On the Dynamics of the Mass Point with Internal Degrees of Freedom

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An equation of motion of the mass point with internal degrees of freedom in scalar potential U depending on relative coordinates and time, velocity and accelerations is obtained both for non-relativistic and relativistic case. In non-relativistic case a generalization of the energy conservation law follows, if $\partial U/\partial t = 0$ fulfilled. A concept of work is generalized to relativistic case, leading to corresponding integral of motion, if $\partial U/\partial \tau = 0$ fulfilled, where τ is proper time of the point. In neglecting an internal degrees of freedom and absence of interaction this integral of motion gives a standard Special Relativity result.

1. Equation of Motion and the Energy Conservation

A long period of supremacy of quantum theories did not crush an interest in classical description of quantum systems. In this connect some conclusions, following from the basic equation of dynamics, the Second Newton's Law, should be noted. As it is well known from the Helmholtz epoch ([1]), the Second Newton's Law for conservative systems

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \tag{I.1}$$

gives a force acting at the mass point in the form $\mathbf{F} = -\nabla U = -\partial U/\partial \mathbf{R}$, where $U = U(\mathbf{R})$ is potential function of coordinate of the mass point. As a result, by applying of Eq.(I.1) to definition of elementary work,

$$dA = (\mathbf{F} \cdot d\mathbf{R}) = (\frac{d\mathbf{P}}{dt} \cdot d\mathbf{R}) = (\mathbf{V} \cdot d\mathbf{P}) \ , \tag{I.2}$$

we obtain a conservation of total mechanical energy

$$E = \frac{m\mathbf{V}^2}{2} + U(\mathbf{R}) \ , \tag{I.3}$$

where $\mathbf{R}$ and $\mathbf{V} = d\mathbf{R}/dt$ are respectively radius vector and velocity of the mass point relative to origin of coordinate system, coupled with absolute rest reference frame (r.f.).

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It is clear from the common considerations that a motion of mass point in the field of some object should be determined by potential function depending not only on relative coordinates $\mathbf{R}$, but also at least on relative velocity $\mathbf{V}$ and accelerations, as well as on time, so that $U = U(t, \mathbf{R}, \mathbf{V}, \mathbf{W}, \dot{\mathbf{W}}, ..., \mathbf{W}^{(N)})$, where $\mathbf{W}^{(k)} = d^k\mathbf{W}/dt^k$, a time dependence being specified by internal dynamics of the mentioned object[1] .

In this case corollaries $\mathbf{F} = -\nabla U$ and (I.3) from equation of motion (I.1) should be changed forasmuch as total differential of the function U is

$$dU = \frac{\partial U}{\partial t}dt + (\frac{\partial U}{\partial \mathbf{R}} \cdot d\mathbf{R}) + (\frac{\partial U}{\partial \mathbf{V}} \cdot d\mathbf{V}) + \sum_{k=0}^{N} (\frac{\partial U}{\partial \mathbf{W}^{(k)}} \cdot d\mathbf{W}^{(k)}) \ . \qquad \text{(I.4)}$$

Indeed, definition of elementary work of the force (I.2) gives more general expression for force, namely

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{R}} + [\mathbf{C} \times \mathbf{V}] \ , \qquad \text{(I.5)}$$

where $\mathbf{C}$ is some pseudo-vector, associated not only with external forces, but probably inherent in mass point. Additional term $[\mathbf{C} \times \mathbf{V}]$ has a sense of gyroscopic force. As far back as Helmholtz in his work "On the conservation of force" ([1]; Addition 3) pointed out at formula (I.6).

Furthermore, when interaction takes place the momentum vector $\mathbf{P}$ has a meaning of dynamical momentum. It can be written as a sum of kinematical momentum $m\mathbf{V}$ and some addition $\mathbf{A}$ (a vector potential), connected both with internal structure of mass point, and with interaction

$$\mathbf{P} = m\mathbf{V} + \mathbf{A} \ . \qquad \text{(I.6)}$$

Then Eqs.(I.2) and (I.6) give

$$dA = (\mathbf{F} \cdot d\mathbf{R}) = (\mathbf{V} \cdot d(m\mathbf{V} + \mathbf{A})) = d\left(\frac{m\mathbf{V}^2}{2} + (\mathbf{A} \cdot \mathbf{V})\right) - (\mathbf{A} \cdot d\mathbf{V}) \ , \qquad \text{(I.7)}$$

or

$$d\left(\frac{m\mathbf{V}^2}{2} + (\mathbf{A} \cdot \mathbf{V})\right) + (\frac{\partial U}{\partial \mathbf{R}} \cdot d\mathbf{R}) - (\mathbf{A} \cdot d\mathbf{V}) =$$
$$= d\left(\frac{m\mathbf{V}^2}{2} + (\mathbf{A} \cdot \mathbf{V}) + U(t, \mathbf{R}, \mathbf{V}, \mathbf{W}, \dot{\mathbf{W}}, ..., \mathbf{W}^{(N)})\right) -$$

[1] Remember W.Weber ([2]- [3]), who tried to explain electrical phenomena as a result of electric interaction of elementary particles, so called *electric atoms*, depending both on their relative disposition $\mathbf{R}$ and on their relative velocity $\mathbf{V}$ and acceleration $\mathbf{W} = d\mathbf{V}/dt$.

$$-\frac{\partial U}{\partial t}dt - (\frac{\partial U}{\partial \mathbf{V}} \cdot d\mathbf{V}) - (\mathbf{A} \cdot d\mathbf{V}) - \sum_{k=0}^{N}(\frac{\partial U}{\partial \mathbf{W}^{(k)}} \cdot d\mathbf{W}^{(k)}) = 0 \ . \tag{I.8}$$

Now, if one suppose

$$\mathbf{A} = -\frac{\partial U}{\partial \mathbf{V}} + [\mathbf{S} \times \mathbf{W}] \ , \tag{I.9}$$

where $\mathbf{S}$ is some pseudo-vector, coupled with both internal structure of the mass point considered and its interaction, $\mathbf{W}$ is an acceleration of this point, then dynamical momentum (I.6) will get an expression

$$\mathbf{P} = m\mathbf{V} - \frac{\partial U}{\partial \mathbf{V}} + [\mathbf{S} \times \mathbf{W}] \ , \tag{I.10}$$

Equation (I.8) reduces to

$$\frac{dE}{dt} = \frac{\partial U}{\partial t} + \sum_{k=0}^{N}(\frac{\partial U}{\partial \mathbf{W}^{(k)}} \cdot \mathbf{W}^{(k+1)}) \ , \tag{I.11}$$

where quantity

$$E = \frac{m\mathbf{V}^2}{2} + (\mathbf{V} \cdot [\mathbf{S} \times \mathbf{W}]) - (\mathbf{V} \cdot \frac{\partial U}{\partial \mathbf{V}}) + U(t, \mathbf{R}, \mathbf{V}, \mathbf{W}, \dot{\mathbf{W}}, ..., \mathbf{W}^{(N)}) \tag{I.12}$$

is a generalization of Eq.(I.3) for total mechanical energy. So, apart from standard kinetic and potential energies an additional energy arises due to both internal degrees of freedom and a dependence of potential energy on relative velocity.

Provided the condition

$$\frac{\partial U}{\partial t} + \sum_{k=0}^{N}(\frac{\partial U}{\partial \mathbf{W}^{(k)}} \cdot \mathbf{W}^{(k+1)}) = 0 \tag{I.13}$$

is fulfilled, the energy (I.12) will be an integral of motion. Condition $dE/dt > 0$ corresponds to absorption of energy by a mass point, and $dE/dt < 0$ corresponds to radiation of energy.

In view of stated above, the equation of motion (I.1) should be written down in the form

$$\frac{d}{dt}(m\mathbf{V} + [\mathbf{S} \times \mathbf{W}]) - [\mathbf{C} \times \mathbf{V}] = \frac{d}{dt}\frac{\partial U}{\partial \mathbf{V}} - \frac{\partial U}{\partial \mathbf{R}} \ . \tag{I.14}$$

Let's note here that derivatives of potential function with respect to accelerations $\mathbf{W}^{(k)}$ do not enter into an equation of motion. Therefore one can be restricted to dependence of potential function only on acceleration $\mathbf{W}$: $U = U(t, \mathbf{R}, \mathbf{V}, \mathbf{W})$. Then the point obeying to equation of motion (I.14) with condition (I.13) represents a Birkhoff's dynamical system ([4]).

We assume in general case that pseudo-vectors $\mathbf{S}$ and $\mathbf{C}$ are stipulated by both internal structure and interaction of the mass point. Therefore they may be written as sums

$$\mathbf{S} = \mathbf{S}_0 + \mathbf{S}^{ext}\ ,\ \mathbf{C} = \mathbf{C}_0 + \mathbf{C}^{ext}\ , \tag{I.15}$$

where $\mathbf{S}_0$ and $\mathbf{C}_0$ are connected exclusively with internal structure, whereas $\mathbf{S}^{ext}$ and $\mathbf{C}^{ext}$ are connected only with interaction. $\mathbf{S}_0$ and $\mathbf{C}_0$ may vary only in direction, rather than in lengths. $\mathbf{S}^{ext}$ and $\mathbf{C}^{ext}$ may depend on those variables as potential function does.

Equation (I.14) may be written in another form supposing

$$U = U_0 - (\mathbf{R}\cdot[\mathbf{V}\times\mathbf{C}]) = U_0 - (\mathbf{R}\cdot[\mathbf{V}\times\mathbf{C}_0]) - (\mathbf{R}\cdot[\mathbf{V}\times\mathbf{C}^{ext}])\ . \tag{I.16}$$

Then Eq.(I.14) reduces to

$$\frac{d}{dt}(m\mathbf{V} + [\mathbf{S}_0\times\mathbf{W}] - [\mathbf{R}\times\mathbf{C}_0]) = -\frac{\partial U_0}{\partial\mathbf{R}} + (\mathbf{R}\cdot[\mathbf{V}\times\frac{\partial\mathbf{C}^{ext}}{\partial\mathbf{R}}]) +$$
$$+\frac{d}{dt}\left(\frac{\partial U_0}{\partial\mathbf{V}} - (\mathbf{R}\cdot[\mathbf{V}\times\frac{\partial\mathbf{C}^{ext}}{\partial\mathbf{V}}]) - [\mathbf{S}^{ext}\times\mathbf{W}] + [\mathbf{R}\times\mathbf{C}^{ext}]\right)\ , \tag{I.17}$$

where

$$(\mathbf{R}\cdot[\mathbf{V}\times\frac{\partial\mathbf{C}^{ext}}{\partial\mathbf{R}}])_i = \varepsilon_{klm}R^kV^l\frac{\partial(\mathbf{C}^{ext})^m}{\partial R^i}\ , \tag{I.18}$$

$$(\mathbf{R}\cdot[\mathbf{V}\times\frac{\partial\mathbf{C}^{ext}}{\partial\mathbf{V}}])_i = \varepsilon_{klm}R^kV^l\frac{\partial(\mathbf{C}^{ext})^m}{\partial V^i}\ . \tag{I.19}$$

For free mass point ($U_0 = 0$, $\mathbf{S}^{ext} = 0$, $\mathbf{C}^{ext} = 0$) Eq.(I.17) leads to a conservation of momentum

$$\mathbf{P} = m\mathbf{V} + [\mathbf{S}_0\times\mathbf{W}] - [\mathbf{R}\times\mathbf{C}_0] = const\ . \tag{I.20}$$

2. The equation of moments for a mass point with internal degrees of freedom

The equation (I.14) is insufficient for description of dynamics of physical system. There is necessary in addition an equation of moments, which for structureless mass point looks like $d\mathbf{L}/dt = \mathbf{M}$, where $\mathbf{L} = [\mathbf{R}\times\mathbf{P}] = m[\mathbf{R}\times\mathbf{V}]$ is angular momentum, $\mathbf{M} = [\mathbf{R}\times\mathbf{F}]$ is total moment of external forces, acting at the system. For individual mass point equation of moments follows from the Eq.(I.1).

For a mass point with internal degrees of freedom, describing by Eq.(I.1), in which force and momentum are specified by equations (I.5) and (I.10), respectively, we have the relation

$$[\mathbf{R}\times\frac{d\mathbf{P}}{dt}] = \frac{d}{dt}[\mathbf{R}\times\mathbf{P}] - [\mathbf{V}\times(-\frac{\partial U}{\partial\mathbf{V}} + [\mathbf{S}\times\mathbf{W}])] = -[\mathbf{R}\times\frac{\partial U}{\partial\mathbf{R}}] + [\mathbf{R}\times[\mathbf{C}\times\mathbf{V}]]\ , \tag{II.1}$$

implying the following equation of moments

$$\frac{d\mathbf{L}}{dt} = \mathbf{M} + \mathbf{T}, \tag{II.2}$$

where

$$\mathbf{L} \doteq [\mathbf{R} \times \mathbf{P}] = m[\mathbf{R} \times \mathbf{V}] - [\mathbf{R} \times \frac{\partial U}{\partial \mathbf{V}}] + [\mathbf{R} \times [\mathbf{S} \times \mathbf{W}]] \tag{II.3}$$

is a dynamical angular momentum,

$$\mathbf{M} \doteq [\mathbf{R} \times \mathbf{F}] = -[\mathbf{R} \times \frac{\partial U}{\partial \mathbf{R}}] + [\mathbf{R} \times [\mathbf{C} \times \mathbf{V}]] \tag{II.4}$$

is a moment of force, acting at the mass point,

$$\mathbf{T} \doteq [\mathbf{V} \times \mathbf{P}] = -[\mathbf{V} \times \frac{\partial U}{\partial \mathbf{V}}] + [\mathbf{V} \times [\mathbf{S} \times \mathbf{W}]] \tag{II.5}$$

is an additional twisting moment, or torque. In standard mechanics the concept ”torque” is applied to the moment of force (II.4). Here we distinguish the moment of force (II.4) and torque (II.5).

It should be noticed, that in the same way both equation $d\mathbf{L}/dt = \mathbf{M}$ follows from Eq.(I.1) for usual mass point and equation (II.2) follows from Eq.(I.14) (i.e. Eq.(I.1), in which $\mathbf{F}$ and $\mathbf{P}$ are specified by equations (I.5) and (I.10)) for a mass point with internal degrees of freedom.

Solution of equation (I.14) may be obtained in principle, if potential function $U = U(t, \mathbf{R}, \mathbf{V}, \mathbf{W})$ and time dependence of pseudo-vectors $\mathbf{S}$ $\mathbf{C}$, coupled with internal structure of mass point, are known. As it is known, one of internal property of particles is spin, associated classically with proper angular momentum of particle. Therefore a temptation arises to connect pseudo-vectors $\mathbf{S}$ and $\mathbf{C}$ with spin. However, having only definition (II.3) for angular momentum it is impossible to define a concept of proper angular momentum. Therefore pseudo-vectors $\mathbf{S}$, $\mathbf{C}$ and their equations of motion should be either postulated here artificially or determined starting from additional arguments. In particular, one may go by the same way as a solid body in mechanics considered as a system of mass point. Then it is possible to define a concept of *particle* with internal degrees of freedom as a system of the same mass points, whose proper angular momentum is determined relative to center of inertia of particle. Such procedure will be made elsewhere. Here it is reasonably to generalize equations and concepts above to relativistic case.

3. Relativistic equation of motion

Relativistic generalization of the second Newton's law for mass point is

$$\frac{d\mathrm{P}}{d\lambda} = \frac{1}{c}\mathrm{F} \tag{III.1}$$

where $\mathrm{P} = \{P^\mu\} = (P^0, \mathbf{P})$, $\mathrm{F} = \{F^\mu\} = (F^0, \mathbf{F})$, $\mu = 0,1,2,3$, are relativistic generalizations of momentum and force, λ is invariant parameter determined by the interval

$$dS^2 = \eta_{\mu\nu} dR^\mu dR^\nu = (dR^0)^2 - d\mathbf{R}^2 = \sigma d\lambda^2 \,, \quad \sigma = \pm 1 \,, \tag{III.2}$$

where $\eta_{\mu\nu} = diag(1,-1,-1,-1)$. Thus, for $\sigma = +1$ parameter $\lambda/c = \tau$ is a proper time of "concomitant observer" K$'$ moving together with event defined by four-dimensional radius-vector $\mathrm{R} = \{R^\mu\} = (R^0, \mathbf{R})$. For $\sigma = -1$ parameter $\lambda = S$ coincides with length of arc of world line of the event R.

Let us emphasize an important fact that standard Special Relativity with interval (III.2) is valid exceptionally for inertial reference frames (r.f.). Usually interval (III.2) is considered as a definition of distance between two points in the Minkowski space $\mathbf{E}^{\mathrm{R}}_{1,3}$. Then coordinates of a point in $\mathbf{E}^{\mathrm{R}}_{1,3}$, defined by radius-vector R, are quantities relative to origin, coinciding with origin of the rest r.f. K, and have absolute character in the meaning of absolute time and absolute space of Newton's mechanics. Relative character in the meaning of Special Relativity they acquire when interval (III.2) is coupled with r.f. K$'$, moving relative to K with velocity $\mathbf{V} = c d\mathbf{R}/dR^0$. In this case radius-vector R is said to be an event R, whose world line is a trajectory of the origin of K$'$, moving with velocity $\mathbf{V}$ in $\mathbf{E}^{\mathrm{R}}_{1,3}$, i.e. in the space of the rest r.f. K.

For inertially moving K$'$ r.h.s. of Eq.(III.1) vanishes, and we obtain conservation of 4-momentum, whence it follows conservation of

$$\mathrm{P}^2 = \eta_{\mu\nu} P^\mu P^\nu = (P^0)^2 - \mathbf{P}^2 = \sigma m_0^2 c^2 \,, \tag{III.3}$$

if 4-momentum is defined as

$$P^\mu = m_0 c U^\mu = m_0 c dR^\mu/d\lambda = m_0 dR^\mu/d\tau \,. \tag{III.4}$$

Relations (III.3)-(III.4) are standard relations of Special Relativity for kinematical momentum, which are extended on any asymptotically free physical systems without any reason. Between other things one may consider an expression (III.2) for relativistic interval as a corollary from relations (III.3), postulating connection between energy and momentum.

If some force be acting on moving K$'$, the latter is no longer inertial one. Then 4-momentum in Eq.(III.1) becomes dynamical momentum, whose definition ought to be analogous to Eq.(I.6)

$$P^\mu = m_0 c \frac{dR^\mu}{d\lambda} + K^\mu \,, \tag{III.5}$$

where K^μ is some addition to kinematical 4-momentum (III.4) due to interaction between moving r.f. K$'$ and rest r.f. K.

In Newton's mechanics an interaction force (I.5) between K$'$ and K is determined by means of elementary work (I.2) which may be written as $dA = -\eta_{ij}F^i dR^j$. This work is a scalar under Galilei transformations, i.e. it is the same in all non-relativistic inertial r.f., but it is not covariant under Lorentz transformations.

Indeed, let $L^{\mu}_{.\nu}$ be matrix elements of the Lorentz transformation $dR'^{\mu} = L^{\mu}_{.\nu}dR^{\nu}$, satisfying to condition $\eta_{\lambda\kappa}L^{\lambda}_{.\mu}L^{\kappa}_{.\nu} = \eta_{\mu\nu}$, so that ([5])

$$L^0_{.0} = \gamma_\sigma = (1 - \mathbf{B}_0^{2\sigma})^{-1/2}\ ,\ L^0_{.i} = \Gamma_\sigma V_{0i}/V_0\ ,\ L^i_{.0} = -\Gamma_\sigma V_0^i/V_0\ ,\ L^i_{.j} = \delta^i_{.j} - \frac{\gamma_\sigma - 1}{\mathbf{V}_0^2}V_0^i V_{0j}\ , \tag{III.6}$$

where

$$\mathbf{B}_0 = \mathbf{V}_0/c\ , \quad V_0 = |\mathbf{V}_0| = \sqrt{\mathbf{V}_0^2}\ , \quad \mathrm{B}_0 = |\mathbf{B}_0| = \sqrt{\mathbf{B}_0^2} = cV_0\ ; \tag{III.7}$$

$$\Gamma_\sigma = \mathrm{B}_0^\sigma \gamma_\sigma\ , \quad \Gamma_+ = \mathrm{B}_0\gamma_+ = \mathrm{B}_0(1 - \mathbf{B}_0^2)^{-1/2}\ , \quad \Gamma_- = \gamma_-/\mathrm{B}_0 = (\mathbf{B}_0^2 - 1)^{-1/2}\ , \tag{III.8}$$

$\mathbf{V}_0$ is velocity of K$'$ relative to K.

Then the Lorentz transformation takes form

$$dR'^0 = \gamma_\sigma \left[dR^0 - \frac{\mathrm{B}_0^\sigma(\mathbf{V}_0 \cdot d\mathbf{R})}{V_0} \right]\ , \tag{III.9}$$

$$d\mathbf{R}' = d\mathbf{R} + \left[(\gamma_\sigma - 1)\frac{(\mathbf{V}_0 \cdot d\mathbf{R})}{c\mathrm{B}_0^\sigma \mathrm{B}_0} - \gamma_\sigma dR^0 \right] \frac{\mathrm{B}_0^\sigma}{c\mathrm{B}_0}\mathbf{V}_0\ . \tag{III.10}$$

Transformation law of relativistic force looks as

$$F'^{\mu} = L^{\mu}_{.\nu}F^{\nu} = L^{\mu}_{.0}F^0 + L^{\mu}_{.i}F^i\ , \tag{III.11}$$

$$F'^0 = \gamma_\sigma \left[F^0 - \frac{\mathrm{B}_0^\sigma(\mathbf{F} \cdot \mathbf{V}_0)}{V_0} \right]\ , \tag{III.12}$$

$$\mathbf{F}' = \mathbf{F} + \left[\frac{(\gamma_\sigma - 1)(\mathbf{F} \cdot \mathbf{V}_0)}{\mathbf{V}_0^2} - \frac{\gamma_\sigma \mathrm{B}_0^\sigma F^0}{V_0} \right] \mathbf{V}_0\ . \tag{III.13}$$

Hence Eqs.(III.10) and (III.13) give transformation law for elementary work

$$dA' = (\mathbf{F}' \cdot d\mathbf{R}') = -\eta_{ij}F'^i dR'^j = -\eta_{ij}L^i_{.\mu}L^j_{.\nu}F^{\mu}dR^{\nu} =$$
$$= dA + \Gamma_\sigma^2 \left[1 - \frac{(\mathbf{V} \cdot \mathbf{V}_0)}{c^2\mathrm{B}_0\mathrm{B}_0^\sigma} \right] cF^0 dt - \frac{\Gamma_\sigma^2}{\mathrm{B}_0\mathrm{B}_0^\sigma}(\mathbf{F} \cdot \mathbf{V}_0)dt + \frac{\gamma_\sigma^2 - 1}{\mathbf{V}_0^2}(\mathbf{F} \cdot \mathbf{V}_0)(\mathbf{V} \cdot \mathbf{V}_0)dt\ , \tag{III.14}$$

whence it follows relativistic transformation of power $N = cdA/dR^0 = (\mathbf{F} \cdot \mathbf{V})$

$$N' = c\frac{dA'}{dR'^0} = (\mathbf{F}' \cdot \mathbf{V}') =$$

$$= \frac{N + L^0_{.0} L^0_{.i} F^0 V^i + c L^0_{.i} L^0_{.0} F^i + L^0_{.i} L^0_{.j} F^i V^j + c[(L^0_{.0})^2 - 1] F^0}{L^0_{.0} + L^0_{.i} V^i / c} =$$

$$= \frac{N + c\Gamma^2_\sigma \left[1 - \frac{(\mathbf{V}\cdot\mathbf{V}_0)}{c^2 \mathrm{B}_0 \mathrm{B}^\sigma_0}\right] F^0 - \frac{\Gamma^2_\sigma}{\mathrm{B}_0 \mathrm{B}^\sigma_0}(\mathbf{F}\cdot\mathbf{V}_0) + \frac{\gamma^2_\sigma - 1}{\mathbf{V}^2_0}(\mathbf{F}\cdot\mathbf{V}_0)(\mathbf{V}\cdot\mathbf{V}_0)}{\gamma_\sigma \left[1 - \frac{\mathrm{B}^\sigma_0 (\mathbf{V}\cdot\mathbf{V}_0)}{V_0}\right]} . \qquad \text{(III.15)}$$

Noncovariance of expression (I.2) and force transformation law (III.13) are inconsistent with principle of relativity, whose successive application means that equations and quantities, such as scalars, 4-vectors, tensors etc., ought to be covariant under Lorentz transformations in any theory. Therefore definition (I.2) should be generalized in the form

$$dW = -\eta_{\mu\nu} F^\mu dR^\nu = -F^0 dR^0 + dA = (-cF^0 + N)dt . \qquad \text{(III.16)}$$

In standard Special Relativity, dealing with interval (III.2), F^0 is defined from Eq.(III.1), where $P^0 = m_0 c dR^0/d\tau = m_0 c^2 \gamma = m_0 c^2 (1 - \mathbf{V}^2/c^2)^{-1/2}$, $\mathbf{V}$ is absolute velocity of the mass point acquiring acceleration $\mathbf{W} = d\mathbf{V}/dt$ under action of the force $\mathbf{F}$. Then taking into account the relation

$$\frac{d\gamma}{dt} = \frac{\gamma^3}{c^2}(\mathbf{V}\cdot\mathbf{W}) , \qquad \text{(III.17)}$$

we obtain

$$\mathbf{F} = \frac{d\mathbf{P}}{d\tau} = \gamma \frac{d(m_0 \gamma \mathbf{V})}{dt} = m_0 \gamma^2 \left[\mathbf{W} + \frac{\gamma^2}{c^2}(\mathbf{V}\cdot\mathbf{W})\mathbf{V}\right] , \qquad \text{(III.18)}$$

$$(\mathbf{F}\cdot\mathbf{V}) = m_0 \gamma^4 (\mathbf{V}\cdot\mathbf{W}) , \qquad \text{(III.19)}$$

$$F^0 = \frac{dP^0}{d\tau} = m_0 c \frac{d\gamma}{d\tau} = m_0 c \gamma \frac{d\gamma}{dt} = \frac{m_0 \gamma^4}{c}(\mathbf{V}\cdot\mathbf{W}) = \frac{1}{c}(\mathbf{F}\cdot\mathbf{V}) = \frac{N}{c} . \qquad \text{(III.20)}$$

Comparison of Eq.(III.16) with Eq.(III.20) shows that scalar dW is identically zero in all inertial r.f. However, should moving r.f. K′ be coupled with considered mass point, the latter ceases to be inertial one. Then in such r.f. an expression (III.2) for length of arc of the world line of the mass point and relations (III.18)-(III.20) become invalid. It means that scalar $dW = = -\eta_{\mu\nu} F^\mu dR^\nu = -\eta_{\mu\nu} F'^\mu dR'^\nu$ does not equal to zero, conserving its covariant expression in all inertial r.f. Relativistic force, acting at rest mass point in K′, may be expressed in terms of potential function $U = U(\lambda, R^\mu, U^\mu, W^\mu, \dot{W}^\mu, ..., (W^{(N)})^\mu)$ by analogy with (I.5) in non-relativistic mechanics, where R^μ, U^μ, $W^\mu = dU^\mu/d\lambda$, $(W^{(k)})^\mu = d^k W^\mu / d\lambda^k$ are relative radius-vector, 4-velocity and 4-accelerations of K′ relative to K. Forasmuch as relation $\eta_{\mu\nu} U^\mu U^\nu = \sigma$ following from (III.2) becomes invalid, arguments of potential function should considered as independent variables, so that total differential of U equals

$$dU = \frac{\partial U}{\partial \lambda} d\lambda + \frac{\partial U}{\partial R^\mu} dR^\mu + \frac{\partial U}{\partial U^\mu} dU^\mu + \sum_{k=0}^{N} \frac{\partial U}{\partial (W^{(k)})^\mu} d(W^{(k)})^\mu . \qquad \text{(III.21)}$$

Formulae (I.10) and (I.5) should be considered as non-relativistic limits of 4-momentum and 4-force

$$P^\mu = m_0 c U^\mu - \eta^{\mu\nu}\frac{\partial U}{c\partial U^\nu} + \frac{1}{2}\eta^{\mu\nu}\varepsilon_{\nu\lambda\kappa\rho}S^{\lambda\kappa}W^\rho \ , \qquad \text{(III.22)}$$

$$F^\mu = -\eta^{\mu\nu}\frac{\partial U}{\partial R^\nu} + \frac{1}{2}\eta^{\mu\nu}\varepsilon_{\nu\lambda\kappa\rho}C^{\lambda\kappa}U^\rho, \qquad \text{(III.23)}$$

respectively, where $S_{\lambda\kappa}$ and $C_{\lambda\kappa}$ are some antisymmetric tensors, characterizing internal structure of the mass point.

Substitution of Eqs.(III.22)-(III.23) into equation (III.1) gives next equation of motion

$$\frac{d}{d\lambda}\left[m_0 c U^\mu + \frac{1}{2}\eta^{\mu\nu}\varepsilon_{\nu\lambda\kappa\rho}S^{\lambda\kappa}W^\rho\right] - \frac{1}{2c}\eta^{\mu\nu}\varepsilon_{\nu\lambda\kappa\rho}C^{\lambda\kappa}U^\rho = \frac{1}{c}\eta^{\mu\nu}\left[\frac{d}{d\lambda}\frac{\partial U}{\partial U^\nu} - \frac{\partial U}{\partial R^\nu}\right] . \qquad \text{(III.24)}$$

Substitution of Eqs.(III.22)-(III.23) into equation (III.16) gives

$$c\eta_{\mu\nu}\frac{dP^\mu}{d\lambda}dR^\nu = c\eta_{\mu\nu}U^\mu dP^\nu = \eta_{\mu\nu}F^\mu dR^\nu = \eta_{\mu\nu}U^\mu F^\nu d\lambda \ , \qquad \text{(III.25)}$$

or

$$\eta_{\mu\nu}U^\mu\frac{d}{d\lambda}\left[m_0 c^2 U^\nu + \frac{c}{2}\eta^{\mu\nu}\varepsilon_{\nu\lambda\kappa\rho}S^{\lambda\kappa}W^\rho\right] = U^\mu\frac{d}{d\lambda}\frac{\partial U}{\partial U^\mu} - U^\mu\frac{\partial U}{\partial R^\mu} \ . \qquad \text{(III.26)}$$

Hence we obtain equation

$$\frac{dE}{d\lambda} = \frac{\partial U}{\partial\lambda} + \sum_{k=0}^{N}\frac{\partial U}{\partial(W^{(k)})^\mu}(W^{(k+1)})^\mu \ , \qquad \text{(III.27)}$$

where quantity

$$E = \frac{m_0 c^2}{2}\eta_{\mu\nu}U^\mu U^\nu + \frac{c}{2}\varepsilon_{\mu\nu\lambda\kappa}U^\mu W^\nu S^{\lambda\kappa} + U - U^\mu\frac{\partial U}{\partial U^\mu} = \frac{m_0 c^2}{2}\sigma \qquad \text{(III.28)}$$

is an integral of motion provided a condition

$$\frac{\partial U}{\partial\lambda} + \sum_{k=0}^{N}\frac{\partial U}{\partial(W^{(k)})^\mu}(W^{(k+1)})^\mu = 0 \qquad \text{(III.29)}$$

is satisfied.

Neglecting internal structure of mass point and its interaction, $U = 0$, from (III.28) we obtain $\eta_{\mu\nu}U^\mu U^\nu = \sigma$ and expression (III.2) for interval of standard Special Relativity. In general case quantity σ does not equal to $+1$ or -1. Specifically, an account of internal structure of free mass point gives

$$\eta_{\mu\nu}dR^\mu dR^\nu + \frac{1}{m_0}\varepsilon_{\mu\nu\lambda\kappa}S^{\lambda\kappa}dR^\mu dU^\nu = \left[1 + \frac{1}{m_0 c}\frac{\varepsilon_{\lambda\kappa\tau\omega}S^{\lambda\kappa}U^\tau W^\omega}{\eta_{\rho\sigma}U^\rho U^\sigma}\right]\eta_{\mu\nu}dR^\mu dR^\nu = \sigma d\lambda^2 \ , \quad \text{(III.30)}$$

i.e. the Minkowski space-time $\mathbf{E}^{\mathrm{R}}_{1,3}$ effectively extends to 8-dimensional phase space with interval (III.30) and degenerate metric, which is equivalent to 4-dimensional conformally flat space with metric

$$g_{\mu\nu} = \left[1 + \frac{1}{m_0 c}\frac{\varepsilon_{\lambda\kappa\tau\omega}S^{\lambda\kappa}U^\tau W^\omega}{\eta_{\rho\sigma}U^\rho U^\sigma}\right]\eta_{\mu\nu} \ , \qquad \text{(III.31)}$$

coordinate dependence of which may be determined, as soon as solution of equation of motion (III.24) for $U = 0$ is found.

The paper above is connected with the problem of motion of spinning particle in external fields, including both description of spin motion and the spin influence on the particle trajectory. Although this problem is about one hundred years, there are many obscure questions, particularly concerning of the spin influence (see, e.g., [6]). We hope that a consideration of this problem *ab ovo*, as it partially done in this paper, will made possible to clarify some of them.

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