l_\infty^\sigma\operatorname{form}(\mathfrak{X}\cup\mathfrak{M})\subseteq\mathfrak{S}_\sigma\operatorname{form}(\mathfrak{X}\cup\mathfrak{M})$. Since G is a monolithic group and $\mathrm{Soc}(G)$ is not a σ-primary group, we have $G\in\operatorname{form}(\mathfrak{X}\cup\mathfrak{M})$ and $\mathrm{Soc}(G)$ is a non-abelian group. But then by lemma 5, it follows that $G\in\mathfrak{X}\cup\mathfrak{M}$. Since $G\in\mathfrak{F}$, we obtain $G\in(\mathfrak{X}\cap\mathfrak{F})\cup(\mathfrak{M}\cap\mathfrak{F})$. Hence,

$$G\in l_\infty^\sigma\operatorname{form}\left((\mathfrak{X}\cap\mathfrak{F})\cup(\mathfrak{M}\cap\mathfrak{F})\right)=(\mathfrak{X}\cap\mathfrak{F})\vee_\infty^\sigma(\mathfrak{M}\cap\mathfrak{F}).$$

The lemma is proved.

Theorem 2. *The lattice l_∞^σ of all totally σ-local formations is distributive.*

P r o o f. Suppose that this theorem is false. Then there exist totally σ-local formations $\mathfrak{M}$, $\mathfrak{X}$, and $\mathfrak{F}$ such that

$$(\mathfrak{X}\cap\mathfrak{F})\vee_\infty^\sigma(\mathfrak{M}\cap\mathfrak{F})\neq\left(\mathfrak{X}\vee_\infty^\sigma\mathfrak{M}\right)\cap\mathfrak{F}.$$

Let $\mathfrak{L}=(\mathfrak{X}\cap\mathfrak{F})\vee_\infty^\sigma(\mathfrak{M}\cap\mathfrak{F})$ and $\mathfrak{H}=\left(\mathfrak{X}\vee_\infty^\sigma\mathfrak{M}\right)\cap\mathfrak{F}$. Since the inclusion $\mathfrak{L}\subseteq\mathfrak{H}$ is obvious, we obtain $\mathfrak{H}\not\subseteq\mathfrak{L}$. Let G be a group of minimal order in $\mathfrak{H}\backslash\mathfrak{L}$. In view that $\mathfrak{L}$ is a σ-local formation, we see that G is a monolithic group with a unique minimal normal subgroup $P=G^{\mathfrak{L}}$.

If P is not σ-primary, then $G\in\mathfrak{L}$ by lemma 14. This contradiction shows that P is a σ_i-group for some $\sigma_i\in\sigma(\mathfrak{H})$. It follows from lemma 13 that $\sigma_i\in\sigma(\mathfrak{L})$, $\mathfrak{L}=LF_\sigma\left(\widetilde{\mathfrak{L}}_\infty^\sigma\right)$, $\mathfrak{H}=LF_\sigma\left(\widetilde{\mathfrak{H}}_\infty^\sigma\right)$, and $\widetilde{\mathfrak{L}}_\infty^\sigma$, $\widetilde{\mathfrak{H}}_\infty^\sigma$ are integrated l_∞^σ-valued definitions such that

$$\widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)=\left(\mathfrak{X}_\infty^\sigma(\sigma_i)\cap\mathfrak{F}_\infty^\sigma(\sigma_i)\right)\vee_\infty^\sigma\left(\mathfrak{M}_\infty^\sigma(\sigma_i)\cap\mathfrak{F}_\infty^\sigma(\sigma_i)\right),\ \widetilde{\mathfrak{H}}_\infty^\sigma(\sigma_i)=\left(\mathfrak{X}_\infty^\sigma(\sigma_i)\vee_\infty^\sigma\mathfrak{M}_\infty^\sigma(\sigma_i)\right)\cap\mathfrak{F}_\infty^\sigma(\sigma_i).$$

Since $\mathfrak{L}$ is a σ-local formation and $\sigma_i\in\sigma(\mathfrak{L})$, we see that G is not a σ_i-group and $\widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)\neq\varnothing$. On the other hand, since P is a σ_i-group, $O_{\sigma_i'}(G)=1$ and $F_{\{\sigma_i\}}(G)=O_{\sigma_i}(G)$.

Since $\mathfrak{L}\subseteq\mathfrak{H}$ we have $\widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)\subseteq\widetilde{\mathfrak{H}}_\infty^\sigma(\sigma_i)$ and since $G\in\mathfrak{H}\backslash\mathfrak{L}$, we claim that $\widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)\subset\widetilde{\mathfrak{H}}_\infty^\sigma(\sigma_i)$. Indeed, by lemma 2, it follows that $G/F_{\{\sigma_i\}}(G)\in\mathfrak{H}_\infty^\sigma(\sigma_i)$. If $\widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)=\widetilde{\mathfrak{H}}_\infty^\sigma(\sigma_i)$, then

$$G/O_{\sigma_i}(G)=G/F_{\{\sigma_i\}}(G)\in\mathfrak{H}_\infty^\sigma(\sigma_i)\subseteq\widetilde{\mathfrak{H}}_\infty^\sigma(\sigma_i)=\widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)$$

and $G\in\mathfrak{L}$ by lemma 11. It is a contradiction. Therefore, $\widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)\subset\widetilde{\mathfrak{H}}_\infty^\sigma(\sigma_i)$. Note also that the condition $\widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)\subset\widetilde{\mathfrak{H}}_\infty^\sigma(\sigma_i)$ implies $\mathfrak{X}_\infty^\sigma(\sigma_i)\neq\varnothing$ and $\mathfrak{M}_\infty^\sigma(\sigma_i)\neq\varnothing$, since otherwise $\widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)=\widetilde{\mathfrak{H}}_\infty^\sigma(\sigma_i)$. Hence $\sigma_i\in$ $\in\sigma(\mathfrak{X})\cap\sigma(\mathfrak{M})$. Thus,

$$G_1=G/F_{\{\sigma_i\}}(G)\in\widetilde{\mathfrak{H}}_\infty^\sigma(\sigma_i)\backslash\,\widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i),\ \widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)\neq\varnothing.$$

It follows from lemma 13 that

$$\sigma\left(\widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)\right)=\sigma\left(\widetilde{\mathfrak{H}}_\infty^\sigma(\sigma_i)\right),\ \widetilde{\mathfrak{L}}_\infty^\sigma(\sigma_i)=LF_\sigma\left(\widetilde{\mathfrak{L}}_\infty^\sigma\sigma_i\right),\ \widetilde{\mathfrak{H}}_\infty^\sigma(\sigma_i)=LF_\sigma\left(\widetilde{\mathfrak{H}}_\infty^\sigma\sigma_i\right),$$

and $\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i$, $\widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i$ are integrated l^{σ}_{∞}-valued definitions such that

$$\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i = \left(\mathfrak{X}^{\sigma}_{\infty}\sigma_i \cap \mathfrak{F}^{\sigma}_{\infty}\sigma_i\right) \vee^{\sigma}_{\infty} \left(\mathfrak{M}^{\sigma}_{\infty}\sigma_i \cap \mathfrak{F}^{\sigma}_{\infty}\sigma_i\right),\ \widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i = \left(\mathfrak{X}^{\sigma}_{\infty}\sigma_i \vee^{\sigma}_{\infty} \mathfrak{M}^{\sigma}_{\infty}\sigma_i\right) \cap \mathfrak{F}^{\sigma}_{\infty}\sigma_i.$$

Since $G_1 \not\in \widetilde{\mathfrak{L}}^{\sigma}_{\infty}(\sigma_i)$, there exist $\alpha_1 \in \sigma(G_1)$ such that

$$G_1/F_{\{\alpha_1\}}(G_1) \notin \widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i(\alpha_1).$$

Note that since $\alpha_1 \in \sigma\left(\widetilde{\mathfrak{H}}^{\sigma}_{\infty}(\sigma_i)\right)$, we have $\alpha_1 \in \sigma\left(\widetilde{\mathfrak{L}}^{\sigma}_{\infty}(\sigma_i)\right)$ and $\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i(\alpha_1) \neq \varnothing$. Obviously,

$$\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i(\alpha_1) = \left(\mathfrak{X}^{\sigma}_{\infty}\sigma_i(\alpha_1) \cap \mathfrak{F}^{\sigma}_{\infty}\sigma_i(\alpha_1)\right) \vee^{\sigma}_{\infty} \left(\mathfrak{M}^{\sigma}_{\infty}\sigma_i(\alpha_1) \cap \mathfrak{F}^{\sigma}_{\infty}\sigma_i(\alpha_1)\right) \subseteq$$
$$\subseteq \widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i(\alpha_1) = \left(\mathfrak{X}^{\sigma}_{\infty}\sigma_i(\alpha_1) \vee^{\sigma}_{\infty} \mathfrak{M}^{\sigma}_{\infty}\sigma_i(\alpha_1)\right) \cap \mathfrak{F}^{\sigma}_{\infty}\sigma_i(\alpha_1).$$

Besides, since $G_1/F_{\{\alpha_1\}}(G_1) \in \widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i(\alpha_1) \setminus \widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i(\alpha_1)$, we have $\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i(\alpha_1) \subset \widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i(\alpha_1)$. Therefore, $\mathfrak{X}^{\sigma}_{\infty}\sigma_i(\alpha_1) \neq \varnothing$ and $\mathfrak{M}^{\sigma}_{\infty}\sigma_i(\alpha_1) \neq \varnothing$. Hence $\alpha_1 \in \sigma\left(\mathfrak{X}^{\sigma}_{\infty}(\sigma_i)\right) \cap \sigma\left(\mathfrak{M}^{\sigma}_{\infty}(\sigma_i)\right)$.

Suppose that $F_{\{\alpha_1\}}(G_1) = 1$ and let N be a minimal normal subgroup of G_1. Then N is not σ-primary. If G_1 is a monolithic group, then since

$$G_1 \in \widetilde{\mathfrak{H}}^{\sigma}_{\infty}(\sigma_i) = \left(\mathfrak{X}^{\sigma}_{\infty}(\sigma_i) \vee^{\sigma}_{\infty} \mathfrak{M}^{\sigma}_{\infty}(\sigma_i)\right) \cap \mathfrak{F}^{\sigma}_{\infty}(\sigma_i),$$

by lemma 14, it follows that

$$G_1 \in \left(\mathfrak{X}^{\sigma}_{\infty}(\sigma_i) \cap \mathfrak{F}^{\sigma}_{\infty}(\sigma_i)\right) \vee^{\sigma}_{\infty} \left(\mathfrak{M}^{\sigma}_{\infty}(\sigma_i) \cap \mathfrak{F}^{\sigma}_{\infty}(\sigma_i)\right) = \widetilde{\mathfrak{L}}^{\sigma}_{\infty}(\sigma_i).$$

This contradiction shows that the group G_1 is not monolithic.

Let $\mathrm{Soc}(G_1) = N_1 \times \ldots \times N_k$, where N_j is a minimal normal subgroup of G_1 and let M_j denote a maximal normal subgroup of G_1 such that M_j contains $N_1 \times \ldots \times N_{j-1} \times N_{j+1} \times \ldots \times N_k$ and does not contain N_j, $j = 1, 2, \ldots, k$. By lemma 6, it follows that G_1/M_j is a monolithic group with a non-σ-primary minimal normal subgroup N_jM_j/N_j and N_jM_j/N_j is G_1-isomorphic to N_j. Set $B_j = G_1/M_j$, $j = 1, 2, \ldots, k$. Since

$$B_j \in \widetilde{\mathfrak{H}}^{\sigma}_{\infty}(\sigma_i) = \left(\mathfrak{X}^{\sigma}_{\infty}(\sigma_i) \vee^{\sigma}_{\infty} \mathfrak{M}^{\sigma}_{\infty}(\sigma_i)\right) \cap \mathfrak{F}^{\sigma}_{\infty}(\sigma_i),$$

we have $B_j \in \widetilde{\mathfrak{L}}^{\sigma}_{\infty}(\sigma_i)$ by lemma 14. It follows from lemma 6 (d) that G_1 is a subdirect product of $B_1, \ldots, B_k$. Hence $G_1 \in \widetilde{\mathfrak{L}}^{\sigma}_{\infty}(\sigma_i)$. This contradiction shows that $F_{\{\alpha_1\}}(G_1) \neq 1$.

On the other hand $F_{\{\alpha_1\}}(G_1) \neq G_1$, since otherwise $G_1/F_{\{\alpha_1\}}(G_1) \simeq 1 \in \mathfrak{L}^{\sigma}_{\infty}\sigma_i(\alpha_1) \neq \varnothing$.

Thus,

$$G_1/F_{\{\alpha_1\}}(G_1) \in \widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i(\alpha_1) \setminus \widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i(\alpha_1),\ \widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i(\alpha_1) \neq \varnothing,\ 1 \neq F_{\{\alpha_1\}}(G_1) \subset G_1.$$

Let $G_2 = G_1/F_{\{\alpha_1\}}(G_1)$. It follows from lemma 13 that

$$\sigma\left(\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i(\alpha_1)\right) = \sigma\left(\widetilde{\mathfrak{H}}^{\sigma}_{\infty}p(\alpha_1)\right),$$

$$\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i(\alpha_1) = LF_{\sigma}\left(\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i\alpha_1\right),\ \widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i(\alpha_1) = LF_{\sigma}\left(\widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i\alpha_1\right),$$

and $\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i\alpha_1$, $\widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i\alpha_1$ are integrated l^{σ}_{∞}-valued definitions such that

$$\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i\alpha_1 = \left(\mathfrak{X}^{\sigma}_{\infty}\sigma_i\alpha_1 \cap \mathfrak{F}^{\sigma}_{\infty}\sigma_i\alpha_1\right) \vee^{\sigma}_{\infty} \left(\mathfrak{M}^{\sigma}_{\infty}\sigma_i\alpha_1 \cap \mathfrak{F}^{\sigma}_{\infty}\sigma_i\alpha_1\right),$$

$$\widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i\alpha_1 = \left(\mathfrak{X}^{\sigma}_{\infty}\sigma_i\alpha_1 \vee^{\sigma}_{\infty} \mathfrak{M}^{\sigma}_{\infty}\sigma_i\alpha_1\right) \cap \mathfrak{F}^{\sigma}_{\infty}\sigma_i\alpha_1.$$

Since $G_2 \not\in \widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i(\alpha_1)$, there exists $\alpha_2 \in \sigma(G_2)$ such that

$$G_2/F_{\{\alpha_2\}}(G_2) \not\in \widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i\alpha_1(\alpha_2).$$

Hence,

$$G_2/F_{\{\alpha_2\}}(G_2)\in \widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i\alpha_1(\alpha_2)\backslash \widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i\alpha_1(\alpha_2).$$

Considering G_2 in the same way as the group G_1, we obtain

$$\alpha_2\in\sigma\left(\mathfrak{X}^{\sigma}_{\infty}\sigma_i(\alpha_1)\right)\cap\sigma\left(\mathfrak{M}^{\sigma}_{\infty}\sigma_i(\alpha_1)\right),\ G_2/F_{\{\alpha_2\}}(G_2)\in \widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i\alpha_1(\alpha_2)\backslash \widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i\alpha_1(\alpha_2),$$

$$\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i\alpha_1(\alpha_2)\neq\varnothing, \text{ and } 1\neq F_{\{\alpha_2\}}(G_2)\subset G_2.$$

Put $G_3=G_2/F_{\{\alpha_2\}}(G_2)$. According to the same argument, we see that the group G_3 satisfies the analogous conditions: there exists

$$\alpha_3\in\sigma\left(\mathfrak{X}^{\sigma}_{\infty}\sigma_i\alpha_1(\alpha_2)\right)\cap\sigma\left(\mathfrak{M}^{\sigma}_{\infty}\sigma_i\,\alpha_1(\alpha_2)\right)$$

such that

$$G_3/F_{\{\alpha_3\}}(G_3)\in \widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i\alpha_1\alpha_2(\alpha_3)\backslash \widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i\alpha_1\alpha_2(\alpha_3),$$

$$\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i\alpha_1\alpha_2(\alpha_3)\neq\varnothing, \text{ and } 1\neq F_{\{\alpha_3\}}(G_3)\subset G_3.$$

Continuing this line of reasoning, we construct the groups

$$G_4=G_3/F_{\{\alpha_3\}}(G_3),\ \ldots,\ G_n=G_{n-1}/F_{\{\alpha_{n-1}\}}(G_{n-1}),\ \ldots$$

such that for any j the following conditions are satisfied:

$$\alpha_{j-1}\in\sigma\left(\mathfrak{X}^{\sigma}_{\infty}\sigma_i\alpha_1\ldots\alpha_{j-3}(\alpha_{j-2})\right)\cap\sigma\left(\mathfrak{M}^{\sigma}_{\infty}\sigma_i\alpha_1\ldots\alpha_{j-3}(\alpha_{j-2})\right),$$

$$G_j=G_{j-1}/F_{\{\alpha_{j-1}\}}(G_{j-1})\in \widetilde{\mathfrak{H}}^{\sigma}_{\infty}\sigma_i\alpha_1\ldots\alpha_{j-2}(\alpha_{j-1})\backslash \widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i\alpha_1\ldots\alpha_{j-2}(\alpha_{j-1}),$$

$$\widetilde{\mathfrak{L}}^{\sigma}_{\infty}\sigma_i\alpha_1\ldots\alpha_{j-2}(\alpha_{j-1})\neq\varnothing, \text{ and } 1\neq F_{\{\alpha_{j-1}\}}(G_{j-1})\subset G_{j-1}.$$

Since $F_{\{\alpha_{j-1}\}}(G_{j-1})\neq 1$, we see that for the constructed sequence of the groups

$$G,\ G_1,\ G_2,\ G_3,\ \ldots,\ G_n,\ \ldots$$

it follows that

$$|G|>|G_1|>|G_2|>|G_3|>\ldots>|G_n|>\ldots.$$

Since the group G is finite, we obtain $G_k = 1$ for some number k. But

$$G_k=G_{k-1}/F_{\{\alpha_{k-1}\}}(G_{k-1}).$$

This implies that $F_{\{\alpha_{k-1}\}}(G_{k-1})=G_{k-1}$, a contradiction.

Thus, our assumption is not true and $\mathfrak{H}\subseteq\mathfrak{L}$. Hence $\mathfrak{H}=\mathfrak{L}$. The theorem is proved.

Note that theorem 2 gives an affirmative answer to question of A. Tzarev on distributivety of the lattice of all totally σ-local formations of finite groups [11, question 3.2].

Let $\mathfrak{F}$ and $\mathfrak{M}$ be totally σ-local formations such that $\mathfrak{M}\subseteq\mathfrak{F}$, then $\mathfrak{F}/^{\sigma}_{\infty}\mathfrak{M}$ denotes the lattice of all totally σ-local formations between $\mathfrak{M}$ and $\mathfrak{F}$.

Corollary 3. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be totally σ-local formations. Then the lattice isomorphism holds*

$$\mathfrak{F}\vee^{\sigma}_{\infty}\mathfrak{H}/^{\sigma}_{\infty}\mathfrak{H}\simeq\mathfrak{F}/^{\sigma}_{\infty}\mathfrak{F}\cap\mathfrak{H}.$$

In the case when $\sigma=\sigma_1$, we get from theorem 2 the following known results.

Corollary 4 [16]. *The lattice l_∞ of all totally local formations is distributive.*

Corollary 5 [2, p. 169]. *The lattice of all soluble totally local formations is distributive.*

Corollary 6 [17]. *The lattice l_∞ of all totally local formations is modular.*

Recall that if $\mathfrak{F}$ and $\mathfrak{M}$ are totally local formations, $\mathfrak{M}\subseteq\mathfrak{F}$, then by the symbol $\mathfrak{F}/_{\infty}\mathfrak{M}$ denotes the lattice of all totally local formations between $\mathfrak{M}$ and $\mathfrak{F}$.

Corollary 7 [17]. *Let $\mathfrak{F}$ and $\mathfrak{M}$ be totally local formations. Then we have*

$$\mathfrak{F}\vee_{\infty}\mathfrak{M}/_{\infty}\mathfrak{M}\simeq\mathfrak{F}/_{\infty}\mathfrak{F}\cap\mathfrak{M}.$$

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Received by editorial board 06.10.2020.