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Majorana-Oppenheimer approach to Maxwell electrodynamics in Riemannian space-time

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The Riemann – Silberstein – Majorana – Oppengeimer approach to the Maxwell electrodynamics in presence of electrical sources and arbitrary media is investigated within the matrix formalism. The symmetry of the matrix Maxwell equation under transformations of the complex rotation group SO(3.C) is demonstrated explicitly. In vacuum case, the matrix form includes four real 4×4 matrices α^b. In presence of media matrix form requires two sets of 4×4 matrices, α^b and β^b – simple and symmetrical realization of which is given. Relation of α^b and β^b to the Dirac matrices in spinor basis is found. Minkowski constitutive relations in case of any linear media are given in a short algebraic form based on the use of complex 3-vector fields and complex orthogonal rotations from SO(3.C) group. The matrix complex formulation in the Esposito's form, based on the use of two electromagnetic 4-vector, is studied and discussed. Extension of the 3-vector complex matrix formalism to arbitrary Riemannian space-time in accordance with tetrad method by Tetrode-Weyl-Fock-Ivanenko is performed.

Keywords: Riemann – Silberstein – Majorana – Oppengeimer approach, Maxwell equations, Minkowski constitutive relations, SO(3.C) group

1. Introduction

Special relativity arose from study of the symmetry properties of the Maxwell equations with respect to motion of references frames: Lorentz [2], Poincar'e [3], Einstein [4] Naturally, an analysis of the Maxwell equations with respect to Lorentz transformations was the first objects of relativity theory: Minkowski [5], Silberstein [6],[7], Marcolongo [8], Bateman [9], and Lanczos [10], Gordon [11], Mandel'stam – Tamm [12],[13],[14].

After discovering the relativistic equation for a particle with spin 1/2 – Dirac [15] – much work was done to study spinor and vectors within the Lorentz group theory: Möglich [16], Ivanenko – Landau [17], Neumann [18], van der Waerden [19], Juvet [24]. As was shown any quantity which transforms linearly under Lorentz transformations is a spinor. For that reason spinor quantities are considered as fundamental in quantum field theory and basic equations for such quantities should be written in a spinor form. A spinor formulation of Maxwell equations was studied by Laporte and Uhlenbeck [25], also see Rumer [33]. In 1931, Majorana [27] and Oppenheimer [26] proposed to consider the Maxwell theory of electromagnetism as the wave mechanics of the photon. They introduced a complex 3-vector wave function satisfying the massless Dirac-like equations. Before Majorana and Oppenheimer, the most crucial steps were made by Silberstein [6], he showed the possibility to have formulated Maxwell equation in term of complex 3-vector entities. Silberstein in his second paper [7] writes that the complex form of Maxwell equations has been known before; he refers there to the second volume of the lecture notes on the differential equations of mathematical physics by B. Riemann that were edited and

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published by H. Weber in 1901 [1]. This not widely used fact is noted by Bialynicki-Birula [118]).

Maxwell equations in the matrix Dirac-like form considered during long time by many authors, the interest to the Majorana-Oppenheimer formulation of electrodynamics has grown in recent years:

Luis de Broglie [28],[29],[35],[41], Mercier [30], Petiau [31], Proca [32], [51], Duffin [34], Kemmer [36],[48],[73], Bhabha [37], Belinfante [38],[39], Taub [40], Sakata – Taketani [43], Schrödinger [42], [46],[47], Tonnelat cite1941-Tonnelat, Heitler [49], [52],[53], Hoffmann [54], Utiyama [55], Mercier [56], Imaeda [57], Fujiwara [58], Gürsey [59], Gupta [60], Lichnerowicz [61], , Ohmura [62], Borgardt [63],[70], Fedorov [64], Kuohsien [65], Bludman [66], Good [67], Moses [68]-[71]-[90], Silveira [72], [100], Lomont [69], Kibble [76], Post [78], Bogush – Fedorov [79], Sachs – Schwebel [81], Ellis [82], Oliver [84], Beckers – Pirotte [85], Casanova [86], Carmeli [87], Bogush [88], Lord [89], Weingarten [91], Mignani – Recami – Baldo [92], [94], [96], Edmonds [97], Strazhev – Tomil'chik [98], Silveira [100], Jena – Naik – Pradhan [101], Venuri [102], Chow [103], Fushchich – Nikitin [104], Cook [106]-[107], Giannetto [109], – Yépez, Brito – Vargas [110], Kidd – Ardini – Anton [111], Recami [112], Krivsky – Simulik [114], Hillion [115], Baylis [116], Inagaki [117], Bialynicki-Birula [118]-[119]-[144], Sipe [120], [121], Esposito [123], Dvoeglazov [124], [125] (see a big list of relevant references therein)-[126], Gersten [122], Gsponer [127], [128],[133],[129],[130],[134], [140],[141],[142],[143], Donev – Tashkova [136], [137], [138], Armour [139].

Our treatment will be with a quite definite accent: the main attention is given to technical aspect of classical electrodynamics based on the theory of rotation complex group SO(3.C) (isomorphic to the Lorentz group – see Kurşunoĝlu [75], Macfarlane cite[80]-[83], Fedorov [99]).

2. Complex matrix form of Maxwell theory in vacuum

Let us start with Maxwell equations in a uniform (ϵ, μ)-media in presence of external sources [45]-[77]-[96]:

$$\begin{gathered} \text{div}\ c\mathbf{B} = 0\ , \qquad \text{rot}\ \mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial ct} \\ \text{div}\ \mathbf{E} = \frac{\rho}{\epsilon\epsilon_0}, \qquad \text{rot}\ c\mathbf{B} = \mu\mu_0 c\mathbf{J} + \epsilon\mu\frac{\partial \mathbf{E}}{\partial ct}\ . \end{gathered} \tag{1}$$

With the use of usual notation for current 4-vector $j^a = (\rho, \mathbf{J}/c)$, $c^2 = 1/\epsilon_0\mu_0$, eqs. (1) read (first, consider the vacuum case):

$$\begin{gathered} \text{div}\ c\mathbf{B} = 0\ , \qquad \text{rot}\ \mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial ct} \\ \text{div}\ \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \text{rot}\ c\mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial ct}\ , \end{gathered} \tag{2}$$

Let us introduce 3-dimensional complex vector $\psi^k = E^k + icB^k$, with the help of which the above equations can be combined into (see Zilberschtein [6]-[7], Bateman [9], Majorana [27], Oppenheimer [26], and many others)

$$\begin{gathered} \partial_1\Psi^1 + \partial_2\Psi^0 + \partial_3\Psi^3 = j^0/\epsilon_0\ , \quad -i\partial_0\psi^1 + (\partial_2\psi^3 - \partial_3\psi^2) = i\ j^1/\epsilon_0 \\ -i\partial_0\psi^2 + (\partial_3\psi^1 - \partial_1\psi^3) = i\ j^2/\epsilon_0\ , \ -i\partial_0\psi^3 + (\partial_1\psi^2 - \partial_2\psi^1) = i\ j^3/\epsilon_0\ . \end{gathered}$$

let $x_0 = ct,\ \partial_0 = c\ \partial_t$. These four relations can be rewritten in a matrix form using a 4-dimensional column Ψ with one additional zero-element [Fuschich – Nikitin [104]:

$$(-i\alpha^0\partial_0 + \alpha^j \partial_j)\Psi = J \;, \qquad \Psi = \left| \begin{array}{c} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{array} \right| \;, \qquad \alpha^0 = \left| \begin{array}{cccc} a_0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \end{array} \right|$$

$$\alpha^1 = \left| \begin{array}{cccc} b_0 & 1 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & -1 \\ b_3 & 0 & 1 & 0 \end{array} \right| , \alpha^2 = \left| \begin{array}{cccc} c_0 & 0 & 1 & 0 \\ c_1 & 0 & 0 & 1 \\ c_2 & 0 & 0 & 0 \\ c_3 & -1 & 0 & 0 \end{array} \right| , \alpha^3 = \left| \begin{array}{cccc} d_0 & 0 & 0 & 1 \\ d_1 & 0 & -1 & 0 \\ d_2 & 1 & 0 & 0 \\ d_3 & 0 & 0 & 0 \end{array} \right| .$$

Here, there arise four ambiguously determined matrices; numerical parameters a_k, b_k, c_k, d_k are arbitrary. Our choice for the matrix form of eight Maxwell equations is the following:

$$(-i\partial_0 + \alpha^j \partial_j)\Psi = J \;, \qquad \Psi = \left| \begin{array}{c} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{array} \right| \;, \qquad J = \frac{1}{\epsilon_0} \left| \begin{array}{c} j^0 \\ i\, j^1 \\ i\, j^2 \\ i\, j^3 \end{array} \right| \tag{3}$$

where

$$\alpha^1 = \left| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right| , \alpha^2 = \left| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right| , \alpha^3 = \left| \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right|$$

$$(\alpha^1)^2 = -I, \qquad (\alpha^2)^2 = -I, \qquad (\alpha^2)^2 = -I$$

$$\alpha^1\alpha^2 = -\alpha^2\alpha^1 = \alpha^3 \;, \qquad \alpha^2\alpha^3 = -\alpha^3\alpha^2 = \alpha^1 \;, \qquad \alpha^3\alpha^1 = -\alpha^1\alpha^3 = \alpha^2 \;.$$

Let us consider the problem of relativistic invariance of this equation. The lack of manifest invariance of 3-vector complex form of Maxwell theory has been intensively discussed in various aspects: for instance, see Esposito [123], Ivezic [128], [133], [129], [130], [134], [140], [141], [142], [143]). Let us start with relations:

$$(-i\partial_0 + \alpha^j \partial_j)\Psi = J \;, \qquad \Psi = \left| \begin{array}{c} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{array} \right| \;, \qquad J = \frac{1}{\epsilon_0} \left| \begin{array}{c} j^0 \\ i\, j^1 \\ i\, j^2 \\ i\, j^3 \end{array} \right| .$$

Arbitrary Lorentz transformation over the function Ψ is given by (take notice that one may introduce four undefined parameters $s_0, ..., s_3$, but we will take $s_0 = 1, s_j = 0$)

$$S = \left| \begin{array}{cccc} s_0 & 0 & 0 & 0 \\ s_1 & . & . & . \\ s_2 & . & O(k) & . \\ s_3 & . & . & . \end{array} \right| \;, \qquad \Psi' = S\Psi \;, \qquad \Psi = S^{-1}\Psi' \;, \tag{4}$$

where $O(k)$ stands for a $(3\times 3$-rotation complex matrix from $SO(3,C)$, isomorphic to the Lorentz group – more detail see in [99] and below in the present text. Equation for a primed function Ψ' is

$$(-i\partial_0 + S\alpha^j S^{-1}\partial_j)\;\Psi' = S\;J \;.$$

When working with matrices α^j we will use vectors $\mathbf{e}_i$ and (3×3)-matrices τ_i, then the structure $S\alpha^j S^{-1}$ is

$$S\alpha^j S^{-1} = \left| \begin{array}{cc} 0 & \mathbf{e}_j O^{-1}(k) \\ -O(k)\mathbf{e}_j^t & O(k)\tau_j O^{-1}(k) \end{array} \right| = \alpha^m O_{mj}(k) \; . \tag{5}$$

Therefore, the matrix Maxwell equation becomes

$$(-i\partial_0 + \alpha^m \; \partial'_m)\Psi' = SJ \; , \qquad O_{mj}\partial_j = \partial'_m \; . \tag{6}$$

Now, one should give special attention to the following: the symmetry properties given by (6) look satisfactory only at real values of parameter a – in this case it describes symmetry of the Maxwell equations under Euclidean rotations. However, if the values of a are imaginary the above transformation S gives a Lorentzian boost; for instance, in the plane $0-3$ the boost is

$$a = ib \; , \qquad S(a = ib) = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & ch\, b & -ish\, b & 0 \\ 0 & ish\, b & ch\, b & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| \tag{7}$$

and the formulas (5) will take the form

$$S\alpha^1 S^{-1} = \text{ch}\, b\, \alpha^1 + i\text{sh}\, b\, \alpha^2$$
$$S\alpha^2 S^{-1} = -i\text{sh}\, b\, \alpha^1 + \text{ch}\, b\, \alpha^2 \; , \qquad S\alpha^3 S^{-1} = \alpha^3 \; . \tag{8}$$

Correspondingly, the Maxwell matrix equation after transformation (1.15a,b) will look asymmetric

$$[\; (-i\partial_0 + \alpha^3\partial_3) + (\text{ch}\, b\, \alpha^1 + i\text{sh}\, b\, \alpha^2)\; \partial_2$$
$$+ (-i\text{sh}\, b\, \alpha^1 + \text{ch}\, b\, \alpha^2)\; \partial_3 \;]\; \Psi' = SJ \; . \tag{9}$$

One can note an identity

$$(\text{ch}\, b - i\text{sh}\, b\, \alpha^3)(-i\partial_0 + \alpha^3\partial_3)$$
$$= -i(\text{ch}\, b\, \partial_0 - \text{sh}\, b\, \partial_3) + \alpha^3(-sh\, b\, \partial_0 + \text{ch}\, b\, \partial_3) = -i\partial'_0 + \alpha^3\partial'_3 \; , \tag{10}$$

where derivatives are changed in accordance with the Lorentzian rule:

$$ch\, b\, \partial_0 - sh\, b\, \partial_3 = \partial'_0 \; , \qquad -sh\, b\, \partial_0 + ch\, b\, \partial_3 = \partial'_3 \; .$$

It remains to determine the action of the operator

$$\Delta = \text{ch}\, b - i\; \text{sh}\, b\, \alpha^3 \tag{11}$$

on two other terms in eq. (9) – one might expect two relations:

$$(\text{ch}\, b - i\text{sh}\, b\, \alpha^3)(\text{ch}\, b\, \alpha^1 + i\text{sh}\, b\, \alpha^2) = \alpha^2$$
$$(\text{ch}\, b - i\text{sh}\, b\, \alpha^3)(-i\text{sh}\, b\, \alpha^1 + \text{ch}\, b\, \alpha^2) = \alpha^3 \; . \tag{12}$$

As easily verified they hold indeed. We should calculate the term $\Delta S\; J$:

$$\Delta S\; J = \left| \begin{array}{c} \text{ch}\, b\; j^0 + \text{sh}\, b\; j^3 \\ i\; j^1 \\ i\; j^2 \\ i(\text{sh}\, b\; j^0 + \text{ch}\, b\; j^3) \end{array} \right| \; ; \tag{13}$$

it is what needed. Thus, the symmetry of the matrix Maxwell equation under the Lorentzian boost in the plane $0-3$ is described by relations:

$$\Delta(b)\,(-i\partial_0 + S\alpha^j S^{-1}\partial_j)\;\Psi' = \Delta S\; J \equiv J'\;,\;(-i\partial'_0 + \alpha^j\partial'_j)\Psi' = J'$$

$$S(b) = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \text{ch}\, b & -i\text{sh}\, b & 0 \\ 0 & i\text{sh}\, b & \text{ch}\, b & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| , \qquad \Delta(b) = \text{ch}\, b - i\;\text{sh}\, b\;\alpha^3\;. \tag{14}$$

For the general case, one can think that for an arbitrary oriented boost the operator Δ should be of the form:

$$\Delta = \Delta_\alpha = ch\;b - i\;sh\;b\;n_j\;\alpha^j\;.$$

To verify this, one should obtain mathematical description of that general boost. We will start with the known parametrization of the real 3-dimension group [99])

$$O(c) = I + 2\;[\;c_0\;\mathbf{c}^{\times} + (\mathbf{c}^{\times})^2\;]\;, \qquad (\mathbf{c}^{\times})_{kl} = -\epsilon_{klj}\;a_j$$

$$O(c) = \left| \begin{array}{ccc} 1 - 2(c_2^2 + c_3^2) & -2c_0c_3 + 2c_1c_2 & +2c_0c_2 + 2c_1c_3 \\ +2c_0c_3 + 2c_1c_2 & 1 - 2(c_3^2 + c_1^2) & -2c_0c_1 + 2c_2c_3 \\ -2c_0c_2 + 2c_1c_3 & +2c_0c_1 + 2c_2c_3 & 1 - 2(c_1^2 + c_2^2) \end{array} \right| . \tag{15}$$

Transition to a general boost is achieved by the change

$$c_0 \;\Longrightarrow\; \text{ch}\;\frac{b}{2}\;, \qquad c_j \;\Longrightarrow\; i\;\text{sh}\;\frac{b}{2}\;\;n_j\;, \qquad n_jn_j = 1$$

thus we arrive at

$$O(b,\mathbf{n})$$

$$= \left| \begin{array}{ccc} 1 - F(n_2^2 + n_3^2) & -i\text{sh}\;b\;n_3 + Fn_1n_2 & i\text{sh}\;b\;n_2 + Fn_1n_3 \\ i\text{sh}\;b\;n_3 + F)n_1n_2 & 1 - F(n_3^2 + n_1^2) & -ish\;b\;n_1 + Fn_2n_3 \\ -i\text{sh}\;b\;n_2 + Fn_1n_3 & i\text{sh}\;b\;n_1 + Fn_2n_3 & 1 - F(n_1^2 + n_2^2) \end{array} \right| . \tag{16}$$

where $F = (1 - \text{ch}\;b)$. We need to examine relation

$$\Delta(b,\mathbf{n})\;(\;-i\partial_0 + \alpha^i O_{ij}(b,\mathbf{n})\partial_j\;)\;\Psi' = \Delta(b,\mathbf{n})SJ\;.$$

After rather long calculation we can indeed prove the general statement: the matrix Maxwell equation

$$(-i\;\partial_0 + \alpha^i\partial_i)\;\Psi = J$$

is invariant under an arbitrary Lorentzian boost:

$$\Delta(-i\;\partial_0 + S\alpha^i S^{-1}\partial_i)\;S\Psi = \Delta SJ \qquad \Longrightarrow \qquad (\partial'_0 + \alpha^i\partial'_i)\;\Psi' = J'$$

$$S(ib,\mathbf{n}) = \left| \begin{array}{cc} 1 & 0 \\ 0 & O(ib,\mathbf{n}) \end{array} \right|$$

$$\begin{array}{ll} t' = \text{ch}\;\beta\;t + \text{sh}\;\beta\;\mathbf{n}\;\mathbf{x}\;, & \mathbf{x}' = +\mathbf{n}\;\text{sh}\;\beta\;t + \mathbf{x} + (\text{ch}\;\beta - 1)\;\mathbf{n}\;(\mathbf{n}\mathbf{x}) \\ \partial'_0 = \text{ch}\;b\;\partial_0 -\;\text{sh}\;b\;(\mathbf{n}\nabla)\;, & \nabla' = -\text{sh}\;b\;\mathbf{n}\;\partial_0 + [\nabla + (\text{ch}\;b - 1)\mathbf{n}(\mathbf{n}\nabla) \\ j'^0 = \text{ch}\;b\;j^0 + \text{sh}\;b\;(\mathbf{nj})\;, & \mathbf{j}' = +\text{sh}\;b\;\mathbf{n}\;j^0 + \mathbf{j} + (ch\;b - 1)\;\mathbf{n}\;(\mathbf{nj})\;. \end{array} \tag{17}$$

Invariance of the matrix equation under Euclidean rotations is achieved in a simpler way:

$$(-i\,\partial_0 + S\alpha^i S^{-1}\partial_i)\; S\Psi = SJ \qquad \Longrightarrow \qquad (-i\,\partial'_0 + \alpha^i \partial'_i)\;\Psi' = J'$$

$$S(a,\mathbf{n}) = \left| \begin{array}{cc} 1 & 0 \\ 0 & O(a,\mathbf{n}) \end{array} \right| \;, \; t' = t \;, \qquad \mathbf{x}' = R(a)\mathbf{x}$$

$$\partial'_0 = \partial_0 \;, \qquad \nabla' = R(a,-\mathbf{n})\nabla \;, \;\; j'^0 = j^0 \;, \qquad \mathbf{j}' = R(a,\mathbf{n})\mathbf{j} \;. \tag{18}$$

3. On the Maxwell-Minkowski electrodynamics in media

In agrement with Minkowski approach [5], in presence of a uniform media we should introduce two electromagnetic tensors F^{ab} and H^{ab} that transform independently under the Lorentz group. At this, the known constitutive (or material) relations change their form in the moving reference frame. In the rest media reference frame the Maxwell equations are

$$F^{ab} = (\mathbf{E},\; c\mathbf{B}), \qquad \text{div}\; \mathbf{B} = 0 \;, \qquad \text{rot}\; \mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial ct}$$

$$H^{ab} = (\mathbf{D},\; \mathbf{H}/c) \;, \qquad \text{div}\; \mathbf{D} = \rho \;, \qquad \text{rot}\; \frac{\mathbf{H}}{c} = \frac{\mathbf{J}}{c} + \frac{\partial \mathbf{D}}{\partial ct} \;. \tag{19}$$

Quantities with simple transformation laws under the Lorentz group are

$$\mathbf{f} = \mathbf{E} + ic\mathbf{B} \;, \qquad \mathbf{h} = \frac{1}{\epsilon_0}\,(\mathbf{D} + i\mathbf{H}/c) \;, \;\; j^a = (j^0 = \rho,\; \mathbf{j} = \mathbf{J}/c) \;; \tag{20}$$

where $\mathbf{f}, \mathbf{h}$ are complex 3-vector under complex orthogonal group $SO(3.C)$, the latter is isomorphic to the Lorentz group. One can combine eqs. (19) into following ones

$$\text{div}\;(\frac{\mathbf{D}}{\epsilon_0} + i\; c\mathbf{B}) = \frac{1}{\epsilon_0}\;\rho$$

$$-i\partial_0(\frac{\mathbf{D}}{\epsilon_0} + ic\mathbf{B}) + \text{rot}\;(\mathbf{E} + i\frac{\mathbf{H}/c}{\epsilon_0}) = \frac{i}{\epsilon_0}\,\mathbf{j} \;. \tag{21}$$

Eqs. (21) can be rewritten in the form

$$\text{div}\;(\frac{\mathbf{h}+\mathbf{h}^*}{2} + \frac{\mathbf{f}-\mathbf{f}^*}{2}) = \frac{1}{\epsilon_0}\;\rho$$

$$-i\partial_0(\frac{\mathbf{h}+\mathbf{h}^*}{2} + \frac{\mathbf{f}-\mathbf{f}^*}{2}) + \text{rot}\;(\frac{\mathbf{f}+\mathbf{f}^*}{2} + \frac{\mathbf{h}-\mathbf{h}^*}{2}) = \frac{i}{\epsilon_0}\,\mathbf{j} \;. \tag{22}$$

It has a sense to define two quantities:

$$\mathbf{M} = \frac{\mathbf{h}+\mathbf{f}}{2} \;, \qquad \mathbf{N} = \frac{\mathbf{h}^* - \mathbf{f}^*}{2} \;, \tag{23}$$

which are different 3-vectors under the group $SO(3.C)$: $\mathbf{M}' = O\;\mathbf{M}$, $\mathbf{N}' = O^*\;\mathbf{N}$. With respect to Euclidean rotations, the identity $O^* = O$ holds; whereas for Lorentzian boosts we have quite other identity $O^* = O^{-1}$. In terms of $\mathbf{M}, \mathbf{N}$, eqs. (22) look

$$\text{div}\;\mathbf{M} + \text{div}\;\mathbf{N} = \frac{1}{\epsilon_0}\;\rho \;, \qquad -i\partial_0\mathbf{M} + \text{rot}\;\mathbf{M} - i\partial_0\mathbf{N} - \text{rot}\;\mathbf{N} = \frac{i}{\epsilon_0}\;\mathbf{j}$$

or in a matrix form

$$(-i\partial_0 + \alpha^i \partial_i)\ M + (-i\partial_0 + \beta^i \partial_i)\ N = J$$

$$M = \begin{vmatrix} 0 \\ \mathbf{M} \end{vmatrix} , \qquad N = \begin{vmatrix} 0 \\ \mathbf{N} \end{vmatrix} , \qquad J = \frac{1}{\epsilon_0} \begin{vmatrix} \rho \\ i\,\mathbf{j} \end{vmatrix} . \tag{24}$$

The matrices α^i and β^i are taken in the form

$$\alpha^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \alpha^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \alpha^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}$$

$$\beta^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \beta^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \beta^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} .$$

All of them after squaring give $-I$, and α_i commute with β_j .

4. Minkowski constitutive relations in a complex 3-vector form

Let us examine how the constitutive relations for an uniform media behave under the Lorentz transformations. One should start with these relation in the rest reference frame

$$\mathbf{D} = \epsilon_0 \epsilon\ \mathbf{E}\ , \qquad \frac{\mathbf{H}}{c} = \frac{1}{\mu_0 \mu} \frac{1}{c^2}\ c\mathbf{B} = \frac{\epsilon_0}{\mu}\ c\mathbf{B}\ . \tag{25}$$

They can be rewritten as

$$\frac{\mathbf{h} + \mathbf{h}^*}{2} = \epsilon\ \frac{\mathbf{f} + \mathbf{f}^*}{2}\ , \qquad \frac{\mathbf{h} - \mathbf{h}^*}{2} = \frac{1}{\mu}\ \frac{\mathbf{f} - \mathbf{f}^*}{2}\ , \tag{26}$$

from whence it follows

$$2\mathbf{h} = (\epsilon + \frac{1}{\mu})\ \mathbf{f} + (\epsilon - \frac{1}{\mu})\ \mathbf{f}^*\ , \ \ 2\mathbf{h}^* = (\epsilon + \frac{1}{\mu})\ \mathbf{f}^* + (\epsilon - \frac{1}{\mu})\ \mathbf{f}\ . \tag{27}$$

This is a complex form of the constitutive relations (25). It should be noted that constitutive relations can be resolved under $\mathbf{f}$, $\mathbf{f}^*$ as well:

$$2\mathbf{f} = (\frac{1}{\epsilon} + \mu)\ \mathbf{h} + (\frac{1}{\epsilon} - \mu)\ \mathbf{h}^*\ , \ \ 2\mathbf{f}^* = (\frac{1}{\epsilon} + \mu)\ \mathbf{h}^* + (\frac{1}{\epsilon} - \mu)\ \mathbf{h}\ ; \tag{28}$$

these are the same constitutive equations (27) in other form. Now let us take into account the Lorentz transformations:

$$\mathbf{f}' = O\ \mathbf{f}\ , \qquad \mathbf{f}^{'*} = O^*\ \mathbf{f}^*\ , \qquad \mathbf{h}' = O\ \mathbf{h}\ , \qquad \mathbf{h}^{'*} = O^*\ \mathbf{h}^*\ ;$$

then eqs. (26) will become

$$\frac{O^{-1}\mathbf{h}' + (O^{-1})^*\mathbf{h}^{*'}}{2} = \epsilon\ \frac{O^{-1}\mathbf{f}' + (O^{-1})^*\mathbf{f}^{*'}}{2}$$

$$\frac{O^{-1}\mathbf{h}' - (O^{-1})^*\mathbf{h}^{*'}}{2} = \frac{1}{\mu}\ \frac{O^{-1}\mathbf{f}' - (O^{-1})^*\mathbf{f}^{*'}}{2}\ .$$

Multiplying both equations by O and summing (or subtracting) the results we get

$$2\mathbf{h}' = (\epsilon + \frac{1}{\mu})\ \mathbf{f}' + (\epsilon - \frac{1}{\mu})\ O(O^{-1})^*\ \mathbf{f}'^*$$
$$2\mathbf{h}'^* = (\epsilon + \frac{1}{\mu})\ \mathbf{f}'^* + (\epsilon - \frac{1}{\mu})\ O^* O^{-1}\ \mathbf{f}'\ . \qquad (29)$$

Analogously, starting from (28) we can produce

$$2\mathbf{f}' = (\frac{1}{\epsilon} + \mu)\ \mathbf{h}' + (\frac{1}{\epsilon} - \mu)\ O(O^{-1})^*\ \mathbf{h}'^*$$
$$2\mathbf{f}'^* = (\frac{1}{\epsilon} + \mu)\ \mathbf{h}'^* + (\frac{1}{\epsilon} - \mu)\ O^* O^{-1}\ \mathbf{h}'\ . \qquad (30)$$

Equations (29)-(30) represent the constitutive relations after changing the reference frame. In this point one should distinguish between two cases: Euclidean rotation and Lorentzian boosts. Indeed, for any Euclidean rotations

$$O^* = O, \qquad \Longrightarrow \qquad O(O^{-1})^* = I, \qquad O^* O^{-1} = I\ ;$$

and therefore eqs. (29)-(30) take the form of (27)-(28); in other words, at Euclidean rotations the constitutive relations do not change their form. However, for any pseudo-Euclidean rotations (Lorentzian boosts)

$$O^* = O^{-1}, \qquad \Longrightarrow \qquad O(O^{-1})^* = 0^2, \qquad O^* O^{-1} = O^{*2}\ ;$$

and eqs. (29)-(30) look

$$2\mathbf{h}' = (\epsilon + \frac{1}{\mu})\mathbf{f}' + (\epsilon - \frac{1}{\mu})O^2\ \mathbf{f}'^*\ ,\ 2\mathbf{h}'^* = (\epsilon + \frac{1}{\mu})\mathbf{f}'^* + (\epsilon - \frac{1}{\mu})O^2\mathbf{f}'$$
$$2\mathbf{f}' = (\frac{1}{\epsilon} + \mu)\mathbf{h}' + (\frac{1}{\epsilon} - \mu)O^{*2}\mathbf{h}'^*\ ,\ 2\mathbf{f}'^* = (\frac{1}{\epsilon} + \mu)\mathbf{h}'^* + (\frac{1}{\epsilon} - \mu)O^{*2}\mathbf{h}'\ . \qquad (31)$$

In complex 3-vector form these relations seem to be shorter than in real 3-vector form:

$$2\mathbf{D}' = \epsilon_0\epsilon\ [\ (I + \frac{OO + O^*O^*}{2})\ \mathbf{E}'\ + \frac{OO - O^*O^*}{2i}\ c\mathbf{B}'\ ]$$
$$+\frac{\epsilon_0}{\mu}\ [\ (I - \frac{OO + O^*O^*}{2})\ \mathbf{E}'\ - \frac{OO - O^*O^*}{2i}\ c\mathbf{B}'\ ]$$
$$2\mathbf{H}'/c = \epsilon_0\epsilon\ [\ (I - \frac{OO + O^*O^*}{2})\ c\mathbf{B}'\ + \frac{OO - O^*O^*}{2i}\ \mathbf{E}'\ ]$$
$$+\frac{\epsilon_0}{\mu}\ [\ (I + \frac{OO + O^*O^*}{2})\ c\mathbf{B}'\ - \frac{OO - O^*O^*}{2i}\ \mathbf{E}'\ ]\ . \qquad (32)$$

They can be written differently

$$\mathbf{D}' = \frac{\epsilon_0}{2}\ \{[\ (\epsilon\ + \frac{1}{\mu}) + (\epsilon\ - \frac{1}{\mu})\ \mathrm{Re}\ O^2\ ]\ \mathbf{E}'\ + (\epsilon\ - \frac{1}{\mu})\ \mathrm{Im}\ O^2\ c\mathbf{B}'\ ]\ \}$$
$$\frac{\mathbf{H}'}{c} = \frac{\epsilon_0}{2}\ \{[(\epsilon\ + \frac{1}{\mu}) - (\epsilon\ - \frac{1}{\mu})\ \mathrm{Re}\ O^2\ ]\ c\mathbf{B}'\ + (\epsilon\ - \frac{1}{\mu})\ \mathrm{Im}\ O^2\ \mathbf{E}'\ ]\ \}\ .$$

The matrix O^2 can be presented differently with the help of double angle variable:

$$O^2 = \left| \begin{array}{ccc} \text{ch } 2b + G\, n_1^2 & G\, n_1 n_2 - i \text{ sh } 2b\, n_3 & G\, n_3 n_1 + i \text{ sh } 2b\, n_2 \\ G\, n_1 n_2 + i \text{ sh } 2b\, n_3 & ch\ 2b + G\, n_2^2 & G\, n_2 n_3 - i \text{ sh } 2b\, n_1 \\ G\, n_3 n_1 - i \text{ sh } 2b\, n_2 & G\, n_2 n_3 + i \text{ sh } 2b\, n_1 & \text{ch } 2b + G\, n_3^2 \end{array} \right| ,$$

where $G = (1 - \text{ch } 2b)$.

The previous result can be easily extended to more generale medias, let us restrict ourselves to linear medias. Indeed, arbitrary linear media is characterized by the following constitutive equations:

$$\mathbf{D} = \epsilon_0\ \epsilon(x)\ \mathbf{E} + \epsilon_0 c\ \alpha(x)\ \mathbf{B}\ , \qquad \mathbf{H} = \epsilon_0 c\ \beta(x)\ \mathbf{E} + \frac{1}{\mu_0}\ \mu(x)\ \mathbf{B}\ , \tag{33}$$

where $\epsilon(x), \mu(x), \alpha(x), \beta(x)$ are 3×3 dimensionless matrices. Eqs. (33) should be rewritten in terms of complex vectors $\mathbf{f}, \mathbf{h}$:

$$\begin{aligned} \frac{\mathbf{h} + \mathbf{h}^*}{2} &= \epsilon(x)\ \frac{\mathbf{f} + \mathbf{f}^*}{2} + \alpha(x)\ \frac{\mathbf{f} - \mathbf{f}^*}{2i} \\ \frac{\mathbf{h} - \mathbf{h}^*}{2i} &= \beta(x)\ \frac{\mathbf{f} + \mathbf{f}^*}{2} + \mu(x)\ \frac{\mathbf{f} - \mathbf{f}^*}{2i}\ . \end{aligned} \tag{34}$$

From (33) it follows

$$\begin{aligned} \mathbf{h} &= [\ (\epsilon(x) + \mu(x)) + i(\beta(x) - \alpha(x))\ ]\ \mathbf{f} \\ &+ [\ (\epsilon(x) - \mu(x)) + i(\beta(x) + \alpha(x))\ ]\ \mathbf{f}^* \\ \mathbf{h}^* &= [\ (\epsilon(x) + \mu(x)) - i(\beta(x) - \alpha(x))\ ]\ \mathbf{f}^* \\ &+ [\ (\epsilon(x) - \mu(x)) - i(\beta(x) + \alpha(x))\ ]\ \mathbf{f}\ . \end{aligned} \tag{35}$$

Under Lorentz transformations, relations (6.17) will take the form

$$\begin{aligned} \mathbf{h}' &= \epsilon_0\, [(\epsilon(x) + \mu(x)) + i(\beta(x) - \alpha(x))]\, \mathbf{f}' \\ &+ [\ (\epsilon(x) - \mu(x)) + i(\beta(x) + \alpha(x))]\, [O(O^{-1})^*]\ \mathbf{f}^{'*} \\ \mathbf{h}^{'*} &= \epsilon_0\, [\ (\epsilon(x) + \mu(x)) - i(\beta(x) - \alpha(x))]\, \mathbf{f}^{'*} \\ &+ [(\epsilon(x) - \mu(x)) - i(\beta(x) + \alpha(x))]\, [O^*(O^{-1})]\mathbf{f}'. \end{aligned} \tag{36}$$

For Euclidean rotation, the constitutive relations preserve their form. For Lorentz boosts we have

$$\begin{aligned} \mathbf{h}' &= [\ (\epsilon(x) + \mu(x)) + i\ (\beta(x) - \alpha(x))\ ]\mathbf{f}' \\ &+ [\ (\epsilon(x) - \mu(x)) + i\ (\beta(x) + \alpha(x))\ ]O^2\mathbf{f}^{'*} \\ \mathbf{h}^{'*} &= [\ (\epsilon(x) + \mu(x)) - i\ (\beta(x) - \alpha(x))\ ]\mathbf{f}^{'*} \\ &+ [\ (\epsilon(x) - \mu(x)) - i\ (\beta(x) + \alpha(x))\ ]O^{*2}\mathbf{f}'. \end{aligned} \tag{37}$$

They are the constitutive equations for arbitrary linear medias in a moving reference frame (similar formulas were produced in quaternion formalism in [108], [113]).

5. Symmetry the matrix Maxwell equation in a uniform media

As noted, Maxwell equations in any media can be presented in the matrix form:

$$(-i\partial_0 + \alpha^i \partial_i)\, M + (-i\partial_0 + \beta^i \partial_i)\, N = J \, . \tag{38}$$

We are to study symmetry properties of this equation under complex rotation group SO(3.C). The terms with α^j matrices were examined in Section **2**), the terms with β^j matrix are new. We restrict ourselves to demonstrating the Lorentz symmetry of eq. (38) under two simplest transformations.

First, let us consider the Euclidean rotation in the plane $(1-2)$, we examine additionally only the term with β-matrices:

$$S\beta^1 S^{-1} = \cos a\; \beta^1 - \sin a\; \beta^2 = \beta^j O_{j1}$$
$$\beta^2 S^{-1} = \sin a\; \beta^1 + \cos a\; \beta^2 = \beta^j O_{j2}$$
$$S\beta^3 S^{-1} = \beta^3 = \beta^j O_{j3} \, . \tag{39}$$

Therefore, we conclude that eq. (38) is symmetrical under Euclidean rotations in accordance with the relations

$$(-i\partial_0 + S\alpha^i S^{-1}\partial_i)\, M' + (-i\partial_0 + S\beta^i S^{-1}\partial_i)\, N' = +SJ \, , \quad \Longrightarrow$$
$$(-i\partial_0 + \alpha^i \partial_i')\, M' + (-i\partial_0 + \beta^i \partial_i')\, N' = +J' \, . \tag{40}$$

For the Lorentz boost in the plane $(0-3)$ we have

$$M' = SM \, , \qquad N' = S^* N = S^{-1} N, \qquad S^* = S^{-1} \, ;$$

and eq. (38) takes the form (note that the additional transformation $\Delta = \Delta_{(\alpha)}$ is combined in terms of α^j; see Sec. **2**)

$$\Delta_{(\alpha)} S \; \left[\, (-i\partial_0 + \alpha^i \partial_i)\, S^{-1} M' + (-i\partial_0 + \beta^i \partial_i)\, S N' \,\right] = \Delta S J$$

or

$$\Delta_{(\alpha)} \; \left[(-i\partial_0 + S\alpha^i S^{-1}\partial_i)\, M' + S^2(-i\partial_0 + S^{-1}\beta^i S\partial_i)\, N'\right] = J' \, ,$$

and further

$$(-i\partial_0' + \alpha^i \partial_i')\, M' + \Delta_{(\alpha)} S^2(-i\partial_0 + S^{-1}\beta^i S\partial_i)\, N' = J' \, . \tag{41}$$

It remains to prove the relationship

$$\Delta_{(\alpha)} S^2\, (-i\partial_0 + S^{-1}\beta^i S\partial_i)\, N' = (-i\partial_0' + \beta^i \partial_i')\, N' \, . \tag{42}$$

By simplicity reason one may expect two identities:

$$\Delta_{(\alpha)} S^2 = \Delta_{(\beta)} \qquad \Longleftrightarrow \qquad \Delta_{(\alpha)} S = \Delta_{(\beta)} S^{-1} \, , \tag{43}$$

and

$$\Delta_{(\beta)}(-i\partial_0 + S^{-1}\beta^i S\partial_i)\, N' = (-i\partial_0' + \beta^i \partial_i')\, N' \, . \tag{44}$$

Let us prove them for a Lorentzian boost in the plane $0-3$:

$$S = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & ch\ b & -i\ sh\ b & 0 \\ 0 & i\ sh\ b & ch\ b & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| , \qquad S^{-1} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & ch\ b & -i\ sh\ b & 0 \\ 0 & i\ sh\ b & ch\ b & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| ;$$

we readily get

$$S^{-1}\beta^1 S = ch\ b\ \beta^1 - i\ sh\ b\ \beta^2 = \beta^j O^{-1}_{j1},$$
$$S^{-1}\beta^2 S = i\ sh\ b\beta^1 + ch\ b\ \beta^2 = \beta^j O^{-1}_{j2}\ , \quad S^{-1}\beta^3 S = \beta^3 = \beta^j O^{-1}_{j3}\ . \tag{45}$$

To verify identity $\Delta_{(\alpha)} S = \Delta_{(\beta)} S^{-1}$,:

$$(ch\ b - ish\ b\ \alpha^3)S = (ch\ b - ish\ b\ \beta^3)S^{-1}\ ,$$

let us calculate separately the left and right parts:

$$(ch\ b - ish\ b\ \alpha^3)S = (ch\ b - ish\ b\ \beta^3)S^{-1} = \left| \begin{array}{cccc} ch\ b & 0 & 0 & -i\ sh\ b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\ sh\ b & 0 & 0 & ch\ b \end{array} \right| .$$

they coincide with each other, so eq. (43) holds. It remains to prove relation (44). Allowing for the properties of β–matrices

$$(\beta^0)^2 = -I,\ (\beta^1)^2 = -I,\ \beta^1\beta^2 = -\ \beta^3, \qquad \beta^2\beta^1 = +\ \beta^3 \ \text{ and so on}$$

we readily find

$$\Delta_{(\beta)}\ (-i\partial_0 + S^{-1}\beta^i S\partial_i)\ N' = (ch\ b - ish\ b\ \beta^3)\ [\ -i\partial_0 + \beta^3\partial_3$$
$$+(ch\ b\ \beta^1 - i\ sh\ b\ \beta^2)\ \partial_1 + (i\ sh\ b\beta^1 + ch\ b\ \beta^2)\ \partial_2\ ]\ N'$$
$$= [\ -i(ch\ b\ \partial_0 - sh\ b\ \partial_3) + \beta^3\ (-sh\ b\ \partial_0 + ch\ b\ \partial_3) + \beta^1\ \partial_1 + \beta^2\ \partial_2\ ]\ N'\ ,$$

that is

$$\Delta_{(\beta)}(-i\partial_0 + S^{-1}\beta^i S\partial_i)\ N' = (-i\partial'_0 + \beta^1\partial_1 + \beta^2\partial_2 + \beta^3\partial'_3)\ N'\ ; \tag{46}$$

the relation (44) holds. Thus, the symmetry of the matrix Maxwell equation in media under the Lorentz group is proved.

6. Maxwell theory, Dirac matrices and electromagnetic 4-vectors

Let us shortly discuss two points relevant to the above matrix formulation of the Maxwell theory.

First, let us write down explicit form for Dirac matrices in spinor basis:

$$\gamma^0 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \qquad \gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$\gamma^1 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \gamma^2 = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \gamma^3 = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}.$$

Taking in mind expressions for α^i, β^i, we immediately see the identities

$$\begin{aligned} &\alpha^1 = i\gamma^0\gamma^2, \qquad \alpha^2 = \gamma^0\gamma^5, \qquad \alpha^3 = i\gamma^5\gamma^2 \\ &\beta^1 = -\gamma^3\gamma^1, \qquad \beta^2 = -\gamma^3, \qquad \beta^3 = -\gamma^1 \,, \end{aligned} \tag{47}$$

so the Maxwell matrix equation in media takes the form

$$\begin{aligned} &(-i\partial_0 + i\gamma^0\gamma^2\partial_1 + \gamma^0\gamma^5\partial_2 + i\gamma^5\gamma^2\partial_3)\, M \\ &+ (-i\partial_0 - \gamma^3\gamma^1\partial_1 - \gamma^3\partial_2 - \gamma^1\partial_3)\, N = J \,. \end{aligned} \tag{48}$$

This Dirac matrix-based form does not seem to be very useful to apply in the Maxwell theory, it does not prove much similarity with ordinary Dirac equation (though that analogy was often discussed in the literature).

Now starting from electromagnetic 2-tensor and dual to it:

$$\tilde{F}_{\rho\sigma} = \frac{1}{2}\, \epsilon_{\rho\sigma\alpha\beta} F^{\alpha\beta} \,, \qquad F_{\alpha\beta} = -\frac{1}{2}\, \epsilon_{\alpha\beta\rho\sigma} \tilde{F}^{\rho\sigma}$$

let us introduce two electromagnetic 4-vectors (below u^α is any 4-vector that in general may not coincide with 4-velocity)

$$e^\alpha = u_\beta F^{\alpha\beta} \,, \qquad b^\alpha = u_\beta \tilde{F}^{\alpha\beta} \,, \qquad u^\alpha u_\alpha = 1 \,; \tag{49}$$

inverse formulas are

$$\begin{aligned} F^{\alpha\beta} &= (e^\alpha\, u^\beta - e^\beta\, u^\alpha) - \epsilon^{\alpha\beta\rho\sigma}\, b_\rho\, u_\sigma \\ \tilde{F}^{\alpha\beta} &= (b^\alpha\, u^\beta - b^\beta\, u^\alpha) + \epsilon^{\alpha\beta\rho\sigma}\, e_\rho\, u_\sigma \,. \end{aligned} \tag{50}$$

Such electromagnetic 4-vector are presented always in the literature on the electrodynamics of moving bodies, from the very beginning of relativistic tensor form of electrodynamics – see Minkowski [5], Gordon [11], Mandel'stam – Tamm [12], [13], [14]; for instance see Yépez – Brito – Vargas [110]. The interest to these field variables gets renewed after Esposito paper [123] in 1998.

In 3-dimensional notation

$$E^1 = -E_1 = F^{10} \,, \qquad cB^1 = cB_1 = \tilde{F}^{10} = -F_{23}, \qquad \text{and so on}$$

the formulas (49) take the form

$$\begin{aligned} e^0 &= \mathbf{u}\, \mathbf{E} \,, \qquad \mathbf{e} = u^0\, \mathbf{E} + c\, \mathbf{u} \times \mathbf{B} \\ b^0 &= c\, \mathbf{u}\, \mathbf{B} \,, \qquad \mathbf{b} = c\, u^0\, \mathbf{B} - \mathbf{u} \times \mathbf{E} \,, \end{aligned} \tag{51}$$

or symbolically $(e, b) = U(u)$ $(\mathbf{E}, \mathbf{B})$; and inverse the formulas (50) look

$$\mathbf{E} = \mathbf{e}\, u^0 - e^0\, \mathbf{u} + \mathbf{b} \times \mathbf{u}$$
$$c\,\mathbf{B} = \mathbf{b}\, u^0 - b^0\, \mathbf{u} - \mathbf{e} \times \mathbf{u}\,. \tag{52}$$

or in symbolical form $(\mathbf{E}, \mathbf{B}) = U^{-1}(u)$ (e, b).

The above possibility is often used to produce a special form of the Maxwell equations. For simplicity, let us consider the vacuum case:

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0\,, \qquad \partial_\alpha F^{\alpha\beta} = \epsilon_0^{-1} j^\beta$$

or differently with the help of the dual tensor:

$$\partial_\beta \tilde{F}^{\beta\alpha} = 0\,, \qquad \partial_\alpha F^{\alpha\beta} = \epsilon_0^{-1} j^\beta \tag{53}$$

These can be transformed to variables b^α, b^α:

$$\partial_\alpha (b^\alpha\, u^\beta - b^\beta\, u^\alpha + \epsilon^{\alpha\beta\rho\sigma}\, e_\rho\, u_\sigma) = 0$$
$$\partial_\alpha (e^\alpha\, u^\beta - e^\beta\, u^\alpha - \epsilon^{\alpha\beta\rho\sigma}\, b_\rho\, u_\sigma) = \epsilon_0^{-1} j^\beta\,. \tag{54}$$

They can be combined into equations for complex field function

$$\Phi^\alpha = e^\alpha + i b^\alpha\,, \qquad \partial_\alpha\, [\, \Phi^\alpha u^\beta - \Phi^\beta u^\alpha + i \epsilon^{\alpha\beta\rho\sigma} \Phi_\rho u_\sigma\,] = \epsilon_0^{-1} j^\beta$$

or differently

$$\partial_\alpha\, [\, \delta^\alpha_\gamma u^\beta - \delta^\beta_\gamma u^\alpha + i \epsilon^{\alpha\beta\rho\sigma} g_{\rho\gamma} u_\sigma\,]\, \Phi^\gamma = \epsilon_0^{-1} j^\beta\,. \tag{55}$$

This is Esposito's representation [123] of the Maxwell equations. One may introduce four matrices, functions of 4-vector u:

$$(\Gamma^\alpha)^\beta{}_\gamma = \delta^\alpha_\gamma u^\beta - \delta^\beta_\gamma u^\alpha + i \epsilon^{\alpha\beta\rho\sigma} g_{\rho\gamma} u_\sigma\,, \tag{56}$$

then eq. (55) becomes

$$\partial_\alpha (\Gamma^\alpha)^\beta{}_\gamma\, \Phi^\gamma = \epsilon_0^{-1} j^\beta\,, \qquad \text{or} \qquad \Gamma^\alpha \partial_\alpha \Phi = \epsilon_0^{-1} j\,. \tag{57}$$

In the 'rest reference frame' when $u^\alpha = (1, 0, 0, 0)$, the matrices Γ^α become simpler and $\Phi = \Psi$:

$$\Gamma^0 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \Gamma^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix}$$
$$\Gamma^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix}, \Gamma^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

and eq. (57) takes the form

$$\left| \begin{array}{cccc} 0 & \partial_1 & \partial_2 & \partial_3 \\ 0 & -\partial_0 & i\partial_3 & -i\partial_2 \\ 0 & -i\partial_3 & -\partial_0 & i\partial_1 \\ 0 & i\partial_2 & -i\partial_1 & -\partial_0 \end{array} \right| \left| \begin{array}{c} 0 \\ E^1 + icB^1 \\ E^2 + icB^2 \\ E^3 + icB^3 \end{array} \right| = \epsilon_0^{-1} \left| \begin{array}{c} \rho \\ j^1 \\ j^2 \\ j^3 \end{array} \right| = \epsilon_0^{-1}\, j \; , \tag{58}$$

or

$$\begin{aligned} \text{div}\;(\mathbf{E} + ic\mathbf{B}) &= \epsilon_0^{-1}\, \rho \; , \\ -\partial_0(\mathbf{E} + ic\mathbf{B}) - i\;\text{rot}\;(\mathbf{E} + ic\mathbf{B}) &= \epsilon_0^{-1}\mathbf{j} \; . \end{aligned}$$

From whence we get equations

$$\begin{aligned} \text{div}\; c\mathbf{B} = 0 \; , \qquad & \text{rot}\;\mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial ct} \\ \text{div}\;\mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad & \text{rot}\; c\mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial ct} \; , \end{aligned}$$

which coincides with eqs. (2).
Relations (58) correspond to a special choice of α-matrices:

$$\beta\;(-i\alpha^0) = \Gamma^0 \; , \;\; \beta\;\alpha^j = \Gamma^j \; , \qquad \text{where} \qquad \beta = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{array} \right| . \tag{59}$$

Esposito's representation of the Maxwell equation at any 4-vector u^α can be easily related to the matrix equation of Riemann – Silberstein – Majorana – Oppenheimer:

$$(-i\alpha^0\partial_0 + \alpha^j\partial_j)\Psi = J \; , \tag{60}$$

indeed

$$\begin{aligned} (-i\alpha^0\partial_0 + \alpha^j\partial_j)U^{-1}\;(U\Psi) = J \\ -i\alpha^0\; U^{-1} = \beta\,\Gamma^0 \; , \qquad \alpha^j\; U^{-1} = \beta\,\Gamma^j \; , \qquad U\Psi = \Phi \\ \beta\;(\Gamma^0\partial_0 + \Gamma^j\partial_j)\;\Phi = J \; , \qquad \beta^{-1}J = \epsilon_0^{-1}(j^a) \\ (\Gamma^0\partial_0 + \Gamma^j\partial_j)\Phi = \epsilon_0^{-1}\; j \; . \end{aligned} \tag{61}$$

Eq. (61) is a matrix representation of the Maxwell equations in Esposito's form

$$\partial_\alpha\;[\;\delta^\alpha_\gamma u^\beta - \delta^\beta_\gamma u^\alpha + i\epsilon^{\alpha\beta\rho\sigma}g_{\rho\gamma}u_\sigma\;]\;\Phi^\gamma = \epsilon_0^{-1}j^\beta \tag{62}$$

Evidently, eqs. (60) and (62) are equivalent to each other. There is no ground to consider the form (62) obtained through the trivial use of identity $I = U^{-1}(u)U(u)$ as having certain especially profound sense. Our point of view contrasts with the claim by Ivezić [128]-[133]-[129]-[130]-[134]- [140]-[141]-[142]-[143] that eq. (62) has a status of a true Maxwell equation in a moving reference frame (at this u^α is identified with 4-velocity).

7. Maxwell equation in a curved space-time, no media case

Now the main question is how the above Maxwell matrix equation (first consider the no-media case)

$$(\alpha^0 \partial_0 + \alpha^j \partial_j) \; \Psi = J \; , \qquad \alpha^0 = -iI$$
$$\Psi = \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right| \; , \qquad J = \frac{1}{\epsilon_0} \left| \begin{array}{c} \rho \\ i\mathbf{j} \end{array} \right|$$

can be generalized to the case of a curved space-time background. We should expect existence of an extended equation in the frame of general Tetrode-Weyl-Fock-Ivanenko tetrad approach [20], [21], [17], [23]. Such an equation might be of the following form

$$\alpha^\rho(x) \; [\; \partial_\rho + A_\rho(x) \;] \; \Psi(x) = J(x)$$
$$\alpha^\rho(x) = \alpha^c \; e^\rho_{(c)}(x) \; , \qquad A_\rho(x) = \frac{1}{2} j^{ab} \; e^\beta_{(a)} \; \nabla_\rho e_{(n)\beta} \; . \tag{63}$$

j^{ab} stands for generators of 3-vector field under complex orthogonal group $SO(3.C)$, their explicit form will be given later. Tetrad represents four covariant vectors related to metric tensor by means of a bilinear function $g_{\alpha\beta}(x) = \eta^{ab} e_{(a)\alpha} e_{(b)\beta}$, so that all tetrads referred by local Lorentz transformations correspond to the same metric $g_{\alpha\beta}(x)$: $e'_{(a)\alpha}(x) = L_a{}^b(x) \; e_{(b)\alpha}(x)$. Eq. (63) can be rewritten as

$$\alpha^c \; (\; e^\rho_{(c)} \partial_\rho + \frac{1}{2} j^{ab} \gamma_{abc} \;) \; \Psi = J(x) \; , \tag{64}$$

where Ricci rotation coefficients are used: $\gamma_{bac} = -\gamma_{abc} = -e_{(b)\beta;\alpha} e^\beta_{(a)} e^\alpha_{(c)}$. With regard to eq. (63), one should expect symmetry properties of the equation under local gauge transformations:

$$\Psi'(x) = S(x)\Psi(x) \; , \qquad e'_{(a)\alpha}(x) = L_a{}^b(x) \; e_{(b)\alpha}(x)$$
$$\alpha^\rho(x) \; [\partial_\rho + A_\rho(x)] \; \Psi(x) = J(x) \; , \; \Longrightarrow$$
$$\alpha'^\rho(x) \; [\partial_\rho + A'_\rho(x)] \; \Psi'(x) = J'(x) \; . \tag{65}$$

We should consider separately Euclidean and Lorentzian tetrad rotations. In the case of Euclidean rotations we may expect the following symmetry:

$\underline{S = S[a(x), \mathbf{n}(x)]}$

$$\Psi' = S\Psi \; , \qquad \Psi = S^{-1}\Psi', \qquad SJ(x) = J'$$
$$S\alpha^\rho S^{-1} \; (\partial_\rho + SA_\rho S^{-1} + S\partial_\rho S^{-1}) \; \Psi'(x) = SJ(x)$$
$$S\alpha^\rho S^{-1} = \alpha'^\rho \; , \qquad SA_\rho S^{-1} + S\partial_\rho S^{-1} = A'_\rho \; . \tag{66}$$

In the case of Lorentzian rotations we may expect other symmetry realized in accordance with relations

$\underline{S = S[ib(x), \mathbf{n}(x)] \; , \;\; \Delta = \Delta[ib(x), \mathbf{n}(x)]}$

$$\Psi' = S\Psi \; , \qquad \Psi = S^{-1}\Psi', \qquad \Delta \; SJ(x) = J'$$
$$\Delta S\alpha^\rho S^{-1} \; (\partial_\alpha + SA_\alpha S^{-1} + S\partial_\alpha S^{-1}) \; \Psi'(x) = \Delta SJ(x)$$
$$\Delta \; S\alpha^\rho S^{-1} = \alpha'^\rho \; , \qquad SA_\alpha S^{-1} + S\partial_\alpha S^{-1} = A'_\alpha \; . \tag{67}$$

Symmetry properties of the local matrices $\alpha^\rho(x)$ can be found quite straightforwardly on the base of analysis performed for the flat Minkowski space. Indeed, for local Euclidean rotations, the rule for $S\alpha^\rho(x)S^{-1}$ is

$$S\alpha^\rho S^{-1} = S\alpha^0 e^\rho_{(0)} S^{-1} + S\alpha^l e^\rho_{(l)} S^{-1}$$
$$= \alpha^0 e^\rho_{(0)} + \alpha^k O_{kl} e^\rho_{(l)} = \alpha^0 e^\rho_{(0)} + \alpha^k e'^\rho_{(k)} = \alpha'^\rho \, .$$

For local Lorentzian rotations, we can easily prove a symmetry relation:

$$\Delta \; S\alpha^\rho(x)S^{-1} = \Delta \; S\alpha^a e^\rho_{(a)} S^{-1}$$
$$= [\Delta \; S\alpha^a S^{-1}] \; e^\rho_{(a)} = \alpha^b L_b{}^a \; e^\rho_{(a)} = \alpha^b e'^\rho_{(b)} = \alpha'^\rho(x) \, .$$

The transformation law for the complex 3-vector connection $A_\rho(x)$ will be proved in Section **8**.

8. On tetrad transformation for complex 3-vector connection

First, let us list six elementary rotations from the local group $SO(3.C)$:

$$S_{23} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos a & -\sin a \\ 0 & 0 & \sin a & \cos a \end{array} \right| , \qquad S_{01} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \text{ch}\, b & -i\text{sh}\, b \\ 0 & 0 & +i\text{sh}\, b & \text{ch}\, b \end{array} \right|$$

$$S^1 = j^{23} = \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau_1 \end{array} \right| , \qquad N^2 = j^{01} = +i \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau_1 \end{array} \right|$$

$$S_{31} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \cos a & 0 & \sin a \\ 0 & 0 & 1 & 0 \\ 0 & -\sin a & 0 & \cos a \end{array} \right| , \qquad S_{02} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \text{ch}\, b & 0 & +i\text{sh}\, b \\ 0 & 0 & 0 & 0 \\ 0 & -i\text{sh}\, b & 0 & \text{ch}\, b \end{array} \right|$$

$$S^2 = j^{31} = \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau_2 \end{array} \right| , \qquad N^2 = j^{02} = +i \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau_2 \end{array} \right|$$

$$S_{12} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \cos a & -\sin a & 0 \\ 0 & \sin a & \cos a & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| , \qquad S_{03} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \text{ch}\, b & -i\text{sh}\, b & 0 \\ 0 & +i\text{sh}\, b & \text{ch}\, b & 0 \\ 0 & 0 & 0 & 1 \end{array} \right|$$

$$S^3 = j^{12} = \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau_3 \end{array} \right| , \qquad N^3 = j^{03} = +i \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau_3 \end{array} \right| \; ;$$

they obey the commutative relations:

$$S^1S^2 - S^2S^1 = S^3 \, , \qquad N^1N^2 - N^2N^1 = -S^3 \, , \qquad S^1N^2 - N^2S^1 = +N^3 \, ;$$

and remaining ones written be cyclic symmetry. Let us turn to some properties of the connection

$A_\alpha(x)$:

$$A_\alpha(x) = \frac{1}{2} j^{ab} \, e^{\beta}_{(a)} \, \nabla_\alpha e_{(b)\beta}$$
$$= S^1 e^{\beta}_{(2)} \, \nabla_\alpha e_{(3)\beta} + S^2 e^{\beta}_{(3)} \, \nabla_\alpha e_{(2)\beta} + S^3 e^{\beta}_{(1)} \, \nabla_\alpha e_{(2)\beta}$$
$$+ N^1 e^{\beta}_{(0)} \, \nabla_\alpha e_{(1)\beta} + N^2 e^{\beta}_{(0)} \, \nabla_\alpha e_{(2)\beta} + N^3 e^{\beta}_{(0)} \, \nabla_\alpha e_{(3)\beta}] \, . \tag{68}$$

Taking in mind the identity $N_k = +iS_k$, and introducing new complex variables

$$A_{(1)\alpha} = e^{\beta}_{(2)} \, \nabla_\alpha e_{(3)\beta} + i \, e^{\beta}_{(0)} \, \nabla_\alpha e_{(1)\beta}$$
$$A_{(2)\alpha} = e^{\beta}_{(3)} \, \nabla_\alpha e_{(1)\beta} + i \, e^{\beta}_{(0)} \, \nabla_\alpha e_{(2)\beta}$$
$$A_{(3)\alpha} = e^{\beta}_{(1)} \, \nabla_\alpha e_{(2)\beta} + i \, e^{\beta}_{(0)} \, \nabla_\alpha e_{(3)\beta} \, ,$$

one can read the above connection as

$$A_\alpha(x) = S^k \, A_{(k)\alpha} \, . \tag{69}$$

With the use of notation

$$A_\alpha(x) = \frac{1}{2} j^{ab} e^{\beta}_{(a)} \nabla_\alpha e_{(b)\beta} = \frac{1}{2} j^{ab} A_{(a)(b)\alpha} \, , \; A_{(a)(b)\alpha} = -A_{(b)(a)\alpha} \tag{70}$$

the above definition for $A_{(k)\alpha}$ can be rewritten differently:

$$A_{(1)\alpha} = A_{(2)(3)\alpha} + iA_{(0)(1)\alpha}$$
$$A_{(2)\alpha} = A_{(3)(1)\alpha} + iA_{(0)(2)\alpha}$$
$$A_{(3)\alpha} = A_{(1)(2)\alpha} + iA_{(0)(3)\alpha} \, .$$

In other words, the 3-quantity $A_{(k)\alpha}$ with respect to 3-index (k) is constructed in terms of "tensor" $A_{(a)(b)\alpha}$ by the same rule that used at constructing 3-dimensional complex vector $-i(\mathbf{E} + ic\mathbf{B})$ in terms of component of tensor F_{ab}.

It is readily verified that such 3-dimensional complex vectors can be built in terms of a skew-symmetric 2-rank real tensor through a simple and symmetrical algebraic construction:

$$\frac{i}{2} \, \bar{\sigma}^a \sigma^b A_{(a)(b)\alpha} = \sigma^k A_{(k)\alpha} \, , \qquad \frac{i}{2} \, \sigma^a \bar{\sigma}^b A_{(a)(b)\alpha} = \sigma^k A^*_{(k)\alpha} \, . \tag{71}$$

From the above it follows covariant formulas for $A_{(k)\alpha}$ and $A^*_{(k)\alpha}$:

$$A_{(k)\alpha} = \frac{i}{4} \, \text{Sp} \, [\bar{\sigma}^a \sigma^b A_{(a)(b)\alpha}] \, , \qquad A^*_{(k)\alpha} = \frac{i}{4} \text{Sp} \, [\sigma_k \sigma^a \bar{\sigma}^b A_{(a)(b)\alpha}] \, . \tag{72}$$

Now, starting from relations between any two tetrads by a local Lorentz transformation:

$$e'^{\alpha}_{(a)} = L_a^{\;b} e^{\alpha}_{(b)} \, , \qquad e^{\alpha}_{(a)} = (L^{-1})_a^{\;b} e'^{\alpha}_{(b)} \, ,$$

let us derive a rule to transform 3-vector connection when changing the tetrad:

$$A_{(a)(b)\alpha} = e^{\beta}_{(a)} \, \nabla_\alpha e_{(b)\beta} = (L^{-1})_a^{\;m} e'^{\beta}_{(m)} \nabla_\alpha (L^{-1})_b^{\;n} e'_{(n)\beta}$$
$$= (L^{-1})_a^{\;m} e'^{\beta}_{(m)} (L^{-1})_b^{\;n} \; \nabla_\alpha e'_{(n)\beta} + (L^{-1})_a^{\;m} e'^{\beta}_{(m)} \; \frac{\partial (L^{-1})_b^{\;n}}{\partial x^\alpha} \; e'_{(n)\beta} \, ,$$

that is

$$A_{(a)(b)\alpha} = (L^{-1})_a{}^m (L^{-1})_b{}^n A'_{(m)(n)\alpha} + (L^{-1})_a{}^m g_{(m)(n)} \frac{\partial (L^{-1})_b{}^n}{\partial x^\alpha} \; . \tag{73}$$

Let us act on this relation from the left by an operator $\frac{i}{4}$ Sp $[\sigma_k \bar{\sigma}^a \sigma^b \; ...]$; it results in

$$A_{(k)\alpha} = \frac{i}{4} \, \text{Sp} \, [\sigma_k \bar{\sigma}^a \sigma^b A_{(a)(b)\alpha}]$$
$$= \frac{i}{4} \, \text{Sp} \, [\sigma_k \bar{\sigma}^a \sigma^b (L^{-1})_a{}^m (L^{-1})_b{}^n \;\; A'_{(m)(n)\alpha}]$$
$$+ \frac{i}{4} \, \text{Sp} \, [\sigma_k \bar{\sigma}^a \sigma^b (L^{-1})_a{}^m \;\; g_{(m)(n)} \;\; \frac{\partial (L^{-1})_b{}^n}{\partial x^\alpha} \;] \; . \tag{74}$$

One may expect eq. (74) to be equivalent to

$$A_{(k)\alpha} = O^{-1}_{kn} A'_{(n)\alpha} + \frac{i}{4} \text{Sp} \, [\sigma_k \bar{\sigma}^a \sigma^b (L^{-1})_a{}^m \;\; g_{(m)(n)} \;\; \frac{\partial}{\partial x^\alpha} (L^{-1})_b{}^n \;] \; ; \tag{75}$$

it is so if an identity holds

$$\frac{i}{4} \text{Sp} \, [\sigma_k \bar{\sigma}^a \sigma^b (L^{-1})_a{}^m (L^{-1})_b{}^n \;\; A'_{(m)(n)\alpha} \;] = O^{-1}_{kl} A'_{(l)\alpha} \; . \tag{76}$$

which is proved by direct calculation (all details are omitted). Now, we are ready to prove the following relationships:

$$O A_\rho O^{-1} + O \partial_\rho O^{-1} = A'_\rho \; . \tag{77}$$

Taking into account the linear decomposition $A_\alpha = A_{(k)\alpha} \tau_k$, eq. (77) can be rewritten as

$$\tau^l O_{lk} A_{(k)\alpha} + O \partial_\alpha O^{-1} = \tau^k A'_{(k)\alpha} \; . \tag{78}$$

Substituting expression for $A_{(k)\alpha}$ through $A'_{(k)\alpha}$ (see (75))

$$A_{(k)\alpha} = O^{-1}_{kn} A'_{(n)\alpha} + \frac{i}{4} \, \text{Sp} \, [\; (\sigma_k \bar{\sigma}^a \sigma^b (L^{-1})_a{}^m \;\; g_{(m)(n)} \;\; \frac{\partial}{\partial x^\alpha} (L^{-1})_b{}^n \;] \; ;$$

we get

$$\tau^l O_{lk} \{ O^{-1}_{kn} A'_{(n)\alpha} + \frac{i}{4} \text{Sp} \, [\; \sigma_k \bar{\sigma}^a \sigma^b (L^{-1})_a{}^m \;\; g_{(m)(n)} \;\; \frac{\partial}{\partial x^\alpha} (L^{-1})_b{}^n \;] \; \}$$
$$+ \; O \partial_\alpha O^{-1} = \tau^k A'_{(k)\alpha} \; .$$

From whence we conclude that an identity must hold:

$$\tau^l O_{lk} \; \frac{i}{4} \text{Sp} \, [\; \sigma_k \bar{\sigma}^a \sigma^b C_{ab,\alpha} \;] \; + \; O \partial_\alpha O^{-1} = 0 \; , \tag{79}$$

where

$$C_{ab,\alpha} = (L^{-1})_a{}^m \;\; g_{(m)(n)} \;\; \frac{\partial}{\partial x^\alpha} (L^{-1})_b{}^n \; .$$

The identity (79) holds indeed which can be proved with the use of simplest transformations – all details are omitted. Thus, generally covariant Maxwell matrix equation in a Riemannian space-time possesses all needed symmetry properties under local tetrad transformations and therefore it is correct.

9. Maxwell equation in a curved space-time, in media

Now we are to extend the Maxwell matrix equation in media to a curved space-time background: starting from the equation

$$(-i\partial_0 + \alpha^i \partial_i)\ M + (-i\partial_0 + \beta^i \partial_i)\ N = J$$
$$M' = SM\ , \qquad N' = S^* N\ ; \tag{80}$$

we may propose the following one

$$\alpha_\rho(x)(i\partial_\rho + A_\rho)\ M + \beta_\rho(x)(i\partial_\rho + B_\rho)\ N = J\ , \tag{81}$$

where $A(x), B_\rho = A^*(x)$ stand connections related to the fields $M(x)$ and $N(x)$ respectively. We should consider separately Euclidean and Lorentzian tetrad rotations.

In the case of Euclidean rotations we may expect the following symmetry:

$$S^* = S\ , \qquad S(x)J(x) = J'(x)$$
$$M'(x) = S(x)M(x)\ , \qquad N'(x) = S(x)N(x)$$
$$S\alpha^\rho S^{-1}\ (\partial_\rho + SA_\rho S^{-1} + S\partial_\rho S^{-1})\ M'(x)$$
$$+S\beta^\rho S^{-1}\ (\partial_\rho + SB_\rho S^{-1} + S\partial_\rho S^{-1})\ N'(x) = SJ(x)$$
$$S\alpha^\rho S^{-1} = \alpha'^{\rho}\ , \qquad S\beta^\rho S^{-1} = \beta'^{\rho}$$
$$SA_\rho S^{-1} + S\partial_\rho S^{-1} = A'_\rho\ , \qquad SB_\rho S^{-1} + S\partial_\rho S^{-1} = B'_\rho\ . \tag{82}$$

In the case of Lorentzian rotations we may expect other symmetry realized in accordance with relations

$$S^* = S^{-1}\ , \qquad \Delta_\alpha(x)\ , \qquad \Delta_\alpha(x)\ S(x)\ J(x) = J'$$
$$M(x)' = S(x)M(x)\ , \qquad N'(x) = S^*(x)N'(x) = S^{-1}(x)N'(x)$$
$$\Delta_\alpha S\alpha^\rho S^{-1}\ (\partial_\alpha + SA_\alpha S^{-1} + S\partial_\alpha S^{-1})\ M'(x)$$
$$+\Delta_\alpha S^2\ S^{-1}\beta^\rho S\ (\partial_\alpha + S^{-1}B_\alpha S + S^{-1}\partial_\alpha S)\ N'(x) = \Delta SJ(x)$$
$$\Delta_\alpha\ S\alpha^\rho S^{-1} = \alpha'^{\rho}\ , \qquad SA_\alpha S^{-1} + S\partial_\alpha S^{-1} = A'_\alpha$$
$$\Delta_\alpha S^2\ S^{-1}\beta^\rho S = \beta'^{\rho}\ , \qquad S^{-1}B_\alpha S + S^{-1}\partial_\alpha S = B'_\alpha\ . \tag{83}$$

In addition to calculation performed in Sections **8,9**, we need to consider only relations involving matrices β^ρ and connection B_ρ. For Euclidean rotation:

$$S\beta^\rho S^{-1} = S\beta^0 e^\rho_{(0)} S^{-1} + S\beta^l e^\rho_{(l)} S^{-1}$$
$$= \beta^0 e^\rho_{(0)} + \beta^k O_{kl} e^\rho_{(l)} = \beta^0 e^\rho_{(0)} + \beta^k e'^{\rho}_{(k)} = \beta'^{\rho}\ .$$

For local Lorentzian rotations

$$\Delta S^2\ S^{-1}\beta^\rho(x)S = \Delta S^2\ S^{-1}\beta^a e^\rho_{(a)} S$$
$$= [\Delta S^2\ (S^{-1}\alpha^a S)\ ]\ e^\rho_{(a)} = \alpha^b L_b{}^a\ e^\rho_{(a)} = \beta^b e'^{\rho}_{(b)} = \beta'^{\rho}(x)\ .$$

Transformation laws for two connections

$$SA_\rho S^{-1} + S\partial_\rho S^{-1} = A'_\rho\ , \qquad S^{-1}B_\rho S + S^{-1}\partial_\rho S = B_\rho\ ,$$

in fact are complex conjugated relations, because of identities

$$S^{-1} = S^{*}, \qquad S = (S^{*})^{-1}, \qquad (B_{\alpha})^{*} = A_{\alpha} \; ,$$

so we need not any additional calculation.

10. Matrix equation in explicit component form

Now we are going to derive tensor generally covariant Maxwell equations when starting with the matrix form

$$-i \; (\; e^{\rho}_{(0)} \partial_{\rho} + \frac{1}{2} j^{ab} \gamma_{ab0} \;) \Psi + \alpha^{k} \; (\; e^{\rho}_{(k)} \partial_{\rho} + \frac{1}{2} j^{ab} \gamma_{abk} \;) \Psi = J(x) \; . \tag{84}$$

Taking in mind

$$\frac{1}{2} j^{ab} \gamma_{ab0} = [s^{1} (\gamma_{230} + i \gamma_{010}) + s^{2} (\gamma_{310} + i \gamma_{020}) + s^{3} (\gamma_{120} + i \gamma_{030}) \;]$$
$$\frac{1}{2} j^{ab} \gamma_{abk} = [s^{1} (\gamma_{23k} + i \gamma_{01k}) + s^{2} (\gamma_{31k} + i \gamma_{02k}) + s^{3} (\gamma_{12k} + i \gamma_{03k}) \;]$$

and introducing notation

$$e^{\rho}_{(0)} \partial_{\rho} = \partial_{(0)} \; , \qquad e^{\rho}_{(k)} \partial_{\rho} = \partial_{(k)}$$
$$(\gamma_{01a}, \gamma_{02a}, \gamma_{03a}) = \mathbf{v}_{a} \; , \qquad (\gamma_{23a}, \gamma_{31a}, \gamma_{12a}) = \mathbf{p}_{a} \; , \qquad a = 0, 1, 2, 3$$

eq. (84) can be transformed to the form

$$(\; \alpha^{k} \; \partial_{(k)} + \mathbf{s v}_{0} + \alpha^{k} \; \mathbf{s p}_{k} \;) \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right|$$
$$-i \; (\; \partial_{(0)} + \mathbf{s p}_{0} - \alpha^{k} \mathbf{s v}_{k}) \left| \begin{array}{c} 0 \\ \mathbf{E} + ic\mathbf{B} \end{array} \right| = \frac{1}{\epsilon_{0}} \left| \begin{array}{c} \rho \\ i \, \mathbf{j} \end{array} \right| \; . \tag{85}$$

Let us divide equation (85) into real and imaginary parts:

$$(\alpha^{k} \partial_{(k)} + \mathbf{s v}_{0} + \alpha^{k} \mathbf{s p}_{k}) \left| \begin{array}{c} 0 \\ \mathbf{E} \end{array} \right| + (\partial_{(0)} + \mathbf{s p}_{0} - \alpha^{k} \mathbf{s v}_{k}) \left| \begin{array}{c} 0 \\ c\mathbf{B} \end{array} \right| = \frac{1}{\epsilon_{0}} \left| \begin{array}{c} \rho \\ 0 \end{array} \right|$$
$$(\alpha^{k} \partial_{(k)} + \mathbf{s v}_{0} + \alpha^{k} \mathbf{s p}_{k}) \left| \begin{array}{c} 0 \\ c\mathbf{B} \end{array} \right| - (\partial_{(0)} + \mathbf{s p}_{0} - \alpha^{k} \mathbf{s v}_{k}) \left| \begin{array}{c} 0 \\ \mathbf{E} \end{array} \right| = \frac{1}{\epsilon_{0}} \left| \begin{array}{c} 0 \\ \mathbf{j} \end{array} \right| \; .$$

From whence we produce explicit equations (for shortness let $c = 1$):

$$\partial_{(k)} E_{k} - [(p_{23} - p_{32}) E_{1} + (p_{31} - p_{13}) E_{2} + (p_{12} - p_{21}) E_{3}]$$
$$+ [(v_{23} - v_{32}) B_{1} + (v_{31} - v_{13}) B_{2} + (v_{12} - v_{21}) B_{3} = \frac{1}{\epsilon_{0}} \rho \; , \tag{86}$$
$$\partial_{(k)} B_{k} - [(p_{23} - p_{32}) B_{1} + (p_{31} - p_{13}) B_{2} + (p_{12} - p_{21}) B_{3}]$$
$$- [(v_{23} - v_{32}) E_{1} + (v_{31} - v_{13}) E_{2} + (v_{12} - v_{21}) E_{(3)} = 0 \; , \tag{87}$$

$$
\begin{aligned}
&(\partial_{(2)}E_3 - \partial_{(3)}E_2) + (v_{20}E_3 v_{30}E_2) \\
&+[-(p_{22}+p_{33})E_1 + p_{12}E_2 + p_{13}E_3] \\
&+\partial_{(0)}B_1 + (p_{20}B_3 - p_{30}B_2) \\
&-[-(v_{22}+v_{33})B_1 + v_{12}B_2 + v_{13}B_3] = 0 \, ,
\end{aligned} \tag{88}
$$

$$
\begin{aligned}
&(\partial_{(2)}B_3 - \partial_{(3)}B_2) + (v_{20}B_3 - v_{30}B_2) \\
&+[-(p_{22}+p_{33})B_1 + p_{12}B_2 + p_{13}B_3] \\
&-\partial_{(0)}E_1 - (p_{20}E_3 - p_{30}E_2) \\
&+[-(v_{22}+v_{33})E_1 + v_{12}E_2 + v_{13}E_3] = \frac{1}{\epsilon_0} j^1 \, ,
\end{aligned} \tag{89}
$$

$$
\begin{aligned}
&(\partial_{(3)}E_1 - \partial_{(1)}E_3) + (v_{30}E_1 - v_{10}E_3) \\
&+[p_{21}E_1 - (p_{11}+p_{33})E_2 + p_{23}E_3] \\
&+\partial_{(0)}B_2 + (p_{30}B_1 - p_{10}B_1) \\
&-[v_{21}B_1 - (v_{11}+v_{33})B_2 + v_{23}B_3] = 0 \, ,
\end{aligned} \tag{90}
$$

$$
\begin{aligned}
&(\partial_{(3)}B_1 - \partial_{(0)}B_3) + (v_{30}B_1 - v_{10}B_3) \\
&+[+p_{31}B_1 - (p_{11}+p_{33})B_2 + p_{23}cB_3] \\
&-\partial_{(0)}E_2 - (p_{30}E_1 - p_{10}E_3) \\
&+[v_{21}E_1 - (v_{11}+v_{33})E_2 + v_{23}E_3] = \frac{1}{\epsilon_0} j^2 \, ,
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
&(\partial_{(1)}E_2 - \partial_{(2)}E_1) + (v_{10}E_2 - v_{20}E_1) \\
&+[p_{31}E_1 + p_{32}E_2 - (p_{11}+p_{22})E_3] \\
&+\partial_{(0)}B_3 + (p_{10}B_2 - p_0 B_1) \\
&-[v_{31}B_1 + v_{32}B_2 - (v_{11}+v_{22})B_3] = 0 \, ,
\end{aligned} \tag{92}
$$

$$
\begin{aligned}
&(\partial_{(1)}B_2 - \partial_{(2)}B_1) + (v_{10}B_2 - v_{20}B_1) \\
&+[+p_{31}B_1 + p_{32}B_2 - (p_{11}+p_{22})B_3] \\
&-\partial_{(0)}E_3 - (p_{10}E_2 - p_{20}E_1) \\
&+[v_{31}E_1 + v_{32}E_2 - (v_{11}+v_{22})E_3] = \frac{1}{\epsilon_0} j^3 \, .
\end{aligned} \tag{93}
$$

We have obtained rather complicated system of eight equations, in next Section we will prove its equivalence to tensor generally covariant Maxwell equations.

11. Relations between matrix and tensor Maxwell equations

In the generally covariant tensor Maxwell equations

$$
\nabla^\alpha F^{\beta\gamma} + \nabla^\beta F^{\gamma\alpha} + \nabla^\gamma F^{\alpha\beta} = 0 \, , \qquad \nabla_\beta F^{\beta\alpha} = \frac{1}{\epsilon_0} \, j^\alpha \tag{94}
$$

let us introduce tetrad field variables, then they take the form

$$\begin{aligned} & \partial_{(n)} F_{(m)(l)} + \gamma_{mbn}\, F^{(b)}_{\ (l)} - \gamma_{lbn}\, F^{(b)}_{\ (m)} \\ & + \partial_{(m)}\, F_{(l)(n)}\ + \gamma_{lbm}\, F^{(b)}_{\ (n)} - \gamma_{nbm}\, F^{(b)}_{\ (l)} \\ & +\ \partial_{(l)} F_{(n)(m)} + \gamma_{nbl}\, F^{(b)}_{\ (m)} - \gamma_{mbl}\, F^{(b)}_{\ (n)} = 0 \,, \end{aligned} \tag{95}$$

$$\partial_{(b)} F^{(b)}_{\ (c)} + e^{\beta}_{(b);\beta}\, F^{(b)}_{\ (c)} + \gamma_{cab}\, F^{(b)(a)} = \frac{1}{\epsilon_0}\, j_{(c)} \,. \tag{96}$$

Now we are to detail eqs. (95) and (96) at

$$n, m, l = 1,2,3, \;\; 0,2,3, \;\; 0,3,1, \qquad 0,1,2\,; \quad \text{and} \quad c = 0\,,\ 1\,,\ 2\,,\ 3\,.$$

Let it be $n, m, l = 1,2,3$:

$$\begin{aligned} & \partial_{(1)} F_{(2)(3)} + \gamma_{2b1}\, F^{(b)}_{\ (3)} - \gamma_{3b1}\, F^{(b)}_{\ (2)} \\ & +\partial_{(2)}\, F_{(3)(1)}\ + \gamma_{3b2}\, F^{(b)}_{\ (1)} - \gamma_{1b2}\, F^{(b)}_{\ (3)} \\ & +\partial_{(3)} F_{(1)(2)} + \gamma_{1b3}\, F^{(b)}_{\ (2)} - \gamma_{2b3}\, F^{(b)}_{\ (1)} = 0 \,, \end{aligned}$$

or

$$\begin{aligned} & \partial_{(1)} F_{(2)(3)} + \gamma_{201} F^{(0)}_{\ (3)} + \gamma_{211} F^{(1)}_{\ (3)} - \gamma_{301} F^{(0)}_{\ (2)} - \gamma_{311} F^{(1)}_{\ (2)} \\ & +\partial_{(2)}\, F_{(3)(1)}\ + \gamma_{302} F^{(0)}_{\ (1)} + \gamma_{322} F^{(2)}_{\ (1)} - \gamma_{102} F^{(0)}_{\ (3)} - \gamma_{122} F^{(2)}_{\ (3)}\ ) \\ & +\ \partial_{(3)} F_{(1)(2)} + \gamma_{103} F^{(0)}_{\ (2)} + \gamma_{133} F^{(3)}_{\ (2)} - \gamma_{203} F^{(0)}_{\ (1)} - \gamma_{233} F^{(3)}_{\ (1)} = 0 \end{aligned}$$

which with notation

$$(F_{(2)(3)}, F_{(3)(1)}, F_{(1)(2)}) = (cB_{(i)})\,, \qquad (F_{(0)(1)}, F_{(0)(2)}, F_{(0)(3)} = (E_{(i)})$$

reads (again let $c = 1$)

$$\begin{aligned} & -\partial_{(k)} B_{(k)} + [\ (p_{23} - p_{32}) B_{(1)} + (p_{31} - p_{13}) B_{(2)} + (p_{12} - p_{21}) B_{(3)}\ ] \\ & \quad -[\ (v_{23} - v_{32}) E_{(1)} + (v_{31} - v_{13}) E_{(2)} + (v_{12} - v_{21}) E_{(3)}\ ] = 0 \,, \end{aligned}$$

which coincides with eq. (87), if

$$E_k = E_{(k)} = \mathbf{E}\,, \qquad B_k = -B_{(k)} = \mathbf{B}\,. \tag{97}$$

Let it be $n, m, l = 0,1,2$:

$$\begin{aligned} & \partial_{(0)} F_{(1)(2)} + \gamma_{1b0}\, F^{(b)}_{\ (2)} - \gamma_{2b0}\, F^{(b)}_{\ (1)} \\ & +\ \partial_{(1)}\, F_{(2)(0)}\ + \gamma_{2b1}\, F^{(b)}_{\ (0)} - \gamma_{0b1}\, F^{(b)}_{\ (2)} \\ & +\partial_{(2)} F_{(0)(1)} + \gamma_{0b2}\, F^{(b)}_{\ (1)} - \gamma_{1b2} F^{(b)}_{\ (0)} = 0 \,, \end{aligned} \tag{98}$$

and further

$$\begin{aligned} \partial_{(0)} cB_{(3)} - v_{10} E_{(2)} - p_{20} cB_{(1)} + v_{20} E_{(1)} + p_{10} cB_{(2)} \\ - \partial_{(1)} \; E_{(2)} \; - p_{31} E_{(1)} + p_{11} E_{(3)} + v_{11} cB_{(3)} - v_{31} cB_{(1)} \\ + \partial_{(2)} E_{(1)} + v_{22} cB_{(3)} - v_{32} cB_{(2)} - p_{32} E_{(2)} + p_{22} E_{(3)} = 0 \; , \end{aligned}$$

which coincides with (92) multiplied by -1.

Let it be $c = 0$ in (96):

$$\partial_{(b)} F^{(b)}_{\;\;(0)} + e^{\beta}_{(b);\beta} \; F^{(b)}_{\;\;(0)} + \gamma_{0ab} \; F^{(b)(a)} = \frac{1}{\epsilon_0} \; \rho \; . \tag{99}$$

Allowing for the identity

$$\begin{aligned} e^{\beta}_{(b);\beta} \; F^{(b)}_{\;\;(0)} = -\gamma_{kc}^{\;\;\;c} F^{(k)}_{\;\;(0)} = -(\gamma_{k00} - \gamma_{k11} - \gamma_{k22} - \gamma_{k33}) F^{(k)}_{\;\;((0)} \\ = -\gamma_{k00} F^{(k)}_{\;\;(0)} + \gamma_{211} F^{(2)}_{\;\;(0)} + \gamma_{311} F^{(3)}_{\;\;(0)} \\ + \gamma_{122} F^{(1)}_{\;\;(0)} + \gamma_{322} F^{(3)}_{\;\;(0)} + \gamma_{133} F^{(1)}_{\;\;(0)} + \gamma_{233} F^{(2)}_{\;\;(0)} \; , \end{aligned}$$

we get

$$\begin{aligned} \partial_{(k)} E_{(k)} - p_{31} E_{(2)} + p_{21} E_{(3)} + p_{32} E_{(1)} - p_{12} E_{(3)} - p_{23} E_{(1)} + p_{13} E_{(2)} - \\ - v_{12} B_{(3)} + v_{13} B_{(2)} + v_{21} B_{(3)} - v_{23} B_{(1)} - v_{31} B_{(2)} + v_{32} B_{(1)} = \frac{1}{\epsilon_0} \; \rho \; , \end{aligned}$$

the latter coincides with (88).

Now, let $c = 3$ in (96):

$$\partial_{(b)} F^{(b)}_{\;\;(3)} + e^{\beta}_{(b);\beta} \; F^{(b)}_{\;\;(3)} + \gamma_{3ab} \; F^{(b)(a)} = \frac{1}{\epsilon_0} \; j_{(3)} \; , \tag{100}$$

from whence it follows

$$\begin{aligned} -\partial_{(0)} E_{(3)} - \partial_{(1)} B_{(2)} + \partial_{(2)} B_{(1)} \\ - v_{10} B_{(2)} + v_{20} B_{(1)} - v_{11} E_{(3)} - p_{31} B_{(1)} - v_{22} E_{(3)} - p_{32} B_{(2)} \\ + v_{31} E_{(1)} + v_{32} E_{(2)} + p_{20} E_{(1)} + p_{22} B_{(3)} - p_{10} E_{(2)} + p_{11} B_{(3)} = -\frac{1}{\epsilon_0} \; j_{(3)} \; , \end{aligned}$$

the latter coincides with (93) multiplied by (-1). In the same manner one can verify all remaining equations. Thus, the matrix and tensor forms of the Maxwell equations are equivalent to each other:

$$\begin{gathered} \alpha^{\alpha}(x) \; [\; \partial_{\rho} + A_{\alpha}(x) \;] \; \Psi = J(x) \\ \nabla^{\alpha} F^{\beta\gamma} + \nabla^{\beta} F^{\gamma\alpha} + \nabla^{\gamma} F^{\alpha\beta} = 0 \; , \qquad \nabla_{\beta} F^{\beta\alpha} = \frac{1}{\epsilon_0} \; j^{\alpha} \; . \end{gathered} \tag{101}$$

12. Relations between matrix and tensor equations in media

Let us find detailed tetrad component form for generally covariant matrix Maxwell equation in presence of a media:

$$\alpha_\rho(x)(\partial_\rho + A_\rho)\, M + \beta_\rho(x)(\partial_\rho + B_\rho)\, N = J$$

$$M = \left| \begin{array}{c} 0 \\ \mathbf{M} \end{array} \right| , \qquad N = \left| \begin{array}{c} 0 \\ \mathbf{N} \end{array} \right| , \qquad J = \frac{1}{\epsilon_0 \epsilon} \left| \begin{array}{c} \rho \\ i\,\mathbf{j} \end{array} \right|$$

$$\mathbf{M} = \frac{\mathbf{h} + \mathbf{f}}{2} = \frac{1}{2}(\frac{\mathbf{D}}{\epsilon_0} + \mathbf{E}) + \frac{i}{2}(c\mathbf{B} + \frac{\mathbf{H}}{\epsilon_0 c})$$

$$\mathbf{N} = \frac{\mathbf{h}^* - \mathbf{f}^*}{2} = \frac{1}{2}(\frac{\mathbf{D}}{\epsilon_0} - \mathbf{E}) + \frac{i}{2}(c\mathbf{B} - \frac{\mathbf{H}}{\epsilon_0 c}) \; . \qquad (102)$$

For a time we will use shortening notation:

$$\frac{\mathbf{D}}{\epsilon_0} \Longrightarrow \mathbf{D} \; , \qquad c\mathbf{B} \Longrightarrow \mathbf{B} \; , \qquad \frac{\mathbf{H}}{\epsilon_0 c} \Longrightarrow \mathbf{H} \; .$$

Eq. (102) can be rewritten as follows:

$$-i(e^{\rho}_{(0)}\partial_\rho + \frac{1}{2}j^{ab}\gamma_{ab0})M + \alpha^k(e^{\rho}_{(k)}\partial_\rho + \frac{1}{2}j^{ab}\gamma_{abk}\;)M$$

$$-i(e^{\rho}_{(0)}\partial_\rho + \frac{1}{2}j^{ab}\gamma_{ab0})N + \beta^k(e^{\rho}_{(k)}\partial_\rho + \frac{1}{2}j^{*ab}\gamma_{abk}\;)N = J(x) \; . \qquad (103)$$

Eq. (103) can be transformed to the form

$$-i[\partial_{(0)} + \mathbf{s}(\mathbf{p}_0 + i\mathbf{v}_0)]M + \alpha^k[\partial_{(k)} + \mathbf{s}(\mathbf{p}_k + i\mathbf{v}_k)]M$$

$$-i[\partial_{(0)} + \mathbf{s}(\mathbf{p}_0 - i\mathbf{v}_0)]N + \beta^k[\partial_{(k)} + \mathbf{s}(\mathbf{p}_k - i\mathbf{v}_k)]N = J(x) \; .$$

Let us divide it into real and imaginary parts:

$$(\alpha^k\partial_{(k)} + \mathbf{s}\mathbf{v}_0 + \alpha^k\mathbf{s}\mathbf{p}_k)\frac{1}{2}\left| \begin{array}{c} 0 \\ \mathbf{D}+\mathbf{E} \end{array} \right|$$

$$+(\partial_{(0)} + \mathbf{s}\mathbf{p}_0 - \alpha^k\mathbf{s}\mathbf{v}_k)\frac{1}{2}\left| \begin{array}{c} 0 \\ \mathbf{B}+\mathbf{H} \end{array} \right|$$

$$+(\beta^k\partial_{(k)} - \mathbf{s}\mathbf{v}_0 + \beta^k\mathbf{s}\mathbf{p}_k)\frac{1}{2}\left| \begin{array}{c} 0 \\ \mathbf{D}-\mathbf{E} \end{array} \right|$$

$$+(\partial_{(0)}\mathbf{s}\mathbf{p}_0 + \beta^k\mathbf{s}\mathbf{v}_k)\frac{1}{2}\left| \begin{array}{c} 0 \\ \mathbf{B}-\mathbf{H} \end{array} \right| = \frac{1}{\epsilon_0}\left| \begin{array}{c} \rho \\ 0 \end{array} \right| . \qquad (104)$$

$$(\alpha^k\partial_{(k)} + \mathbf{s}\mathbf{v}_0 + \alpha^k\mathbf{s}\mathbf{p}_k)\frac{1}{2}\left| \begin{array}{c} 0 \\ \mathbf{B} \\ +\mathbf{H} \end{array} \right|$$

$$-(\partial_{(0)} + \mathbf{s}\mathbf{p}_0 - \alpha^k\mathbf{s}\mathbf{v}_k)\frac{1}{2}\left| \begin{array}{c} 0 \\ \mathbf{D}+\mathbf{E} \end{array} \right|$$

$$+(\;\beta^k\partial_{(k)} - \mathbf{s}\mathbf{v}_0 + \beta^k\mathbf{s}\mathbf{p}_k)\frac{1}{2}\left| \begin{array}{c} 0 \\ \mathbf{B}-\mathbf{H} \end{array} \right|$$

$$-(\partial_{(0)} + \mathbf{s}\mathbf{p}_0 + \beta^k\mathbf{s}\mathbf{v}_k)\frac{1}{2}\left| \begin{array}{c} 0 \\ \mathbf{D}-\mathbf{E} \end{array} \right| = \frac{1}{\epsilon_0}\left| \begin{array}{c} 0 \\ \mathbf{j} \end{array} \right| . \qquad (105)$$

From these one can derive the following explicit equations:
Let us detail eqs. (104) – we will specify only two cases:

$$\partial_{(k)} D_k - (p_{23} - p_{32}) D_1 - (p_{31} - p_{13}) D_2 - (p_{12} - p_{21}) D_3$$
$$+ (v_{23} - v_{32}) H_1 + (v_{31} - v_{13}) H_2 + (v_{12} - v_{21}) H_3 = \rho \; ,$$
$$\partial_{(2)} E_3 - \partial_{(3)} E_2 + v_{20} E_3 - v_{30} E_2 - (p_{22} + p_{33}) E_1 + p_{12} E_2 + p_{13} E_3 +$$
$$p_{20} B_3 - p_{30} B_2 + (v_{22} + v_{33}) B_1 - v_{12} B_2 - v_{13} B_3 = 0 \; .$$

Now let us consider two equations from (105):

$$\partial_{(k)} B_k - (p_{23} - p_{32}) c B_1 - (p_{31} - p_{13}) B_2 - (p_{12} - p_{21}) B_3$$
$$- (v_{23} - v_{32}) E_1 - (v_{31} - v_{13}) E_2 - (v_{12} - v_{21}) E_3 = 0$$
$$\partial_{(2)} H_3 - \partial_{(3)} H_2 + v_{20} H_3 - v_{30} H_2 - (p_{22} + p_{33}) H_1 + p_{12} H_2 + p_{13} H_3$$
$$- p_{20} D_3 + p_{30} D_2 - (v_{22} + v_{33}) D_1 + v_{12} D_2 + v_{13} D_3 = j^1 \; .$$

Evidently, these equations (and their cyclic counterparts) are equivalent to tensor generally covariant Maxwell equations

$$\nabla^\alpha F^{\beta\gamma} + \nabla^\beta F^{\gamma\alpha} + \nabla^\gamma F^{\alpha\beta} = 0 \; , \qquad \nabla_\beta H^{\beta\alpha} = j^\alpha \tag{106}$$

in tetrad representation

$$(F_{(2)(3)}, F_{(3)(1)}, F_{(1)(2)}) = (cB_{(i)}) \; , \qquad F_{(0)(i)} = E_{(i)}$$
$$(H_{(2)(3)}, H_{(3)(1)}, H_{(1)(2)}) = (H_{(i)}/c) \; , \qquad H_{(0)(i)} = cD_{(i)} \; .$$

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