БЕЛОРУССКИЙ ГОСУДАРСТВЕННЫЙ УНИВЕРСИТЕТ

Н. Г. Абрашина-Жадаева
И. А. Тимощенко

VECTOR AND TENSOR ANALYSIS THROUGH EXAMPLES AND EXERCISES

ВЕКТОРНЫЙ И ТЕНЗОРНЫЙ АНАЛИЗ В ПРИМЕРАХ И ЗАДАЧАХ

Допущено
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по специальности «Компьютерная физика»

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кафедра фундаментальной и прикладной математики
Гродненского государственного университета
имени Янки Купалы (заведующий кафедрой
доктор физико-математических наук, профессор *Е. А. Ровба*);
заведующий отделом вычислительной математики
Института математики НАН Беларуси кандидат
физико-математических наук *Г. Ф. Громыко*;
доцент кафедры теории и практики перевода № 1
Минского государственного лингвистического университета
кандидат филологических наук *В. Г. Минина*

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В учебном пособии излагаются основы векторного и тензорного анализа. Приводятся базовые теоретические сведения, многочисленные упражнения для самостоятельного изучения материала. Основное внимание сосредоточено на методах решения задач.

Предназначено для студентов высших учебных заведений по специальности «Компьютерная физика».

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Preface

This textbook is written based on lecture courses and practical classes that were held at the Faculty of Physics of the Belarusian State University under the curriculum "Mathematical Analysis" and "Fundamentals of Vector and Tensor Analysis". It is intended for foreign students of physical, mathematical, engineering-physical and engineering-technical specialties and for those who independently study mathematics in English.

The training of future engineers, teachers of the specialties "Physics", "Chemistry", "Biology" etc., is closely related to acquiring mathematical knowledge and practical skills. Therefore, the authors of the book sought to present the basics of mathematical information that a qualified natural scientist must have in an accessible and convenient form. The goal is to assist with self-mastering important topics of higher mathematics for students, as well as in preparing classes on these topics for teachers.

The book is also for those readers who want both to understand the basics of higher mathematics and to learn how to apply them. The main emphasis in the textbook is not on the theoretical aspect, but on explanations of the fundamentals of the subject with the help of examples. The authors, using illustrative examples, show the meaning of the most difficult concepts, methods of their application, usefulness, and significance. Plenty of exercises serves as a support for mastering skills.

Everything valuable in the book belongs to the mathematical community, all errors, of course, belong to the authors. Comments and suggestions can be sent to zhadaeva@bsu.by or timoshchenkoia@bsu.by.

Chapter 1

Introduction to tensor algebra

1.1. Index notation

In tensor calculus according to the method of root letters and indices any tensor is specified by using a *root letter* and an ordered set of *indices*, which take a certain range, e. g., p, v^i, w_{kl}, $t^{ki}{}_l$ $(i, k, l = 1, \ldots, n)$. The number, sequence and position (upper or lower) of indices define algebraic and transformational properties of objects. If the index is assigned a certain value, then this index is called *fixed*.

To simplify formula manipulation the *Einstein summation convention* is assumed:

1) upper and lower identical indices are to be summed over their range. The sum sign $\sum$ is omitted. Repeated indices are called *dummy or summation indices*. Pair of dummy indices can occur in a formula only once, but can be easily renamed without changing the result of an expression:

$$a_i b^{im} = a_k b^{km};$$

2) an index which is not a dummy index is called the *free index*. Free indices in the different parts of an expression are to be the same:

$$\begin{aligned} \text{correct expression}: &\quad a^{ik} v_i = b^k; \\ \text{incorrect expression}: &\quad a^{ik} v_i = b^j. \end{aligned}$$

Kronecker delta:

$$\delta_i{}^k = \delta^k{}_i = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k. \end{cases}$$

Example 1. Write the short expression $a_{ik} x^k = b_i$, $i, k = 1, 2, 3$, in the full form.

Solution. Let $i = 1$. Taking into account the summation convention we can expand

$$b_1 = a_{1k} x^k = a_{11} x^1 + a_{12} x^2 + a_{13} x^3.$$

For the index values $i = 2$ and $i = 3$ the expansion is similar. Thus we obtain the following full form of $a_{ik}x^k = b_i$:

$$\begin{cases} a_{11}x^1 + a_{12}x^2 + a_{13}x^3 = b_1, \\ a_{21}x^1 + a_{22}x^2 + a_{23}x^3 = b_2, \\ a_{31}x^1 + a_{32}x^2 + a_{33}x^3 = b_3. \end{cases}$$

Example 2. Factorize the expression $a_k - c_k{}^j a_j$.

Solution. Using Kronecker delta we can express the single element a_k as $a_k = \delta_k{}^j a_j$. Therefore

$$a_k - c_k{}^j a_j = \delta_k{}^j a_j - c_k{}^j a_j = (\delta_k{}^j - c_k{}^j) a_j.$$

Exercises

Write expressions in the full form.

1.1. $a_i = b_i{}^k c_k,\ i, k = 1, 2, 3.$

1.2. $d = a_{ik} b^i b^k,\ i, k = 1, 2.$

1.3. $d = a^k{}_k,\ k = 1, 2, 3.$

1.4. $c^i{}_k{}^k = a^i,\ i, k = 1, 2, 3.$

1.5. $c^i{}_{kj} a^j = a^i b_k.\ i, j, k = 1, 2.$

Simplify expressions.

1.6. $a^i \delta_i{}^k = b^k.$

1.7. $\delta_i{}^l \delta_k{}^m a_l b_m - a_i b_k.$

1.8. $\delta_i{}^k \delta_j{}^l a^{ij} = \delta_m{}^l b^k c^m.$

Factorize expressions.

1.9. $a^i{}_k b^k - b^i.$

1.10. $a^i{}_k b_i c^k - a^i{}_i.$

1.11. $a^i{}_{jk} b^j - a^i{}_{rs} b^r b^s c_k.$

1.12. $a^i{}_{jk} b^j b^k - a^l{}_{rs} c_l b^i d^{rs}.$

1.13. Show that $\delta_k{}^k = n,\ k = \overline{1, n}.$

Find the value of expressions $(i, j, \ldots = \overline{1, n})$.

1.14. $\delta_j{}^i \delta_k{}^j \delta_l{}^k \delta_m{}^l.$

1.15. $\delta_j{}^i \delta_l{}^j \delta_k{}^l \delta_i{}^k.$

1.2. Conjugate linear spaces

Let V^n be a linear space of the dimension n containing elements $\vec{x}$, $\vec{y}$, etc. Any ordered linearly independent set of n elements $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$, $\vec{e}_i \in V^n$, $i = \overline{1, n}$, is called the *basis* of the space V^n. Any element $\vec{x}$ in V^n can be uniquely expressed as a linear combination of vectors $\vec{e}_i$:

$$\vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + \ldots + x^n\vec{e}_n = x^i\vec{e}_i.$$

Coefficients x^i of this linear combination are called *components* or *coordinates* of the element $\vec{x}$ in the basis $(\vec{e}_i)$.

Let V_n be another linear space of the dimension n containing elements $\underset{\rightarrow}{f}$, $\underset{\rightarrow}{g}$, etc. Suppose there is a functional that maps each pair of elements $\vec{x} \in V^n$ and $\underset{\rightarrow}{f} \in V_n$ into a real number $\langle \vec{x}, \underset{\rightarrow}{f} \rangle \in \mathbb{R}$ having the following properties:

a) $\langle \vec{x}, \underset{\rightarrow}{f} \rangle$ is *linear* with respect to each argument, that is $\forall \alpha, \beta \in \mathbb{R}$

$$\langle \alpha\vec{x} + \beta\vec{y}, \underset{\rightarrow}{f} \rangle = \alpha\langle \vec{x}, \underset{\rightarrow}{f} \rangle + \beta\langle \vec{y}, \underset{\rightarrow}{f} \rangle;$$

$$\langle \vec{x}, \alpha\underset{\rightarrow}{f} + \beta\underset{\rightarrow}{g} \rangle = \alpha\langle \vec{x}, \underset{\rightarrow}{f} \rangle + \beta\langle \vec{x}, \underset{\rightarrow}{g} \rangle;$$

b) $\langle \vec{x}, \underset{\rightarrow}{f} \rangle$ is *homogeneous*, that is if $\langle \vec{x}, \underset{\rightarrow}{f} \rangle = 0$ for all $\underset{\rightarrow}{f}$, then $\vec{x}$ is the null element of the space V^n; and if $\langle \vec{x}, \underset{\rightarrow}{f} \rangle = 0$ for all $\vec{x}$, then $\underset{\rightarrow}{f}$ is the null element of the space V_n;

c) $\langle \vec{x}, \underset{\rightarrow}{f} \rangle$ is *symmetric*, that is $\langle \vec{x}, \underset{\rightarrow}{f} \rangle = \langle \underset{\rightarrow}{f}, \vec{x} \rangle$ for all $\vec{x}$ and $\underset{\rightarrow}{f}$.

A real number $\langle \vec{x}, \underset{\rightarrow}{f} \rangle$ is called the *bundle*.

Spaces V^n and V_n, for elements of which the bundle is defined, are called *conjugate* or *dual* spaces.

If a set of elements $(\underset{\rightarrow}{e}^1, \underset{\rightarrow}{e}^2, \ldots \underset{\rightarrow}{e}^n)$ is a basis in space V_n, then we can express any element $\underset{\rightarrow}{f} \in V^n$ as

$$\underset{\rightarrow}{f} = f_i\,\underset{\rightarrow}{e}^i.$$

Bases $(\vec{e}_i)$ and $(\underset{\rightarrow}{e}^k)$ satisfying conditions

$$\langle \vec{e}_i, \underset{\rightarrow}{e}^k \rangle = \delta_i^k$$

are called *dual* or *reciprocal* bases. In such bases we can calculate components x^i and f_i as follows:

$$x^i = \langle \vec{x}, \underset{\rightarrow}{e}^i \rangle, \quad f_i = \langle \underset{\rightarrow}{f}, \vec{e}_i \rangle, \quad \text{and} \quad \langle \vec{x}, \underset{\rightarrow}{f} \rangle = x^i f_i.$$

A function $f : V^n \mapsto \mathbb{R}$ that maps an element of a linear space V^n into a real number satisfying $\forall \vec{x}, \vec{y} \in V^n$ and $\forall \alpha \in \mathbb{R}$ conditions

$$f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y}) \quad \text{and} \quad f(\alpha\vec{x}) = \alpha f(\vec{x})$$

is called the *linear function*.

A set V^* of all linear functions defined in the space V^n is also a linear space of the dimension n, which is the conjugate space to V^n, if the bundle is $\langle \underset{\to}{f}, \vec{x}\rangle = f(\vec{x})$.

Example 1. Let V_3 be a linear space of linear functions defined in a space V^3. Find the reciprocal basis to

$$\vec{e}_1 = (1,0,1), \quad \vec{e}_2 = (0,1,1), \quad \vec{e}_3 = (1,2,0).$$

Calculate coordinates of the element $\vec{x} = (2,3,2)$ in given basis.

Solution. The coordinates of vectors $\vec{e}_i, i = 1,2,3$, are given in some other basis $(\hat{\vec{e}}_i)$. Consider the basis $(\underset{\to}{\hat{e}}^i)$, $\underset{\to}{\hat{e}}^i \in V_3$, that is reciprocal to the basis $(\hat{\vec{e}}_i)$. Let a linear function has coordinates $\underset{\to}{f} = (a_1, a_2, a_3)$ in reciprocal basis and $\underset{\to}{f} = a_i\, \underset{\to}{\hat{e}}^i$. Then the bundle of $\underset{\to}{f} \in V_3$ and $\vec{x} \in V^3$ is equal to $\langle \underset{\to}{f}, \vec{x}\rangle = a_i x^i$. We are to find three elements $\underset{\to}{e}^i$ that satisfy $\langle \underset{\to}{e}^i, \vec{e}_k\rangle = \delta^i{}_k$. Let

$$\vec{e}_i = \underset{i}{x}^k\, \hat{\vec{e}}_k, \qquad \underset{\to}{e}^i = \overset{i}{a}_k\, \underset{\to}{\hat{e}}^k.$$

The coordinates $\underset{i}{x}^k$ are known. Then we write the condition $\langle \underset{\to}{e}^i, \vec{e}_k\rangle = \delta^i{}_k$ as

$$\langle \underset{\to}{e}^1, \vec{e}_1\rangle = \overset{1}{a}_k\, \underset{1}{x}^k = \overset{1}{a}_1 \cdot 1 + \overset{1}{a}_2 \cdot 0 + \overset{1}{a}_3 \cdot 1 = \delta^1{}_1 = 1;$$
$$\langle \underset{\to}{e}^1, \vec{e}_2\rangle = \overset{1}{a}_k\, \underset{2}{x}^k = \overset{1}{a}_1 \cdot 0 + \overset{1}{a}_2 \cdot 1 + \overset{1}{a}_3 \cdot 1 = \delta^1{}_2 = 0;$$
$$\langle \underset{\to}{e}^1, \vec{e}_3\rangle = \overset{1}{a}_k\, \underset{3}{x}^k = \overset{1}{a}_1 \cdot 1 + \overset{1}{a}_2 \cdot 2 + \overset{1}{a}_3 \cdot 0 = \delta^1{}_3 = 0.$$

This is a system of linear equations for variables $\overset{1}{a}_1, \overset{1}{a}_2, \overset{1}{a}_3$, which solution is

$$\overset{1}{a}_1 = \frac{2}{3}, \quad \overset{1}{a}_2 = -\frac{1}{3}, \quad \overset{1}{a}_3 = \frac{1}{3} \quad \text{or} \quad \underset{\to}{e}^1 = \left(\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}\right).$$

Doing the same for $\underset{\to}{e}^2$ and $\underset{\to}{e}^3$ we obtain

$$\underset{\to}{e}^2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \quad \text{and} \quad \underset{\to}{e}^3 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right).$$

To find coordinates of $\vec{x}$ in the basis $(\vec{e}_i)$ we use the formula $x^i = \langle \vec{x}, \underset{\to}{e}^i \rangle$:

$$x^1 = \langle \underset{\to}{e}^1, \vec{x} \rangle = \frac{2}{3} \cdot 2 - \frac{1}{3} \cdot 3 + \frac{1}{3} \cdot 2 = 1;$$
$$x^2 = \langle \underset{\to}{e}^2, \vec{x} \rangle = -\frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 2 = 1;$$
$$x^3 = \langle \underset{\to}{e}^3, \vec{x} \rangle = \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 3 - \frac{1}{3} \cdot 2 = 1.$$

Thus $\vec{x} = 1 \cdot \vec{e}_1 + 1 \cdot \vec{e}_2 + 1 \cdot \vec{e}_3$.

Exercises

Let V_3 be a linear space of linear functions defined in a space V^3. Find the reciprocal basis to the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and calculate coordinates of the element $\vec{x} = (2, 3, 2)$ in given basis.

1.16. $\vec{e}_1 = (1, 2, -3)$, $\vec{e}_2 = (2, -2, 0)$, $\vec{e}_3 = (-2, 0, 1)$, $\vec{x} = (1, 2, 3)$.

1.17. $\vec{e}_1 = (2, -1, 0)$, $\vec{e}_2 = (-1, 0, 1)$, $\vec{e}_3 = (0, 2, -1)$, $\vec{x} = (3, -2, 1)$.

1.18. $\vec{e}_1 = (0, -1, 1)$, $\vec{e}_2 = (2, 0, 1)$, $\vec{e}_3 = (-1, -1, 2)$, $\vec{x} = (1, 1, 1)$.

1.19. $\vec{e}_1 = (1, 1, 1)$, $\vec{e}_2 = (1, 0, 0)$, $\vec{e}_3 = (0, 1, 0)$, $\vec{x} = (-1, 1, -1)$.

1.20. Let $\vec{x} = (x^1, x^2) \in V^2$ and $\underset{\to}{y} = (y_1, y_2) \in V_2$. Define the bundle as

$$\langle \vec{x}, \underset{\to}{y} \rangle = a^i{}_k x^k y_i,$$

where $(a^i{}_k)$ is a matrix 2×2. Prove that spaces V^2 and V_2 are dual.

Let a bundle for elements of spaces V^2 and V_2 be defined as it done in problem 1.20. Find the reciprocal basis to $(\vec{e}_1, \vec{e}_2)$. Calculate coordinates of elements $\vec{x}$ and $\underset{\to}{y}$ in given bases and their bundle.

1.21. $\vec{e}_1 = (1, 2)$, $\vec{e}_2 = (1, 1)$, $(a^i{}_k) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, $\vec{x} = (2, 3)$, $\underset{\to}{y} = (-1, 2)$.

1.22. $\vec{e}_1 = (1, 0)$, $\vec{e}_2 = (0, 1)$, $(a^i{}_k) = \begin{pmatrix} 3 & -4 \\ 3 & -2 \end{pmatrix}$, $\vec{x} = (1, 1)$, $\underset{\to}{y} = (-2, 1)$.

1.23. $\vec{e}_1 = (-2, 3)$, $\vec{e}_2 = (1, -1)$, $(a^i{}_k) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\vec{x} = (-1, 0)$, $\underset{\to}{y} = (1, -2)$.

1.24. $\vec{e}_1 = (1, -1)$, $\vec{e}_2 = (-2, 3)$, $(a^i{}_k) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\vec{x} = (-2, 1)$, $\underset{\to}{y} = (1, -3)$.

1.25. Consider a space of polar vectors $\vec{x}, \vec{y}, \ldots$ and a space of lamellar vectors $\underset{\to}{f}, \underset{\to}{g}, \ldots$. Polar vector is defined by an ordered pair of points O and M, the point O being common point for all vectors. A lamellar vector $\underset{\to}{f}$ is defined by an ordered pair of parallel planes ω and μ (or by a pair of parallel lines in the two dimensional case) and the plane ω contains the point O. Define the bundle of polar and lamellar vectors by following rule:

a) if $O \in \omega$ and $M \in \omega$, then $\langle \vec{x}, \underset{\to}{f} \rangle = \langle \underset{\to}{f}, \vec{x} \rangle = 0$;

b) if $M \notin \omega$, then $\langle \vec{x}, \underset{\to}{f} \rangle = \langle \underset{\to}{f}, \vec{x} \rangle$ is the coefficient of proportionality between line segments OM and ON (fig. 1.1).

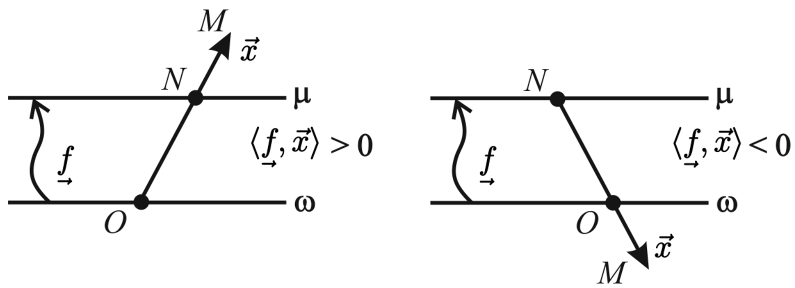

Fig. 1.1

The point $N \in \mu$ and the points M and N are on the same straight line OM. In addition, $\langle \underset{\to}{f}, \vec{x} \rangle > 0$ if the points M and N are on the same side of the plane ω and $\langle \underset{\to}{f}, \vec{x} \rangle > 0$ otherwise. Prove that spaces of polar and lamellar vectors are dual.

1.26. Let P be some point in a three dimensional space and C^1_P be a set of all differentiable functions at P. Consider a set

$$T_1(P) = \{\mathrm{d}f(P) \mid f \in C^1_P\}.$$

The set $T_1(P)$ is a linear space. Define a bundle of an element $\vec{l} \in V^3$ and $\mathrm{d}f(P) \in T_1(P)$ as[1]:

$$\langle \vec{l}, \mathrm{d}f(P) \rangle = |\vec{l}| \frac{\partial f}{\partial l}.$$

Prove that spaces $T_1(P)$ and V^3 are dual.

[1]The definition of directional derivative can be found in section 3.1.

1.3. Changing of bases

Suppose that in a linear space V^n there are an arbitrary "old" basis $(\vec{e}_i), i = 1, \ldots, n$, and some "new" basis $(\vec{e}_{i'}),\ i' = 1', \ldots, n' = n$. In the tensor calculus the new basis elements are denoted by the same root letter "e" but with primed indices. Since $(\vec{e}_{i'})$ is a basis, we can expand its elements $\vec{e}_i$ as

$$\vec{e}_{i'} = A^i{}_{i'}\vec{e}_i,$$

where n^2 numbers $A^i{}_{i'}$ form a matrix, called the *transformation matrix* from the "old" basis to the "new" one. The element of the i^{th} row and the i'^{th} column of the matrix $(A^i{}_{i'})$ is the i^{th} coordinate of the element $\vec{e}_{i'}$ in the basis $\vec{e}_i$. The matrix $(A^i{}_{i'})$ is nondegenerate that is $\det(A^i{}_{i'}) \neq 0$.

An "old" basis can be expressed through a "new" basis by *inverse transformation matrix* $(A^{i'}{}_i)$:

$$\vec{e}_i = A^{i'}{}_i\vec{e}_{i'},$$

where

$$A^{k'}{}_i A^i{}_{j'} = \delta^{k'}{}_{j'}, \qquad A^{k'}{}_i A^j{}_{k'} = \delta_i^{\,j}.$$

Coordinates of a vector $\vec{x} \in V^n$ in an "old" and a "new" bases are connected by the relation

$$x^{i'} = A^{i'}{}_i x^i, \qquad x^i = A^i{}_{i'} x^{i'}.$$

Let $\underset{\rightarrow}{e}^i$ and $\underset{\rightarrow}{e}^{i'}$ be an "old" and a "new" reciprocal bases in a dual space V_n correspondingly. Then

$$\underset{\rightarrow}{e}^{i'} = A_i{}^{i'}\underset{\rightarrow}{e}^i, \qquad \underset{\rightarrow}{e}^i = A_{i'}{}^i\underset{\rightarrow}{e}^{i'};$$

$$f_{i'} = A_{i'}{}^i f_i, \qquad f_i = A_i{}^{i'} f_{i'},$$

where $(A_{i'}{}^i)$ is the inverse transposed transformation matrix and $(A_i{}^{i'})$ is the transposed transformation matrix:

$$A_i{}^{k'} A_{j'}{}^i = \delta^{k'}{}_{j'}, \quad A_i{}^{k'} A_{k'}{}^j = \delta_i^{\,j} \quad \text{and} \quad A^i{}_{k'} A_i{}^{j'} = \delta^{j'}{}_{k'}.$$

You can see that coordinates of elements of a dual space V_n are changing in the same way as basis elements of space V^n, and vise versa, coordinates of elements of an initial space V^n are changing in the same way as basis elements of the dual space V_n. Arithmetic vectors (x^i) and (f_i) (i. e. ordered set of numbers) are called *contravariant* vectors and *covariant* vectors or briefly *vectors* and *covectors*.

Example 1. Let $\vec{e}_i$ and $\vec{e}_{i'}$ be an "old" and a "new" bases in a space V^2 correspondingly and $\vec{e}_1 = (1,2), \vec{e}_2 = (1,3)$ and $\vec{e}_{1'} = (1,1), \vec{e}_{2'} = (3,2)$. Let V_2 be the dual space of linear functions. Find coordinates of elements $\vec{x} \in V^2$ and $\underset{\to}{f} \in V_2$ in the "new" basis, if their coordinates in the "old" bases are $(x^i) = (3,1), (f_i) = (2,0)$. Show that $x^i f_j = x^{i'} f_{j'}$.

Solution. First of all we are to find the "old" reciprocal basis. Following Example 1 of the section 1.2 we obtain

$$\underset{\to}{e}^1 = (3,-1), \quad \underset{\to}{e}^2 = (-2,1).$$

To calculate the transformation matrix we can use its definition namely that the element $A^i_{i'}$ is the i^{th} coordinate of the vector $\vec{e}_{i'}$ in the basis $\vec{e}_i$. Therefore $A^i_{i'}$ is the bundle

$$A^i{}_{i'} = \langle \underset{\to}{e}^i, \vec{e}_{i'} \rangle.$$

Thus we obtain

$$A = (A^i{}_{i'}) = \begin{pmatrix} 2 & 7 \\ -1 & -4 \end{pmatrix}.$$

Then transposed and inverse transformation matrices are

$$A^{\mathrm{T}} = (A_{i'}{}^i) = \begin{pmatrix} 2 & -1 \\ 7 & -4 \end{pmatrix} \quad \text{and} \quad A^{-1} = (A^{i'}{}_i) = \begin{pmatrix} 4 & 7 \\ -1 & -2 \end{pmatrix}.$$

As far the transformation matrix is found we can easily calculate coordinates

$$x^{i'} = A^{i'}{}_i x^i, \quad x^{1'} = 19, \ x^{2'} = -5;$$
$$f_{i'} = A_{i'}{}^i f_i, \quad f_{1'} = 4, \ f_{2'} = 14;$$

and the bundle

$$x^i f_i = 3 \cdot 2 + 1 \cdot 0 = 6, \quad x^{i'} f_{i'} = 19 \cdot 4 - 5 \cdot 14 = 6.$$

Exercises

Let $\vec{e}_i$ and $\vec{e}_{i'}$ be an "old" and a "new" bases in a space V^2 correspondingly. Let V_2 be the dual space of linear functions. Find coordinates of elements $\vec{x} \in V^2$ and $\underset{\to}{f} \in V_2$ in the "new" basis and show that $x^i f_j = x^{i'} f_{j'}$.

1.27. $\vec{e}_1 = (2,-1), \ \vec{e}_{1'} = (-1,1), \ (x^i) = (1,1),$
$\vec{e}_2 = (1,0), \ \vec{e}_{2'} = (1,-2), \ (f_i) = (-2,5).$

1.28. $\vec{e}_1 = (1,0)$, $\vec{e}_{1'} = (1,1)$, $(x^i) = (-1,3)$,
$\vec{e}_2 = (0,1)$, $\vec{e}_{2'} = (-1,1)$, $(f_i) = (5,1)$.

1.29. $\vec{e}_1 = (3,2)$, $\vec{e}_{1'} = (2,-1)$, $(x^i) = (3,1)$,
$\vec{e}_2 = (-1,2)$, $\vec{e}_{2'} = (-3,1)$, $(f_i) = (-1,3)$.

1.30. $\vec{e}_1 = (1,2)$, $\vec{e}_{1'} = (3,-2)$, $(x^i) = (10,5)$,
$\vec{e}_2 = (1,-3), \vec{e}_{2'} = (-2,3), (f_i) = (-5,15)$.

1.31. Prove that for all x^i and f_i there are equalities $A^{i'}{}_i x^i = A_i{}^{i'} x^i$ and $A_{i'}{}^i f_i = A^i{}_{i'} f_i$.

1.32. Let (x^1, x^2, x^3) be curvilinear coordinates[2] in a three dimensional space and P is an arbitrary point. Then vectors

$$\vec{e}_1 = \frac{\partial \vec{r}}{\partial x^1}, \quad \vec{e}_2 = \frac{\partial \vec{r}}{\partial x^2}, \quad \vec{e}_3 = \frac{\partial \vec{r}}{\partial x^3}$$

form a basis in V^3. Let $(x^{1'}, x^{2'}, x^{3'})$ be another curvilinear coordinates that induce the basis $(\vec{e}_{i'})$. Find the transformation matrix from the basis $\vec{e}_i$ to the basis $\vec{e}_{i'}$.

1.4. Definition of a tensor. Tensor algebra

An object characterized in each basis of a linear space V^n by a set of n^{p+q} numbers $a^{i_1 \dots i_p}_{j_1 \dots j_q}$ $(i_1, \dots, i_p, j_1, \dots, j_q = 1, \dots, n)$ that at basis change are transformed by the linear homogeneous rule

$$a^{i'_1 \dots i'_p}_{j'_1 \dots j'_q} = A^{i'_1}{}_{i_1} \dots A^{i'_p}{}_{i_p} A_{j'_1}{}^{j_1} \dots A_{j'_q}{}^{j_q} a^{i_1 \dots i_p}_{j_1 \dots j_q}$$

is called the *tensor* of the type (p,q) in the space V^n. The number $r = p + q$ defines the tensor *valence* (*rank*), with the numbers p and q being *contravariant* and *covariant* valences of the tensor. The numbers $a^{i_1 \dots i_p}_{j_1 \dots j_q}$ are called tensor *components* in given basis.

Two tensors are called *one-type* if their contravariant and covariant valences are equal. Tensors having indices of both types are called *mixed*. Two one-type tensors are *equal* if their respective components are equal in some basis. The tensor is called the *null-tensor* if all its components in some basis are zero.

If we want to write down the components of some tensor, it is convenient to use the matrix notation. In the matrix representation of the second

[2] The definition of curvilinear coordinates can be found in section 5.4.

valence tensor we consider the first index to correspond to the matrix row number, and the second index shows the column number. Components of the third valence tensor can be written in the form of a three-dimensional matrix. Fixing the last (third) index, we get a square matrix $n \times n$, which is called the two-dimensional section which first index corresponds to the row number and the second one corresponds to the column number. The resulting sections are aligned in the row matrix $n \times n^2$. For example, for $n = 2$, tensor components $a^i{}_{jk}$ are written as

$$\left(\begin{array}{cc|cc} a^1{}_{11} & a^1{}_{21} & a^1{}_{12} & a^1{}_{22} \\ a^2{}_{11} & a^2{}_{21} & a^2{}_{12} & a^2{}_{22} \end{array} \right).$$

Similarly, writing down a tensor of valence 4 we fix last two indices (the third and the forth) and obtain n^2 two-dimensional sections aligned in the rectangular matrix $n^2 \times n^2$:

$$\left(\begin{array}{cc|cc} a^{11}{}_{11} & a^{12}{}_{11} & a^{11}{}_{12} & a^{12}{}_{12} \\ a^{21}{}_{11} & a^{22}{}_{11} & a^{21}{}_{12} & a^{22}{}_{12} \\ \hline a^{11}{}_{21} & a^{12}{}_{21} & a^{11}{}_{22} & a^{12}{}_{22} \\ a^{21}{}_{21} & a^{22}{}_{21} & a^{21}{}_{22} & a^{22}{}_{22} \end{array} \right).$$

Algebraic operations

1. *Summation.* Sum of two one-type tensors $a^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ and $b^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ is the tensor $c^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ of the same type and valence, which components in each basis are defined as

$$c^{i_1 \ldots i_p}_{j_1 \ldots j_q} = a^{i_1 \ldots i_p}_{j_1 \ldots j_q} + b^{i_1 \ldots i_p}_{j_1 \ldots j_q}.$$

2. *Multiplication.* Multiplication of a tensor $a^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ of type (p, q) and a tensor $b^{k_1 \ldots k_r}_{l_1 \ldots l_s}$ of type (r, s) is the tensor of type $(p + r, q + s)$, which components in each basis are defined as

$$c^{i_1 \ldots i_p k_1 \ldots k_r}_{j_1 \ldots j_q l_1 \ldots l_s} = a^{i_1 \ldots i_p}_{j_1 \ldots j_q} b^{k_1 \ldots k_r}_{l_1 \ldots l_s}.$$

The multiplication is not commutative.

3. *Contraction and transvection.* Contraction of a tensor $a^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ of a type (p, q) over upper index i_l, $1 \leqslant l \leqslant p$, and lower index j_m, $1 \leqslant m \leqslant q$, is the tensor of the type $(p - 1, q - 1)$, which components in each basis are defined as

$$c^{i_1 \ldots i_{l-1} i_{l+1} \ldots i_p}_{j_1 \ldots j_{m-1} j_{m+1} \ldots j_q} = a^{i_1 \ldots i_{l-1} k\, i_{l+1} \ldots i_p}_{j_1 \ldots j_{m-1} k\, j_{m+1} \ldots j_q}.$$

Contraction is applicable only to tensors which contravariant and covariant valences are greater or equal to one. Contraction can be repeated and the maximal number of these operations is equal to $\min(p,q)$. If $p=q$, the resulting tensor obtained by the maximal number of contractions is a tensor of valence zero, i. e. a scalar.

Transvection is the combination of tensor multiplication and contraction, for example

$$a^{ik}b_l \quad : \quad a^{ik}b_k \quad \text{or} \quad a^{ik}b_i.$$

4. *Building an isomer.* An isomer is formed by interchanging upper or lower indices. The resulting tensor has the same valence and type.

Direct tensor criterion: for any nondegenerate transformation of the basis the tensor components are transformed according to rules conserving the results of addition, multiplication, contraction, and also equality of tensors.

Inverse tensor criterion: in order for a set of n^{p+q} numbers $a^{i_1\ ldotsi_p}_{j_1\ ldotsj_q}$ to be a tensor of type (p,q), it is necessary and sufficient that the result of its multiplication or transvection with any tensor of fixed valence and type is the corresponding tensor.

A tensor $a^{k\dots}_{\dots i\dots j\dots}$ is called *symmetrical* with respect to a pair of upper or lower indices i and j, if it is invariant for interchanging of these indices, that is

$$a^{k\dots}_{\dots i\dots j\dots} = a^{k\dots}_{\dots j\dots i\dots}, \quad i,j=1,\dots,n.$$

A tensor $b^{\dots i\dots j\dots}_{k\dots}$ is called *antisymmetrical* (*alternating*) with respect to the pair of upper or lower indices i and j, if interchanging of these indices results only in change of the sign

$$b^{\dots i\dots j\dots}_{k\dots} = -b^{\dots j\dots i\dots}_{k\dots}, \quad i,j=1,\dots,n.$$

Symmetry and antisymmetry properties of an tensor are defined only by the position of indices but not by their name.

A tensor is called *symmetrical with respect to a group* of one-type indices, if it is invariant for any permutation of indices within this group. A tensor is called *antisymmetrical with respect to a group* of one-type indices, if permutation of any pair of indices within this group results only in changing the sign.

A tensor that is symmetrical (antisymmetrical) with respect to all indices of a certain type is called *completely symmetrical* (*completely antisymmetrical*). An antisymmetrical tensor of valence 2 is often called *bivector*.

5. *Symmetrization.* The symmetrization over m indices is the sum of all different isomers obtained by permuting these indices divided by $p!$. For example

a) over two indices:

$$a^{(ik)} = \frac{1}{2}(a^{ik} + a^{ki})$$

or

$$a^{(i|kj|m)} = \frac{1}{2}(a^{ikjm} + a^{mjki});$$

b) over three indices:

$$a_{(ikl)} = \frac{1}{3!}(a_{ikl} + a_{kli} + a_{lik} + a_{kil} + a_{ilk} + a_{lki}).$$

The symmetrization is denoted by round brackets ( ).

6. *Alternating.* The alternating over m indices is the sum of all different isomers, taken with positive sing if permutation is even and with negative sign if it is odd, divided by $p!$. For example

a) over two indices:

$$a^{[ik]} = \frac{1}{2}(a^{ik} - a^{ki}),$$

or

$$a^{[i|kj|m]} = \frac{1}{2}(a^{ikjm} - a^{mjki});$$

b) over three indices:

$$a_{[ikl]} = \frac{1}{3!}(a_{ikl} + a_{kli} + a_{lik} - a_{kil} - a_{ilk} - a_{lki}).$$

The alternating is denoted by square brackets [ ].

Tensors $a^{(ik)}$ or $a_{(ikl)}$ are completely symmetric, tensors $a^{[ik]}$ or $a_{[ikl]}$ are completely antisymmetric, and

$$a^{ik} = a^{(ik)} + a^{[ik]};$$

$$a_{ikl} \neq a_{(ikl)} + a_{[ikl]}.$$

Example 1. Let $a^i{}_k$ be a mixed tensor. Prove that $a^k{}_k$ is a scalar.

Solution. Consider the transformation rule for an object $a^k{}_k$:

$$a^{k'}{}_{k'} = A^{k'}{}_k A_{k'}{}^r a^k{}_r.$$

Since $A^{k'}{}_k A_{k'}{}^r = \delta^r{}_k$, then $a^{k'}{}_{k'} = \delta^r{}_k a^k{}_r = a^k{}_k$. Thus the object $a^k{}_k$ is invariant for basis changing, therefore $a^k{}_k$ is a scalar.

Example 2. Let ${b^i}_j = {a^i}_{jk} u^k$ be a tensor for any contravariant vector u^i. Prove that ${a^i}_{jk}$ is a tensor.

Solution. As far as ${b^i}_j$ is a tensor, it components are transformed as

$$ {b^{i'}}_{j'} = {A^{i'}}_i {A_{j'}}^j {b^i}_j. $$

The transformation rule for vector u^i is

$$ u^{k'} = {A^{k'}}_k u^k. $$

On the other hand in a "new" basis

$$ {b^{i'}}_{j'} = {a^{i'}}_{j'k'} u^{k'}. $$

Then

$$ {b^{i'}}_{j'} = {a^{i'}}_{j'k'} u^{k'} = {a^{i'}}_{j'k'} {A^{k'}}_k u^k = {A^{i'}}_i {A_{j'}}^j {b^i}_j = {A^{i'}}_i {A_{j'}}^j {a^i}_{jk} u^k $$

or

$$ {a^{i'}}_{j'k'} {A^{k'}}_k u^k = {A^{i'}}_i {A_{j'}}^j {a^i}_{jk} u^k. $$

Factorizing this expression we obtain

$$ ({a^{i'}}_{j'k'} {A^{k'}}_k - {A^{i'}}_i {A_{j'}}^j {a^i}_{jk}) u^k = 0. $$

Since this equality is true for all vectors u^k, the expression in brackets is equal to zero, i. e.

$$ {a^{i'}}_{j'k'} {A^{k'}}_k = {A^{i'}}_i {A_{j'}}^j {a^i}_{jk}. $$

Multiplying both parts of above expression by ${A_{r'}}^k$ and taking into account that ${A^{k'}}_k {A_{r'}}^k = {\delta_{r'}}^{k'}$, we obtain

$$ {a^{i'}}_{j'k'} {A^{k'}}_k {A_{r'}}^k = {a^{i'}}_{j'k'} {\delta_{r'}}^{k'} = {a^{i'}}_{j'r'} = {A^{i'}}_i {A_{j'}}^j {A_{r'}}^k {a^i}_{jk}. $$

Thus, the a^i_{jk} is transformed according to tensor of type $(1, 2)$ transformation rule.

Example 3. A tensor ${a^i}_j$ of type $(1, 1)$, a contravariant vector x^k and a covariant vector y_k in a certain basis have the following components:

$$ ({a^i}_j) = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}, \quad (x^i) = (1, 2), \quad (y_i) = (-2, 1). $$

Find: a) ${a^i}_j x^j$; b) ${a^i}_j y_i$; c) ${a^i}_j x^j y_i$; d) ${a^i}_i$.

Solution. Direct calculations give us

a) ${a^1}_j x^j = {a^1}_1 x^1 + {a^1}_2 x^2 = 1 \cdot 1 + (-2) \cdot 2 = -3,$

${a^2}_j x^j = {a^2}_1 x^1 + {a^2}_2 x^2 = (-3) \cdot 1 + 4 \cdot 2 = 5;$

b) ${a^i}_1 y_i = {a^1}_1 y_i + {a^2}_1 y_2 = 1 \cdot (-2) + (-3) \cdot 1 = -5,$

${a^i}_2 y_i = {a^1}_2 y_i + {a^2}_2 y_2 = (-2) \cdot (-2) + 4 \cdot 1 = 8;$

c) ${a^i}_j x^j y_i = {a^1}_1 x^1 y_1 + {a^1}_2 x^2 y_1 + {a^2}_1 x^1 y_2 + {a^2}_2 x^2 y_2 = 11;$

d) ${a^i}_i = {a^1}_1 + {a^2}_2 = 1 + 4 = 5.$

Example 4. Show that transvection of a symmetrical tensor a_{ij} and an antisymmetrical tensor b^{ij} is zero, i. e. $a_{ij}b^{ij} = 0$.

Solution. Rename summation indices in the expression $a_{ij}b^{ij}$ as $i \to j$, and $j \to i$:

$$a_{ij}b^{ij} = a_{ji}b^{ji}.$$

Taking into account that $a_{ij} = a_{ji}$ and $b^{ij} = -b^{ji}$, we obtain

$$a_{ij}b^{ij} = a_{ji}b^{ji} = -a_{ij}b^{ij}.$$

Since the bundle $a_{ij}b^{ij}$ is a scalar, then from the equality of a number to itself with opposite sign it follows from that it is equal to zero, that is, $a_{ij}b^{ij} = 0$.

Exercises

1.33. Write the transformation rule for different tensors of valence 3.

1.34. How many independent components has: a) a symmetric tensor of valence 2; b) an antisymmetric tensor of valence 2?

1.35. A tensor c_{ijkl}, $i = 1, 2, 3$, of valence 4 meets following symmetry conditions: $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{jilk}$. Show that these conditions result in reducing of the number of independent components from 81 to 36. Show that if in addition the tensor satisfies condition $c_{ijkl} = c_{klij}$ its number of independent components is equal to 21.

1.36. A tensor c_{iklm}, $i = 1, 2, 3, 4$, of valence 4 satisfies following antisymmetry conditions: $c_{iklm} = -c_{ikml} = -c_{kilm}$. Show that these conditions result in reducing of the number of independent components from 256 to 36. Show that if in addition the tensor satisfies condition $c_{iklm} = c_{lmik}$ it number of independent components is equal to 21.

1.37. Prove that if corresponding components of tensors are equal in a certain basis they are equal in any other basis.

1.38. Prove that if a tensor is symmetrical (antisymmetrical) in a certain basis, it is symmetrical (antisymmetrical) in a certain basis in any other basis.

1.39. A tensor of the type $(1, 1)$ has in a certain basis following components:

$$\delta_i^{\ k} = \delta^k_{\ i} = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k. \end{cases}$$

Does the tensor $\delta_i^{\ k}$ have different components in another basis?

1.40. Let $a^i_{\ j}$ and $b^i_{\ j}$ be tensors. Prove that the object $c^i_{\ j} = a^i_{\ j} + b^j_{\ i}$, is not a tensor.

1.41. Let $a^i_{\ j}$ be a mixed tensor and let u^i and v_i be contravariant and covariant vectors respectively. Prove that $a^i_{\ j}u^j$ is a contravariant vector and $a^i_{\ j}v_i$ is a covariant vector.

Multiply following tensors:

1.42. $(a^i) = (1, 1), \quad (b^i) = (1, -1)$.

1.43. $(a^i) = (1, 1), \quad (b_i) = (-1, 1)$.

1.44. $(a_i) = (4, -2), \quad (b_{jk}) = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$.

1.45. $(a^i_{\ jk}) = \left(\begin{array}{cc|cc} 3 & 4 & 2 & 5 \\ 5 & 7 & 1 & 3 \end{array}\right), (b_l) = (-1,\ 1)$.

1.46. $(a_i) = (-1, 1), \quad (b_{jkl}) = \left(\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{array}\right)$.

1.47. A covariant tensor a_{ij} and a contravariant vectors x^k and y^k have components:

$$(a_{ij}) = \begin{pmatrix} 1 & -2 & -3 \\ 5 & 2 & 4 \\ 3 & 0 & 7 \end{pmatrix}, \quad (x^i) = (-1, 2, -5), \quad (y^i) = (2, 3, -1).$$

Compute: a) $a_{ij}x^j$; b) $a_{ij}x^i$; c) $a_{ij}x^iy^j$; d) $a_{ij}x^jy^i$.

1.48. A mixed tensor a^i_j, a contravariant vector x^k and a covariant vector y_k have components:

$$(a^i_{\ j}) = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{pmatrix}, \quad (x^i) = (1, 1, 0), \quad (y_i) = (0, 1, 1).$$

Compute: a) $a^i_{\ j}x^j$; b) $a^i_{\ j}y_i$; c) $a^i_{\ j}x^jy_i$; d) $a^i_{\ i}$.

1.49. Is it possible to build transvection of: a) a vector and a covector; b) a vector and a vector; c) pair of covectors?

1.50. Let $a^i{}_j$ and $b^k{}_l$ be tensors. With the help of multiplication and transvection build various scalars from them.

1.51. Let a_{ij} and b^{klm} be tensors. With the help of multiplication and transvection build from them various tensors of valences 1, 3 and 5.

1.52. A mixed tensor $a^{ij}{}_k$ has components:

a) $\left(\begin{array}{cc|cc} 1 & 5 & 2 & 6 \\ 3 & 7 & 4 & 8 \end{array}\right)$; b) $\left(\begin{array}{cc|cc} 0 & -2 & 2 & 5 \\ 1 & 3 & 3 & 2 \end{array}\right)$.

Calculate components of contravariant vectors $a^{ij}{}_i$ and $a^{ij}{}_j$.

1.53. A mixed tensor $a^{ij}{}_{kl}$ has components:

$$\left(\begin{array}{cc|cc} -3 & -4 & 3 & 4 \\ -5 & -7 & 5 & 7 \\ \hline -2 & -5 & 2 & 5 \\ -1 & -3 & 1 & 3 \end{array}\right).$$

Calculate: a) $a^{ij}{}_{il}$; b) $a^{ij}{}_{kj}$; c) $a^{ij}{}_{ki}$; d) $a^{ij}{}_{jl}$; e) $a^{ij}{}_{ij}$; f) $a^{ij}{}_{ji}$.

1.54. Let $a_{i_1 \ldots i_n}$ be a covariant tensor.

a) How many isomers can be build from this tensor?

b) For $n = 3$ the tensor a_{ijk} has components $\left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array}\right)$. Write down the matrix of all isomers.

Write down components of following tensors in space V^2 in the matrix form.

1.55. $x^i y^k$.

1.56. $x^{(i} y^{k)}$.

1.57. $x^{[i} y^{k]}$.

1.58. $x^i a_{jk}$.

1.59. $x^i a_{ik}$.

1.60. $x^i a^k{}_k$.

1.61. $x^{(i} a^{k)}{}_k$.

1.62. $x^{[i} a^{k]}{}_k$.

1.63. $a^i{}_i a^k{}_k$.

1.64. $a^i{}_{(i} a^k{}_{k)}$.

1.65. $a^i{}_{[i} a^k{}_{k]}$.

1.66. $\delta^i{}_j a^k{}_i$.

1.67. $\delta^i{}_j a^j{}_k$.

1.68. $\delta^i{}_k a^k{}_i$.

1.69. $\delta^i{}_j a^k{}_l$.

1.70. $a^i{}_j \delta^k{}_l$.

Symmetrize and alternate following tensors.

1.71. $(a_{ij}) = \begin{pmatrix} 1 & 2 & 7 \\ 4 & 0 & 3 \\ 5 & 1 & 6 \end{pmatrix}.$

1.72. $(a_{ij}) = \begin{pmatrix} 4 & 1 & 2 \\ 5 & 3 & 1 \\ 0 & 6 & 3 \end{pmatrix}.$

1.73. $(a_{ij}) = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 5 \\ 1 & 0 & 1 \end{pmatrix}.$

1.74. $(a_{ij}) = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 4 & -4 \\ -3 & 2 & 6 \end{pmatrix}.$

1.75. Let a_{ijk} be a tensor with components $(a_{ijk}) = \left(\begin{array}{cc|cc} 3 & 4 & 2 & 5 \\ 5 & 7 & 1 & 3 \end{array}\right).$ Find components of the following tensors: a) $a_{(ij)k}$; b) $a_{(i|j|k)}$.

1.76. Find tensors $a^{[ij]}{}_{kl}$, $a^{ij}{}_{[kl]}$, $a^{[ij]}{}_{[kl]}$, if

$$(a^{ij}_{kl}) = \left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ \hline -4 & -3 & -2 & -1 \\ 5 & 6 & 7 & 8 \end{array}\right).$$

Find tensors $a_{(ijk)}$ and $a_{[ijk]}$, if tensor a_{ijk} has the following components:

1.77. $\left(\begin{array}{ccc|ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 2 & 1 & 5 & 9 & 8 & 7 & 6 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}\right).$

1.78. $\left(\begin{array}{ccc|ccc|ccc} 2 & 4 & 6 & 4 & 6 & 2 & 6 & 8 & 4 \\ 6 & 2 & 8 & 2 & 6 & 4 & 4 & 6 & 2 \\ 4 & 6 & 2 & 8 & 2 & 4 & 2 & 4 & 6 \end{array}\right).$

Find out whether the following tensors are symmetric or antisymmetric and with respect to what indices.

1.79. $\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$

1.80. $\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & -1 & 2 & -1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 & -1 & 2 & 1 \\ -1 & 2 & -1 & 1 & -2 & 1 & 0 & 0 & 0 \end{array}\right).$

1.81. $\left(\begin{array}{ccc|ccc|ccc} 0 & -1 & 1 & 0 & 2 & -2 & 0 & -1 & 1 \\ 1 & 0 & -1 & -2 & 0 & 2 & 1 & 0 & -1 \\ -1 & 1 & 0 & 2 & -2 & 0 & -1 & 1 & 0 \end{array}\right).$

Calculate invariants $a^k{}_k$, $a^i{}_{[i}a^k{}_{k]}$ and $a^i{}_{[i}a^j{}_j a^k{}_{k]}$, if the tensor $a^i{}_j$ has components:

1.82. $(a^i{}_j) = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$. **1.83.** $(a^i{}_j) = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -2 \\ 0 & -2 & 5 \end{pmatrix}$.

1.84. Let a_{ijk} be a tensor that is symmetrical over the first and the second indices. Show that the antisymmetric over the last two indices tensor $c_{ijk} = a_{i[jk]}$ looses symmetry over the first two, i. e. $c_{ijk} \neq c_{jik}$.

1.85. Prove that if a tensor is symmetrical over a certain pair of indices, the symmetrization over this pair does not change the tensor and alternating gives null-tensor.

1.86. Prove that if a tensor is antisymmetrical over a certain pair of indices, the alternating over this pair does not change the tensor and symmetrization gives null-tensor.

1.87. Prove that $a_{(ij)}b^{ij} = a_{ij}b^{(ij)} = a_{(ij)}b^{(ij)}$.

1.88. Prove that $a_{[ij]}b^{ij} = a_{ij}b^{[ij]} = a_{[ij]}b^{[ij]}$.

1.89. Let a tensor a_{ijk} be symmetrical over the first to indices. Prove that $a_{(ikl)} = \dfrac{1}{3}(a_{ikl} + a_{kli} + a_{lik})$.

1.90. Let a tensor a_{ijk} be antisymmetrical over the first to indices. Prove that $a_{[ikl]} = \dfrac{1}{3}(a_{ikl} + a_{kli} + a_{lik})$.

1.91. Prove that in a linear space V^3 any completely antisymmetric tensor of valence greater that 3 is a null-tensor.

1.92. Prove that $a_{[ij}b_{kl]} = \dfrac{1}{4}(a_{ij}b_{kl} - a_{ik}b_{jl} - a_{lj}b_{ki} + a_{lk}b_{ji})$.

1.93. Let $a_{ijk} + a_{ikj}$ be a tensor. Prove that if the object a_{ijk} is symmetrical over the second and the third indices, then a_{ijk} is a tensor.

1.94. Let $a_{ik}u^i u^k$ be a scalar for any contravariant vector u^i. Prove that $a_{(ik)}$ is a tensor.

1.95. Let $b^j{}_{ik}w^{ik}$ be a contravariant vector for any contravariant bivector w^{ik}. Prove that $b^j{}_{[ik]}$ is a tensor.

1.96. Prove that if covariant tensor a_{ijk} satisfies conditions $a_{ijk} = a_{jik}$ and $a_{ijk}u^i u^j u^k$ for any contravariant vector u^k, then $a_{ijk} + a_{jki} + a_{kij} = 0$.

1.97. Prove that if a covariant tensor a_{ijkl} satisfies conditions $a_{ijkl}u^i v^j u^k v^l = 0$ for any contravariant vectors u^k and v^k, then $a_{ijkl} + {}+a_{kjil} + a_{ilkj} + a_{klij} = 0$, and if in addition $a_{ijkl} + a_{jikl} = 0$, $a_{ijkl} + a_{ijlk} = 0$, $a_{ijkl} + a_{jkil} + a_{kijl} = 0$ then a_{ijk} is a null-tensor.

1.98. Prove that if a tensor $a_{jk}{}^{i}$ satisfies conditions$a_{jk}{}^{i}u^{j}u^{k}v_{i} = 0$ for any vectors u^{k} and v_{i} such that $u^{k}v_{k} = 0$, then $a_{(jk)}{}^{i} = s_{(j}\delta_{k)}{}^{i}$, where s_{j} is a vector.

1.99. Prove that if the equality $u^{j}{}_{i}v_{j} = \sigma v_{i}$ is true for any covariant vector v_{i}, then $u^{j}{}_{i} = \sigma\delta^{j}{}_{i}$, where σ does not depend on v_{i}.

1.100. Let tensor a_{ijkl} satisfy conditions $a_{ijkl} = a_{ijlk}$ and $a_{ijkl} + a_{iklj} + {}+a_{iljk} = 0$. Prove that

a) if $a_{ijkl} - a_{ikjl} = 0$, then $a_{ijkl} = 0$;

b) if $a_{ijkl} + a_{ikjl} = 0$, then $a_{ijkl} = 0$.

1.101. Prove that if a_{ij} is a nondegenerate covariant symmetrical tensor ($\det(a_{ij}) \neq 0$), then an object b^{ij}, defined by equations $a_{ij}b^{ik} = \delta_{i}^{k}$, is contravariant symmetrical tensor.

1.5. Tensor product of linear spaces

Let V and $\widetilde{V}$ be linear spaces with elements $x \in V$ and $\tilde{x} \in \widetilde{V}$. An ordered pair $x \otimes \tilde{x}$ is called *dyad*.

Consider elements $\underset{k}{x} \in V$, and $\underset{k}{\tilde{x}} \in \widetilde{V}$, $k = 1, 2, 3, \ldots$. and build a set T containing elements

$$t = \underset{1}{x} \otimes \underset{1}{\tilde{x}} + \underset{2}{x} \otimes \underset{2}{\tilde{x}} + \ldots + \underset{k}{x} \otimes \underset{k}{\tilde{x}},$$

which $\forall\alpha \in \mathbb{R}$ satisfy following conditions:

1) a sum $\underset{1}{x} \otimes \underset{1}{\tilde{x}} + \underset{2}{x} \otimes \underset{2}{\tilde{x}} + \ldots + \underset{k}{x} \otimes \underset{k}{\tilde{x}}$ does not depend on the order of summands;

2) $(x + y) \otimes \tilde{x} = x \otimes \tilde{x} + y \otimes \tilde{x}$;

3) $(\alpha x) \otimes \tilde{x} = x \otimes (\alpha\tilde{x})$.

A *sum* of two elements $t = \underset{1}{x} \otimes \underset{1}{\tilde{x}} + \underset{2}{x} \otimes \underset{2}{\tilde{x}} + \ldots + \underset{k}{x} \otimes \underset{k}{\tilde{x}}$ and $s = \underset{1}{y} \otimes \underset{1}{\tilde{y}} + \underset{2}{y} \otimes \underset{2}{\tilde{y}} + \ldots + \underset{m}{y} \otimes \underset{m}{\tilde{y}}$ is the element

$$t + s = \underset{1}{x} \otimes \underset{1}{\tilde{x}} + \underset{2}{x} \otimes \underset{2}{\tilde{x}} + \ldots + \underset{k}{x} \otimes \underset{k}{\tilde{x}} + \underset{1}{y} \otimes \underset{1}{\tilde{y}} + \underset{2}{y} \otimes \underset{2}{\tilde{y}} + \ldots + \underset{m}{y} \otimes \underset{m}{\tilde{y}}.$$

A *product* of an element t and a number α is the element

$$\alpha t = (\alpha \underset{1}{x}) \otimes \underset{1}{\tilde{x}} + (\alpha \underset{2}{x}) \otimes \underset{2}{\tilde{x}} + \ldots + (\alpha \underset{k}{x}) \otimes \underset{k}{\tilde{x}}.$$

The set T with above operation of summation and multiplication on a number is a linear space called *tensor product of linear spaces* V and $\widetilde{V}$ and is denoted as

$$T = V \otimes \widetilde{V}.$$

Elements of the space T are called *tensors* of valence 2 over spaces V and $\widetilde{V}$. The dimension of tensor product is

$$\dim (V \otimes \widetilde{V}) = \dim V \dim \widetilde{V}.$$

Let (e_i) and $(\tilde{e}_i)$ be a bases in spaces V and $\widetilde{V}$ respectively. Then a set of dyads $(e_i \otimes \tilde{e}_j)$ forms a *basis* in he space $T = V \otimes \widetilde{V}$.

Consider $T = T^1{}_1 = V^n \otimes V_n$, where V^n and V_n are dual linear spaces. Then an expansion of a tensor $t \in T^1{}_1$ over the basis $(e_i \otimes \tilde{e}_j)$ can be written as

$$t = t^i{}_k \vec{e}_i \otimes \underset{\rightarrow}{e}^k,$$

where numbers $t^i{}_k$ are called *components* of tensor t. Similarly

$$t = t_i{}^k \underset{\rightarrow}{e}^i \otimes \vec{e}_k, \quad \text{if} \quad t \in V_n \otimes V^n;$$

$$t = t^{ik} \vec{e}_i \otimes \vec{e}_k, \quad \text{if} \quad t \in V^n \otimes V^n;$$

$$t = t_{ik}\, \underset{\rightarrow}{e}^i \otimes \underset{\rightarrow}{e}^k, \quad \text{if} \quad t \in V_n \otimes V_n.$$

Let $\overset{1}{V}$, $\overset{2}{V}$, $\overset{3}{V}$ be arbitrary linear spaces. If we add to properties 1)–3) the following one

4) $\forall \overset{(j)}{x} \in \overset{j}{V} : (\overset{(1)}{x} \otimes \overset{(2)}{x}) \otimes \overset{(3)}{x} = \overset{(1)}{x} \otimes (\overset{(2)}{x} \otimes \overset{(2)}{x})$,

then

$$(\overset{1}{V} \otimes \overset{2}{V}) \otimes \overset{3}{V} = \overset{1}{V} \otimes (\overset{2}{V} \otimes \overset{3}{V}) = \overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V}.$$

A *tensor product* of spaces $\overset{1}{V} \otimes \overset{2}{V} \otimes \ldots \otimes \overset{r}{V}$ is called the space $T = (\overset{1}{V} \otimes \overset{2}{V} \otimes \ldots \otimes \overset{r-1}{V}) \otimes \overset{r}{V}$ with elements

$$t = \underset{1}{\overset{(1)}{x}} \otimes \ldots \otimes \underset{1}{\overset{(r)}{x}} + \ldots + \underset{k}{\overset{(1)}{x}} \otimes \ldots \otimes \underset{k}{\overset{(r)}{x}}, \quad \underset{m}{\overset{(j)}{x}} \in \overset{j}{V},\ j = \overline{1, r},\ m = \overline{1, k}.$$

If elements $\overset{(j)}{e}_i$ form a basis in spaces $\overset{j}{V}$ respectively, then a set of polyads

$$\overset{1}{e}_{i_1} \otimes \ldots \otimes \overset{r}{e}_{i_r}$$

form a basis in the space $\overset{1}{V} \otimes \overset{2}{V} \otimes \ldots \otimes \overset{r}{V}$.

Let V^n and V_n be a dual linear spaces of dimension n. An element of the linear space

$$T^p_q = \underbrace{(V^n \otimes \ldots \otimes V^n)}_{p \text{ times}} \otimes \underbrace{(V_n \otimes \ldots \otimes V_n)}_{q \text{ times}}.$$

is called *tensor of type* (p, q) and numbers $t^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ in expansion

$$t = t^{i_1 \ldots i_p}_{j_1 \ldots j_q} \vec{e}_{i_1} \otimes \ldots \otimes \vec{e}_{i_p} \otimes \underset{\to}{e}^{j_1} \otimes \ldots \otimes \underset{\to}{e}^{j_q}.$$

are called elements of the tensor t in given basis.

Algebraic operations

1. *Summation.* The sum of two one-type tensors $a \in T^p_q$ and $b \in T^p_q$ is the tensor

$$c = a + b \in T^p_q$$

of the same type and valence.

2. *Multiplication.* Multiplication of tensors $a \in T^p_q$ and $b \in T^r_s$ is the tensor

$$c = a \otimes b \in T^{p+r}_{q+s}.$$

3. *Contraction.* Let us illustrate by examples:

$$a = \vec{x} \otimes \vec{y} \otimes \underset{\to}{f}; \quad a^{ik}_{k} \ : \ \langle \vec{y}, \underset{\to}{f} \rangle \vec{x}.$$

$$b = \vec{x} \otimes \underset{\to}{g} \otimes \underset{\to}{f} \otimes \vec{y}; \quad b^{i}{}_{jk}{}^{k} \ : \ \langle \vec{y}, \underset{\to}{f} \rangle \vec{x} \otimes \underset{\to}{g},$$

$$b^{i}{}_{ik}{}^{l} \ : \ \langle \vec{x}, \underset{\to}{g} \rangle \underset{\to}{f} \otimes \vec{y},$$

$$b^{i}{}_{ik}{}^{k} : \langle \vec{x}, \underset{\to}{g} \rangle \left\langle \vec{y}, \underset{\to}{f} \right\rangle.$$

Example 1. Find components of a tensor $c = \vec{a} \otimes \vec{b}$ in a basis $(\vec{e}_1, \vec{e}_2)$, if $\vec{a} = \vec{e}_1 + \vec{e}_2$ and $\vec{b} = \vec{e}_1 - \vec{e}_2$.

Solution. Taking into account properties 1)–3), we can write:

$$c = \vec{a} \otimes \vec{b} = (\vec{e}_1 + \vec{e}_2) \otimes (\vec{e}_1 - \vec{e}_2) =$$

$$= \vec{e}_1 \otimes \vec{e}_1 - \vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1 - \vec{e}_2 \otimes \vec{e}_2 = c^{ik} \vec{e}_i \otimes \vec{e}_k.$$

Thus

$$(c^{ik}) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Exercises

1.102. Specify which of the below expressions make sense, if $\vec{a}_1$, $\vec{a}_2, \vec{a}_3$ are vectors and $\underset{\to}{b}^1, \underset{\to}{b}^2, \underset{\to}{b}^3$ are covectors:

a) $\vec{a}_1 \otimes \vec{a}_2 + \vec{a}_2 \otimes \vec{a}_3$;

b) $\vec{a}_1 \otimes \vec{a}_2 \otimes \vec{a}_3 + \vec{a}_2 \otimes \vec{a}_3$;

c) $\vec{a}_1 \otimes \underset{\to}{b}^1 - 2\underset{\to}{b}^1 \otimes \vec{a}_1$;

d) $\vec{a}_1 \otimes \underset{\to}{b}^2 + \underset{\to}{b}^1 \otimes \underset{\to}{b}^2$;

e) $\vec{a}_1 \otimes \underset{\to}{b}^2 + \vec{a}_2 \otimes \underset{\to}{b}^1$;.

f) $\underset{\to}{b}^1 \otimes \vec{a}_1 \otimes \vec{a}_2 + \vec{a}_2 \otimes \vec{a}_3 \otimes \underset{\to}{b}^2$;

g) $\vec{a}_1 \otimes \vec{a}_2 + \vec{a}_3 \otimes \vec{a}_3 - \vec{a}_1 \otimes \vec{a}_1$;

h) $\underset{\to}{b}^1 \otimes \underset{\to}{b}^2 - 3\underset{\to}{b}^2 \otimes \underset{\to}{b}^3$.

1.103. Find components of below tensors in a basis $(\vec{e}_1, \vec{e}_2)$, if $\vec{a}_1 = \vec{e}_2$, $\vec{a}_2 = \vec{e}_1 + \vec{e}_2$, $\vec{a}_3 = 2\vec{e}_1 + 4\vec{e}_2$ and $\underset{\to}{b}^1 = -\underset{\to}{e}^1 + \underset{\to}{e}^2$, $\underset{\to}{b}^2 = -4\underset{\to}{e}^1 + 6\underset{\to}{e}^2$, $\underset{\to}{b}^3 = 14\underset{\to}{e}^1 - 9\underset{\to}{e}^2$:

a) $\vec{a}_1 \otimes \vec{a}_2 + \vec{a}_2 \otimes \vec{a}_3$;

b) $\vec{a}_1 \otimes \underset{\to}{b}^2 + \vec{a}_2 \otimes \underset{\to}{b}^1$;

c) $\underset{\to}{b}^1 \otimes \vec{a}_2 + 2\underset{\to}{b}^3 \otimes \vec{a}_1$;

d) $\underset{\to}{b}^1 \otimes \underset{\to}{b}^2 + 3\underset{\to}{b}^2 \otimes \underset{\to}{b}^3$.

1.104. Let a vector $\vec{x} \in V^3$ and a covector $\underset{\to}{y} \in V_3$ have components $\vec{x} = \vec{e}_1$, $\underset{\to}{y} = \underset{\to}{e}^2$ in an "old" basis and let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -2 \\ 0 & -2 & 5 \end{pmatrix}$ be a transformation matrix to a "new" basis. Find components of tensor $\vec{x} \otimes \underset{\to}{y}$ in "old" and "new" bases.

1.105. Write a tensor $a^{ik} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ in dyad form.

1.106. Write a tensor $a^{ij}_k = \left(\begin{array}{cc|cc} 1 & -1 & 2 & -2 \\ 1 & -1 & 2 & -2 \end{array}\right)$ in triad form.

1.6. The metric tensor

A dot product $\vec{a} \cdot \vec{b}$ of vectors $\vec{a}$ and $\vec{b}$ in a linear space V^n is called the abstract operation that satisfies conditions:

1) *invariance* — $\vec{a} \cdot \vec{b} \in \mathbb{R}$;

2) *symmetry* — $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$;

3) *bilinearity* — $(\mu\vec{a} + \gamma\vec{c}) \cdot \vec{b} = \mu(\vec{a} \cdot \vec{b}) + \gamma(\vec{c} \cdot \vec{b})$ and $\vec{a} \cdot (\mu\vec{b} + \gamma\vec{d}) = \mu(\vec{a} \cdot \vec{b}) + \gamma(\vec{a} \cdot \vec{d})$;

4) *homogeneity* — if $\vec{a} \cdot \vec{b} = 0$ for all $\vec{a}$, then $\vec{b} = \vec{0}$; and if $\vec{a} \cdot \vec{b} = 0$ for all $\vec{b}$, then $\vec{a} = \vec{0}$.

This definition is similar to the definition of a bundle with the essential difference that the scalar product is defined for two elements of one linear space, while the bundle is specified for elements belonging to dual spaces.

Let $g : V^n \to V_n$ be a non-degenerate single-valued linear mapping of an element $\vec{x} \in V^n$ to an element $\underset{\to}{x} \in V_n$: $x_i = g_{ik}x^k,\ \det(g_{ik}) \neq 0.$

Define the dot product $\vec{a} \cdot \vec{b}$ as the bundle of one of vectors $\vec{a}$ or $\vec{b}$ with a corresponding vector $\underset{\to}{b}$ or $\underset{\to}{a}$:

$$\vec{a} \cdot \vec{b} = \langle \vec{a}, \underset{\to}{b} \rangle = \langle \underset{\to}{a}, \vec{b} \rangle = g_{ik}a^i b^k.$$

Symmetry property of the dot product yields $g_{ki} = g_{ik}$.

Similarly, if g^{-1} is an inverse linear mapping $g^{-1} : V_n \to V^n$, then $x^i = g^{ik}x_k$ and

$$\underset{\to}{a} \cdot \underset{\to}{b} = g^{ik}a_i b_k.$$

The tensor g_{ik} is also symmetrical and

$$g_{ik}g^{kj} = \delta_i^j.$$

The tensors g_{ik} and g^{ik} are called *fundamental metric tensors* of the space V^n.

Lowering of index operation:

$$x_i = g_{ik}x^k.$$

Raising of index operation:

$$x^i = g^{ik}x_k.$$

The operation of lowering and raising of indices can be applied to tensors of an arbitrary valence:

$$P_{ikl} = g_{im}P^m{}_{kl} = g_{km}P_i{}^m{}_l, \quad Q^{ik} = g^{im}Q_m{}^k = g^{im}g^{kn}Q_{mn}.$$

Norm of a vector x^i:

$$\|x^i\|^2 = g_{ik}x^i x^k.$$

If the quadratic form $g_{ik}x^i x^k$ is positive (negative) definite then the space is called *Euclidean* space and denoted as E^n. A space with the indefinite quadratic form $g_{ik}x^i x^k$ is called *Pseudo–Euclidean* space.

If the dot product is defined independently then the components of the metric tensor g_{ik} in the basis $(\vec{e}_i)$ are defined as

$$g_{ik} = \vec{e}_i \cdot \vec{e}_k, \quad g^{ik} = \underset{\to}{e}^i \cdot \underset{\to}{e}^k.$$

The basis in which $g_{ik} = 0$ at $i \neq k$ is called *orthogonal*; if in addition

$$g_{ik} = \begin{cases} \pm 1, & i = k, \\ 0, & i \neq k \end{cases}$$

the basis is called *orthonormal*.

In a orthonormal basis of Pseudo–Euclidean Minkowski space the metric tensor is:

$$(g_{ik}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Vectors $\vec{x}$ having $\|\vec{x}\|^2 < 0$ are called *time-like*; vectors $\vec{x}$ having $\|\vec{x}\|^2 > 0$ are called *space-like* and finally vectors $\vec{x}$ that norm $\|\vec{x}\|^2 = 0$ for $\vec{x} \neq \vec{0}$ are called *isotropic*.

Changes of basis that conserve the form $g_{ik} = \begin{cases} \pm 1, & i = k, \\ 0, & i \neq k \end{cases}$ of the metric tensor are called *orthogonal transformations*. Orthogonal continuous transformations of a basis are rotations in euclidean spaces and they are called Lorentz transformation in Minkowsky space.

Example 1. A fundamental tensor of a space V^2 has in some basis the following components:

$$(g_{ik}) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Determine whether the space V^2 is Euclidean or Pseudo–Euclidean. Find dot product of vectors $(x^i) = (-1, 3)$ and $(y_i) = (2, -5)$, their norm, components of a covariant bivector $w_{ik} = x_{[i}y_{k]}$ and mixed tensor $v^i_k = x^i y_k + y^i x_k$.

Solution. The components of metric tensor g_{ik} are the coefficients of quadratic form $g_{ik}x^i x^k$. Therefore according to Sylvester's criterion the quadratic form $g_{ik}x^i x^k$ is positive-definite and thus the space V^2 is Euclidean.

The contravariant metric space components g^{ik} can be found by any of methods of matrix inversion known from linear algebra:

$$(g^{ik}) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then lowering the index of x^i and raising of the index of y_i yields:

$$
\begin{aligned}
&x_i = g_{ik}x^k :\\
&x_1 = g_{1k}x^k = g_{11}x^1 + g_{12}x^2 = 1\cdot(-1) + 1\cdot 3 = 2;\\
&x_2 = g_{2k}x^k = g_{21}x^1 + g_{22}x^2 = 1\cdot(-1) + 2\cdot 3 = 5.\\
&y^i = g^{ik}y_k :\\
&y^1 = g^{1k}y_k = g^{11}y_1 + g^{12}y_2 = 2\cdot 2 + (-1)\cdot(-5) = 9;\\
&y^2 = g^{2k}y_k = g^{21}y_1 + g^{22}y_2 = (-1)\cdot 2 + 1\cdot(-5) = -7.
\end{aligned}
$$

Dot product and norms are

$$\vec{x}\cdot\vec{y} = g_{ik}x^i y^k = g_{11}x^1y^1 + g_{12}x^1y^2 + g_{21}x^2y^1 + g_{22}x^2y^2 = -17,$$

$$\|\vec{x}\|^2 = g_{ik}x^i x^k = 13, \quad \|\vec{y}\|^2 = g_{ik}y^i y^k = 53.$$

To find the bivector $w_{ik} = x_{[i}y_{k]} = \frac{1}{2}(x_i y_k - x_k y_i)$ it is sufficient to calculate the only component w_{12}:

$$w_{12} = \frac{1}{2}(x_1y_2 - x_2y_1) = \frac{1}{2}(2\cdot(-5) - 5\cdot 2) = -10.$$

Then

$$(w_{ik}) = \begin{pmatrix} 0 & -10 \\ 10 & 0 \end{pmatrix}.$$

And components of the tensor $v^i_k = x^i y_k + y^i x_k$ are

$$(v^i_k) = \begin{pmatrix} 16 & 50 \\ -8 & -50 \end{pmatrix}.$$

Example 2. Let $(\vec{e}_1, \vec{e}_2)$ be a basis on Euclidean plane E^2, $|\vec{e}_1| = |\vec{e}_2| = 1$, and the angle between basis vectors be α $(0 < \alpha < \frac{\pi}{2})$. Construct the reciprocal basis $(\underset{\to}{e}^1, \underset{\to}{e}^2)$. Illustrate geometrical sense of contravariant and covariant components of a vector $\vec{x}$.

Solution. The bases $(\vec{e}_1, \vec{e}_2)$ and $(\underset{\to}{e}^1, \underset{\to}{e}^2)$ are reciprocal if

$$\vec{e}_1\cdot\underset{\to}{e}^1 = 1, \quad \vec{e}_1\cdot\underset{\to}{e}^2 = 0, \quad \vec{e}_2\cdot\underset{\to}{e}^1 = 0, \quad \vec{e}_2\cdot\underset{\to}{e}^2 = 1.$$

Therefore $\vec{e}_1 \perp \underset{\to}{e}^2$ and $\vec{e}_2 \perp \underset{\to}{e}^1$, and their norms are

$$|\underset{\to}{e}^1| = \frac{1}{\sin\alpha}, \quad |\underset{\to}{e}^2| = \frac{1}{\sin\alpha}, \quad \underset{\to}{e}^1\cdot\underset{\to}{e}^2 = -\frac{\cos\alpha}{\sin\alpha}.$$

To find coordinates of an arbitrary vector $\vec{x}$ geometrically in the unit basis $(\vec{e}_1, \vec{e}_2)$ we are to place the basis in the beginning of the vector, then to draw through the end of the vector $\vec{x}$ two lines that are *parallel* to the basis vectors and finally to find the length of the corresponding segments cut off by above lines on the extension of the basis vectors. In this way the coordinate x^1 of the vector $\vec{x}$ in the basis $(\vec{e}_1, \vec{e}_2)$ is length of the segment OA, and the coordinate x^2 is length of the segment OD as it is shown on fig. 1.2. In the non-unity basis $(\underset{\rightarrow}{e}^1, \underset{\rightarrow}{e}^2)$ the coordinate x_1 of the vector $\vec{x}$ of is the length of the segment OC divided by norm of the vector $\underset{\rightarrow}{e}^1$:

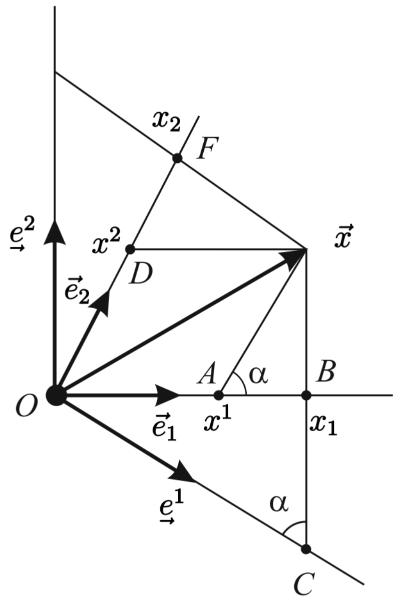

Fig. 1.2

$$x_1 = \frac{OC}{|\underset{\rightarrow}{e}^1|} = OC \sin\alpha = OB.$$

Similarly we find that the coordinate x_2 is length of the segment OF.

Thus, covariant coordinates of a vector in an unity basis are the segments cut off by lines that are *perpendicular* to the basis vectors. If the angle α between the basis vectors tends to $\pi/2$, then x^i tends to x_i and the difference between covariant and contravariant coordinates vanishes.

From geometrical approach it is easy to calculate that

$$x_1 = OA + AB = x^1 + x^2 \cos\alpha;$$

$$x_2 = OD + DF = x^1 \cos\alpha + x^2.$$

To find coordinates x_i algebraically we are preliminary to find covariant metric tensor components:

$$(g_{ik}) = \begin{pmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{pmatrix} = \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}.$$

Then lowering the index of the vector x^i we obtain the same result:

$$x_1 = g_{1k} x^k = x^1 + x^2 \cos\alpha;$$

$$x_2 = g_{2k} x^k = x^1 \cos\alpha + x^2.$$

Example 3. Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be an arbitrary basis in E^3. Find the reciprocal basis $(\underset{\to}{e}^1, \underset{\to}{e}^2, \underset{\to}{e}^3)$.

Solution. The condition for bases in E^3 to be reciprocal is

$$\vec{e}_1 \cdot \underset{\to}{e}^1 = 1, \quad \vec{e}_2 \cdot \underset{\to}{e}^1 = 0, \quad \vec{e}_3 \cdot \underset{\to}{e}^1 = 0 \quad \text{and so forth.}$$

Then it is clear that $\underset{\to}{e}^1 \perp \vec{e}_2$ and $\underset{\to}{e}^1 \perp \vec{e}_3$, therefore the vector $\underset{\to}{e}^1$ is collinear to the vector $\vec{e}_2 \times \vec{e}_3$:

$$\underset{\to}{e}^1 = k(\vec{e}_2 \times \vec{e}_3).$$

The coefficient k can be found from the condition $\vec{e}_1 \cdot \underset{\to}{e}^1 = 1$:

$$\vec{e}_1 \cdot k(\vec{e}_2 \times \vec{e}_3) = 1 \quad \Rightarrow \quad k = \frac{1}{\vec{e}_1\vec{e}_2\vec{e}_3}.$$

The other vectors can be found by the same way. Finally the reciprocal basis is

$$\underset{\to}{e}^1 = \frac{\vec{e}_2 \times \vec{e}_3}{\vec{e}_1\vec{e}_2\vec{e}_3}, \quad \underset{\to}{e}^2 = \frac{\vec{e}_3 \times \vec{e}_1}{\vec{e}_1\vec{e}_2\vec{e}_3}, \quad \underset{\to}{e}^3 = \frac{\vec{e}_1 \times \vec{e}_2}{\vec{e}_1\vec{e}_2\vec{e}_3}.$$

Exercises

Let g_{ik} be a fundamental metric tensor. Determine whether the space V^2 is Euclidean or Pseudo–Euclidean. Find dot product of vectors x^i and y_i, their norm, components of a covariant bivector $w_{ik} = x_{[i}y_{k]}$ and mixed tensor $v^i_k = x^i y_k + y^i x_k$.

1.107. $(g_{ik}) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad (x^i) = (2, 1), \quad (y_i) = (-2, 3).$

1.108. $(g_{ik}) = \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix}, \quad (x^i) = (2, 1), \quad (y_i) = (1, -3).$

1.109. $(g_{ik}) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \ (x^i) = (1, -1, 1), \ (y_i) = (-1, 0, 2).$

1.110. $(g_{ik}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ (x^i) = (1, -1, 1, 1), \ (y_i) = (1, 1, 0, 0).$

Find components ${a^i}_j$, $a_i^{\ j}$, a^{ij} of the following tensors:

1.111. $(a_{ij}) = \begin{pmatrix} 9 & -5 \\ -5 & 3 \end{pmatrix}$, $\quad (g_{ij}) = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$.

1.112. $(a_{ij}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\quad (g_{ij}) = \begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix}$.

1.113. $(a_{ij}) = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{pmatrix}$, $\quad (g_{ij}) = \begin{pmatrix} 21 & -10 & -4 \\ -10 & 5 & 2 \\ -4 & 2 & 1 \end{pmatrix}$.

Find components a_{ijkl} and a^{ijkl} of the following tensors:

1.114. $(a^{ij}_{kl}) = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 2 \end{array}\right)$, $\quad (g_{ij}) = \begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix}$.

1.115. $(a^{ij}_{kl}) = \left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ \hline -4 & -3 & -2 & -1 \\ 5 & 6 & 7 & 8 \end{array}\right)$, $\quad (g_{ij}) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

1.116. Find components $g_i^{\ k}$ and ${g^k}_i$ of the metric tensor.

1.117. Show that the metric tensor $(g_{ik}) = \operatorname{diag}(1,1)$ of Euclidean plane is invariant to transformation

$$\begin{aligned} \vec{e}_{1'} &= \vec{e}_1 \cos\varphi + \vec{e}_2 \sin\varphi, \\ \vec{e}_{2'} &= -\vec{e}_1 \sin\varphi + \vec{e}_2 \cos\varphi. \end{aligned}$$

1.118. Show that the metric tensor $(g_{ik}) = \operatorname{diag}(-1,1)$ of Pseudo–Euclidean plane is invariant to transformation

$$\begin{aligned} \vec{e}_{1'} &= \vec{e}_1 \cosh\varphi + \vec{e}_2 \sinh\varphi, \\ \vec{e}_{2'} &= \vec{e}_1 \sinh\varphi + \vec{e}_2 \cosh\varphi. \end{aligned}$$

Let $(\vec{i}, \vec{j}, \vec{k})$ be an orthonormal basis in E^3. Find the reciprocal basis to the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.

1.119. $\vec{e}_1 = \vec{i} - 2\vec{j}, \quad \vec{e}_2 = -\vec{i} + \vec{j}$.

1.120. $\vec{e}_1 = -\vec{i} + \vec{j}, \quad \vec{e}_2 = 2\vec{i} + 3\vec{j}$.

1.121. $\vec{e}_1 = 3\vec{i} - 4\vec{j}, \quad \vec{e}_2 = -\vec{i} + 2\vec{j} + 2\vec{k}, \quad \vec{e}_3 = \vec{i} + \vec{j} + \vec{k}.$

1.122. $\vec{e}_1 = \vec{i} - 2\vec{j} - \vec{k}, \quad \vec{e}_2 = 4\vec{i} + 2\vec{j} - \vec{k}, \quad \vec{e}_3 = -3\vec{i} + \vec{j} + \vec{k}.$

1.7. Tensors in Euclidean space E^3

In Euclidean spaces covariant and contravariant components are equal in orthonormal bases. In this case there is no difference between upper and lower indices and all indices are considered to be lower for convenience. Therefore the summation convention assumes summation over two identical indices.

Levi-Civita's symbol if a completely antisymmetrical object ε_{ijk} that

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if any tow indices are equal;} \\ +1, & \text{if } ijk \text{ is an even permutation of } 1,2,3; \\ -1, & \text{if } ijk \text{ is an odd permutation of } 1,2,3. \end{cases}$$

Properties of Levi-Civita's symbol:

1) $\varepsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix};$

2) $\varepsilon_{ijk} \cdot \varepsilon_{mnl} = \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{il} \\ \delta_{jm} & \delta_{jn} & \delta_{jl} \\ \delta_{km} & \delta_{kn} & \delta_{kl} \end{vmatrix};$

3) $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl};$

4) $\varepsilon_{ijk}\varepsilon_{jkl} = 2\delta_{il};$

5) $\varepsilon_{ijk}\varepsilon_{ijk} = 6.$

Vector algebraic operation can be written in an orthonormal basis $\vec{e}_i$ as follows:

1) basis decomposition:

$$\vec{a} = a_i\vec{e}_i;$$

2) dot product:

$$\vec{a} \cdot \vec{b} = a_k b_k;$$

3) cross product:

$$\vec{a} \times \vec{b} = \vec{e}_i \varepsilon_{ijk} a_j b_k;$$

4) scalar triple product:

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \varepsilon_{ijk} a_i b_j c_k.$$

Example 1. Write down components of the vector $a_i = \varepsilon_{ijk} b_{jk}$.

Solution. The first component of the vector a_i is equal to $a_1 = \varepsilon_{1jk} b_{jk}$. Since Levi-Civita's symbol equals zero if any two indices are identical there are only two nontrivial summands:

$$a_1 = \varepsilon_{123} b_{23} + \varepsilon_{132} b_{32} = b_{23} - b_{32}.$$

Similarly we obtain

$$a_2 = \varepsilon_{213} b_{13} + \varepsilon_{231} b_{31} = b_{31} - b_{13},$$

$$a_3 = \varepsilon_{312} b_{12} + \varepsilon_{321} b_{21} = b_{12} - b_{21}.$$

Example 2. Find an antisymmetric tensor b_{jk} from the equation

$$b_i = \frac{1}{2} \varepsilon_{ijk} b_{jk}.$$

Solution. First we transvect the vector b_i with ε_{pqi} and apply property 3) of Levi-Civita's symbol:

$$\varepsilon_{pqi} b_i = \frac{1}{2} \varepsilon_{pqi} \varepsilon_{ijk} b_{jk} = \frac{1}{2} (\delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj}) b_{jk} = \frac{b_{pq} - b_{qp}}{2}.$$

Since the tensor b_{pq} is antisymmetric, i. e. $b_{pq} = -b_{qp}$, then $b_{pq} = \varepsilon_{pqi} b_i$. The antisymmetric tensor $b_{pq} = \varepsilon_{pqi} b_i$ is called *dual* tensor to the vector b_i.

Example 3. Expand the double cross product $\vec{a} \times (\vec{b} \times \vec{c})$.

Solution. Write down the product $\vec{a} \times (\vec{b} \times c)$ in orthonormal basis and use property 3) of Levi-Civita's symbol as follows:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{e}_i \varepsilon_{ijk} a_j (\vec{b} \times \vec{c})_k = \vec{e}_i \varepsilon_{ijk} a_j \varepsilon_{klm} b_l c_m = \vec{e}_i \varepsilon_{ijk} \varepsilon_{klm} a_j b_l c_m =$$

$$= \vec{e}_i (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_j b_l c_m = \vec{e}_i a_j b_i c_j - \vec{e}_i a_j b_l c_i =$$

$$= (\vec{e}_i b_i)(a_j c_j) - (\vec{e}_i c_i)(a_j b_j) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}).$$

Example 4. Show that the determinant of a matrix 3×3 can be written as

$$\det(a_{ij}) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}.$$

Solution. It is known that

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \varepsilon_{ijk}a_i b_j c_k.$$

If we assume $a_i = a_{1i},\ b_i = a_{2i},\ c_i = a_{3i}$, we obtain required expression.

Example 5. Write down in a brief form the solution of an inhomogeneous system of linear equations $a_{ik}x_k = b_i$, $i, k = \overline{1,3}$, if $\det(a_{ik}) \neq 0$.

Solution. According to Cramer's rule and Example 4 the solution can be written as

$$x_1 = \frac{1}{\det(a_{ik})}\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = \frac{\varepsilon_{ijk}b_i a_{2j}a_{3k}}{\varepsilon_{pqr}a_{1p}a_{2q}a_{3r}},$$

$$x_2 = \frac{\varepsilon_{ijk}a_{1i}b_j a_{3k}}{\varepsilon_{pqr}a_{1p}a_{2q}a_{3r}}, \quad \text{and} \quad x_3 = \frac{\varepsilon_{ijk}a_{1i}a_{2j}b_k}{\varepsilon_{pqr}a_{1p}a_{2q}a_{3r}}.$$

Exercises

1.123. Write down components of a tensor $b_{pq} = \varepsilon_{pqi}b_i$.

1.124. Let $a = \det(a_{ik})$. Show that $a\varepsilon_{ijk} = \varepsilon_{mnp}a_{im}a_{jn}a_{kp}$ and $a = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{mnp}a_{im}a_{jn}a_{kp}$.

1.125. Let $a = \det(a_{ik})$. Show that the algebraic adjunct of an element a_{ik} can be written as $A_{rs} = \frac{\partial a}{\partial a_{rs}} = \frac{1}{2}\varepsilon_{rjk}\varepsilon_{snp}a_{jn}a_{kp}$, and then $a = \frac{1}{3}A_{rs}a_{sr}$.

1.126. Show that in an orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ Levi-Civita's symbol is equal to

$$\varepsilon_{ijk} = (\vec{e}_1 \times \vec{e}_3)\cdot\vec{e}_3.$$

1.127. Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ an arbitrary basis in a space E^3 and $g = \det(g_{ik})$. Prove that

a) $\frac{1}{g} = \det(g^{ik})$, b) $g = |(\vec{e}_1 \times \vec{e}_3)\cdot\vec{e}_3|$.

1.128. Prove that $g = \det(g_{ik})$ is not a scalar.

1.129. Expand the double cross product $(\vec{a}\times\vec{b})\times\vec{c}$.

1.130. Prove Lagrange's identity: $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$. Find $(\vec{a}\times\vec{b})^2$.

Prove the following identities:

1.131. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{b}(\vec{a}\vec{c}\vec{d}) - \vec{a}(\vec{b}\vec{c}\vec{d}) = \vec{c}(\vec{a}\vec{b}\vec{d}) - \vec{d}(\vec{a}\vec{b}\vec{c})$.

1.132. $\big((\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\big)\cdot(\vec{c}\times\vec{a}) = \big(\vec{a}\vec{b}\vec{c}\big)^2$.

1.133. $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$.

1.134. $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) + (\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) = 0$.

1.135. Write the given expression in a vector form:

a) $\varepsilon_{inl}\varepsilon_{irs}\varepsilon_{lmp}\varepsilon_{stp}a_n a_r b_m c_t$; b) $\varepsilon_{inl}\varepsilon_{krs}\varepsilon_{lmp}\varepsilon_{stp}a_r a'_n a_r b_k b'_i c_t c'_m$.

1.136. Let a cartesian coordinate system (x_1, x_2, x_3) be firstly rotated by the angle $\pi/6$ around the x^3 axis, and then be rotated by the angle $\pi/2$ around $x^{1'}$ axis in such a way that the axis $x^{2'}$ coincides with the axis x^3. Find the transformation matrix.

1.137. Let $(\vec{e}_i)$ and $(\vec{e}_{i'})$ be orthonormal bases. Euler angles are introduced as follows (fig. 1.3):

a) an angle θ is the angle between vectors $\vec{e}_3$ and $\vec{e}_{3'}$;

b) an angle φ is the angle between vectors $\vec{e}_1$ and $\vec{u}$, where the vector $\vec{u}$ is the unit vector along the line of intersection of planes containing vectors $(\vec{e}_1, \vec{e}_2)$ and $(\vec{e}_{1'}, \vec{e}_{2'})$. Three vectors $\vec{u}$, $\vec{e}_3$ and $\vec{e}_{3'}$ have to form a right-hand triple;

c) an angle ψ is the angle between vectors $\vec{u}$ and $\vec{e}_{1'}$.

Find the transformation matrix from the basis $(\vec{e}_i)$ to $(\vec{e}_{i'})$.

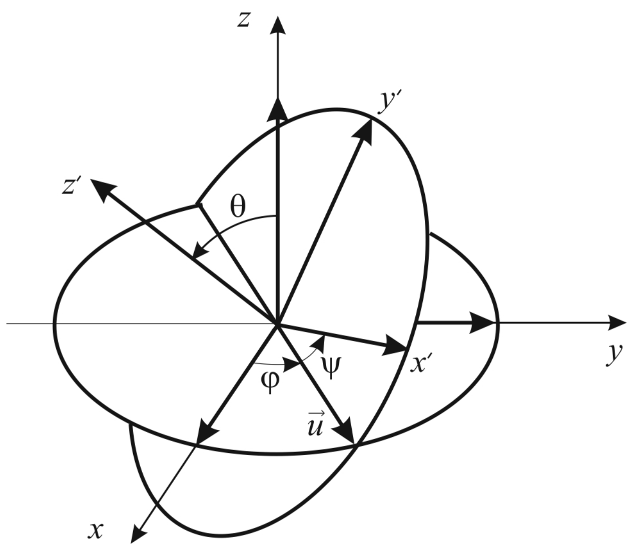

Fig. 1.3

Chapter 2

Differential geometry

2.1. Vector function of a real variable

Let T be a connected set of $\mathbb{R}$ (segment, interval, etc.) A *vector function* of a real variable is the mapping of each number $t \in T$ onto a vector $\vec{r}(t) \in E^3$.

Hodograph of vector function $\vec{r}(t)$ is the locus of the end of $\vec{r}(t)$ for different values of parameter t, the beginning of $\vec{r}(t)$ being constant.

The vector $\vec{r}_0$ is called *limit of vector function* $\vec{r}(t)$ at a point t_0 (as $t \to t_0$), if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \in T$, $0 < |t - t_0| < \delta$, implies $|\vec{r}(t) - \vec{r}_0| < \varepsilon$ and written

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}_0.$$

If vector functions $\vec{r}(t)$, $\vec{R}(t)$ and a scalar function $\lambda(t)$ have limits $\vec{r}_0$, $\vec{R}_0$ and λ_0 as $t \to t_0$ correspondingly there exist the following limits:

1) $\lim\limits_{t \to t_0}(\vec{r}(t) \pm \vec{R}(t)) = \vec{r}_0 \pm \vec{R}_0$;
2) $\lim\limits_{t \to t_0} \lambda(t)\vec{r}(t) = \lambda_0\vec{r}_0$;
3) $\lim\limits_{t \to t_0} \vec{r}(t) \cdot \vec{R}(t) = \vec{r}_0 \cdot \vec{R}_0$;
4) $\lim\limits_{t \to t_0} \vec{r}(t) \times \vec{R}(t) = \vec{r}_0 \times \vec{R}_0$.

A vector function $\vec{r}(t)$ is called *continuous at the point* $t_0 \in T$, if it is defined in neighborhood of this point and

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0).$$

A vector function $\vec{r}(t)$ is called *continuous on a set* T, if it is continuous in every point of T.

If vector functions $\vec{r}(t), \vec{R}(t)$ and a scalar function $\lambda(t)$ are continuous on a set T, then functions $\vec{r}(t) \pm \vec{R}(t)$, $\lambda(t)\vec{r}(t)$, $\vec{r}(t) \cdot \vec{R}(t)$ and $\vec{r}(t) \times \vec{R}(t)$ are continuous on the set T.

A vector function $\vec{r}(t)$ is called *differentiable at the point* $t_0 \in T$, if there exists the limit

$$\lim_{t\to t_0} \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0},$$

which is called the derivative of the vector function at this point and written as $\vec{r}\,'(t_0)$. A vector function $\vec{r}(t)$ is called *differentiable on a set* T, if it is differentiable at every point of T.

If vector functions $\vec{r}(t), \vec{R}(t)$ and a scalar function $\lambda(t)$ are differentiable on a set T, the following functions are differentiable and

1) $(\vec{r}(t) \pm \vec{R}(t))' = \vec{r}\,'(t) \pm \vec{R}'(t)$;

2) $(\lambda(t)\vec{r}(t))' = \lambda'(r)\vec{r}(t) + \lambda(r)\vec{r}\,'(t)$;

3) $(\vec{r}(t) \cdot \vec{R}(t))' = \vec{r}\,'(t) \cdot \vec{R}(t) + \vec{r}(t) \cdot \vec{R}'(t)$;

4) $(\vec{r}(t) \times \vec{R}(t))' = \vec{r}\,'(t) \times \vec{R}(t) + \vec{r}(t) \times \vec{R}'(t)$.

The *second derivative* of a vector function $\vec{r}$ is the first derivative of $\vec{r}\,'(t)$ and denoted as $\vec{r}\,''(t)$. Similarly the third derivative $\vec{r}\,'''(t) = (\vec{r}\,''(t))'$, etc.

The regularity class $C^n(T)$ is a set of all vector functions having n continuous derivatives on the set T.

In any basis $(\vec{i}, \vec{j}, \vec{k})$ in E^3 a vector function is defined by three functions $x(t), y(t), z(t)$ as

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k},$$

and if basis does not depend on t, the combination of properties 1) and 2) of derivative yields

$$\lim_{t\to t_0} \vec{r}(t) = \vec{i} \lim_{t\to t_0} x(t) + \vec{j} \lim_{t\to t_0} y(t) + \vec{k} \lim_{t\to t_0} z(t),$$

$$\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}.$$

If a vector function $\vec{r}(t)$ has n continuous derivatives in a neighborhood of the point t_0, the *Taylor's formula* with the remainder in Peano's form is true:

$$\vec{r}(t) = \vec{r}(t_0) + \vec{r}\,'(t_0)(t - t_0) + \frac{1}{2!}\vec{r}\,''(t_0)(t - t_0)^2 + \ldots +$$

$$+\frac{1}{n!}\vec{r}^{\,(n)}(t_0)(t - t_0)^n + \vec{o}\big((t - t_0)^n\big),$$

$$\vec{o}\left((t-t_0)^n\right) = o\left((t-t_0)^n\right)\vec{i} + o\left((t-t_0)^n\right)\vec{j} + o\left((t-t_0)^n\right)\vec{k}.$$

Riemann's integral of vector function is defined in the same way as for real function and possesses the same properties. Indefinite and definite integrals of vector function can be calculated coordinate-wise:

$$\int \vec{r}(t)dt = \vec{i}\int x(t)dt + \vec{j}\int y(t)dt + \vec{k}\int z(t)dt.$$

Example 1. Find the hodograph of the vector function

$$\vec{r}(t) = \frac{2t}{t^2+2}\vec{i} + \frac{2t}{t^2+2}\vec{j} + \frac{t^2-2}{t^2+2}\vec{k}.$$

Solution. The vector function $\vec{r}(t)$ has components:

$$x = \frac{2t}{t^2+2}, \quad y = \frac{2t}{t^2+2}, \quad z = \frac{t^2-2}{t^2+2}.$$

To eliminate parameter t from these equations we raise each equality to the second power and sum them. In the result we obtain the equation of the sphere

$$x^2 + y^2 + z^2 = 1.$$

Moreover, components x and y satisfy equation $x - y = 0$, that is the equation on the plane, passing through the origin of coordinates. Thus the hodograph is the circle that is intersection locus of above sphere and plane.

Example 2. Find the trajectory of a particle if its radius vector satisfies the differential equation

$$\vec{r}\,' = \vec{\omega} \times \vec{r},$$

where $\vec{\omega}$ ia a constant vector.

Solution. First, calculate dot product of both part of the differential equation and the vector $\vec{r}$:

$$\vec{r} \cdot \vec{r}\,' = \vec{r} \cdot (\vec{\omega} \times \vec{r}) = 0.$$

Therefore the vector $\vec{r}$ is orthogonal to the velocity of the particle $\vec{r}\,'$ at any time. Since

$$2\vec{r} \cdot \vec{r}\,' = (\vec{r} \cdot \vec{r})' = 0,$$

the norm $|\vec{r}(t)|$ is constant. This means that the trajectory belongs to a sphere. Then the end of the radius vector moves in the direction of the velocity $\vec{r}'$ vector, which is perpendicular to the constant vector $\vec{\omega}$ at any time and thus trajectory belongs to the plane with normal vector $\vec{\omega}$. Finally the trajectory is an intersection locus of described sphere and plane, that is the circle being perpendicular to the vector $\vec{\omega}$. The radius and position of this circle depends on initial conditions.

Exercises

Evaluate the limit of vector functions.

2.1. $\lim\limits_{t\to 0}\vec{r}(t)$, where $\vec{r}(t)=\dfrac{\sin t}{t}\vec{i}+\dfrac{\cos t-1}{2t}\vec{j}+e^{t^2}\vec{k}$.

2.2. $\lim\limits_{t\to \pi}\vec{r}(t)$, where $\vec{r}(t)=\dfrac{\sin t}{t-\pi}\vec{i}+\dfrac{t}{\pi}\vec{j}+\dfrac{1+\cos t}{t}\vec{k}$.

2.3. $\lim\limits_{t\to 1}\vec{r}(t)$, where $\vec{r}(t)=\dfrac{e^t-e}{t-1}\vec{i}+\dfrac{\ln t}{1-t}\vec{j}+(t-1)\sin\dfrac{1}{t-1}\vec{k}$.

2.4. $\lim\limits_{t\to \infty}\vec{r}(t)$, where $\vec{r}(t)=t^2\left(\cosh\dfrac{1}{t}-1\right)\vec{i}+\dfrac{t}{\ln(\cosh 2t)}\vec{j}+\dfrac{t^3-1}{t^3+2}\vec{k}$.

Find the derivative of vector functions.

2.5. $\vec{r}(t)=\vec{i}a\cos t+\vec{j}b\sin t+\vec{k}ct$.

2.6. $\vec{r}(t)=\vec{i}a\cosh t+\vec{j}b\sinh t+\vec{k}ct^2$.

2.7. $\vec{r}(t)=\vec{i}e^t\cos t+\vec{j}e^t\sin t+\vec{k}e^t$.

2.8. $\vec{r}(t)=\vec{i}(t-\sin t)+\vec{j}(1-\cos t)+\vec{k}4\sin\dfrac{t}{2}$.

Find the integral of vector functions.

2.9. $\int\vec{r}(t)dt$, where $\vec{r}(t)=\vec{i}te^t+\vec{j}\sin^2 t-\dfrac{\vec{k}}{1+t^2}$.

2.10. $\int\vec{r}(t)dt$, where $\vec{r}(t)=\vec{i}\dfrac{t}{1+t^2}+\vec{j}te^{t^2}+\vec{k}\cos t$.

2.11. $\int\limits_0^1\vec{r}(t)dt$, where $\vec{r}(t)=\vec{i}\dfrac{te^{-t/2}}{2}+\vec{j}\dfrac{e^{t/2}}{2}+\vec{k}e^t$.

2.12. $\int\limits_0^\pi\vec{r}(t)dt$, where $\vec{r}(t)=\vec{i}2t+\vec{j}\pi t\sin t+\vec{k}\pi$.

Find hodographs of vector functions.

2.13. $\vec{r}(t) = 2\vec{i} + t^2\vec{j} - t^2\vec{k}$.

2.14. $\vec{r}(t) = \dfrac{t^2+1}{(t+1)^2}\vec{i} + \dfrac{2t}{(t+1)^2}\vec{j}$.

2.15. $\vec{r}(t) = \vec{i}\cos t + \vec{j}\sin t + \vec{k}$.

2.16. $\vec{r}(t) = t\vec{i} + \dfrac{t^2}{3}\vec{j} + \dfrac{t^3}{9}\vec{k}$.

Find derivatives (here $\vec{r} = \vec{r}(t)$).

2.17. $\vec{r}^{\,2}$.

2.18. $\vec{r}^{\,\prime 2}$.

2.19. $\vec{r}^{\,\prime} \times \vec{r}^{\,\prime\prime}$.

2.20. $\vec{r}^{\,\prime}\vec{r}^{\,\prime\prime}\vec{r}^{\,\prime\prime\prime}$.

2.21. $(\vec{r}^{\,\prime} \times \vec{r}^{\,\prime\prime}) \times \vec{r}^{\,\prime\prime\prime}$.

2.22. $\sqrt{\vec{r}^{\,2}}$.

2.23. A vector function $\vec{r}\,(t)$ is a solution of the differential equation $\vec{r}^{\,\prime\prime} = \vec{r}^{\,\prime} \times \vec{a}$, where $\vec{a}$ is a constant vector. Express quantities a) $(\vec{r}^{\,\prime} \times \vec{r}^{\,\prime\prime})^2$; b) $\vec{r}^{\,\prime}\vec{r}^{\,\prime\prime}\vec{r}^{\,\prime\prime\prime}$ by means of vectors $\vec{a}$ and $\vec{r}^{\,\prime}$.

2.24. Show that the vector function $\vec{r}\,(t) = \vec{a}\cos\omega t + \vec{b}\sin\omega t$, where $\vec{a}$ and $\vec{b}$ are constant vectors, satisfies the equations: a) $\vec{r} \times \vec{r}^{\,\prime} = \omega(\vec{a} \times \vec{b})$; b) $\vec{r}^{\,\prime\prime} + \omega^2\vec{r} = \vec{0}$.

2.25. Show that the vector function $\vec{r}\,(t) = \vec{a}e^{\omega t} + \vec{b}e^{-\omega t}$, where $\vec{a}$ and $\vec{b}$ are constant vectors, satisfies the equation $\vec{r}^{\,\prime\prime} - \omega^2\vec{r} = \vec{0}$.

2.26. A trajectory of a particle in the cylindrical coordinates is given by

$$\vec{r}(t) = \rho(t)\vec{e}_\rho(\varphi(t)) + z(t)\vec{k},$$

where $\vec{e}_\rho(\varphi) = \vec{i}\cos\varphi + \vec{j}\sin\varphi$. Find the square of the instantaneous velocity $\vec{r}^{\,\prime 2}$.

2.27. A trajectory of a particle in the spherical coordinates is given by

$$\vec{r}(t) = r(t)\vec{e}_r(\theta(t), \varphi(t)),$$

where $\vec{e}_r(\theta, \varphi) = (\vec{i}\cos\varphi + \vec{j}\sin\varphi)\sin\theta + \vec{k}\cos\theta$. Find the square of the instantaneous velocity $\vec{r}^{\,\prime 2}$.

2.28. Let $(\vec{i}, \vec{j}, \vec{k})$ and $(\vec{e}_x(t), \vec{e}_y(t), \vec{e}_z(t))$ be orthonormal bases of stationary and moving reference frames respectively. Then consider the radius vector of a particle with respect to the stationary frame of reference as $\vec{r} = \vec{r}_0 + \vec{\rho}$, where the vector $\vec{r}_0$ describes the motion of the origin of the moving frame of reference and the $\vec{\rho}(t) = x(t)\vec{e}_x(t) + y(t)\vec{e}_y(t) + z(t)\vec{e}_z(t)$ is the radius vector of the particle with respect to the moving frame of reference. Using Poison's formulas

$$\frac{\mathrm{d}\vec{e}_i(t)}{\mathrm{d}t} = \vec{\omega}(t) \times \vec{e}_i(t), \quad |\vec{e}_i(t)| = 1, \quad i = x, y, z,$$

find the acceleration of the particle.

Find the trajectory of a particle, if the radius vector satisfies the following differential equation.

2.29. $\vec{r}\,' = \vec{e} \times (\vec{r} \times \vec{e})$, where $\vec{e}$ is a constant unit vector.

2.30. $\vec{r}\,' = a\vec{e} + \vec{e} \times \vec{r}$, where $a = \mathrm{const}$ and $\vec{e}$ is a constant unit vector.

2.31. $\vec{r}\,' = \dfrac{1}{2}\vec{r}\,^2\vec{e} - \vec{r}\,(\vec{r} \cdot \vec{e})$, where $\vec{e}$ is a constant unit vector.

2.32. $\vec{r}\,'' = \vec{r}\,' \times \vec{a}$, where $\vec{a}$ is a constant vector.

2.33. Is it possible to state that an arbitrary nonzero vector function $\vec{r}\,(t)$ satisfies the equalities: a) $|\vec{r}\,'(t)| = |\vec{r}\,(t)|'$; b) $\vec{r}\,(t) \cdot \vec{r}\,'(t) = |\vec{r}\,(t)||\vec{r}\,(t)|'$?

2.34. Prove that if $\vec{e}(t)$ is an unit vector function collinear to a vector function $\vec{a}(t)$ at any point t, then

$$\vec{e} \times \mathrm{d}\vec{e} = \frac{\vec{a} \times \mathrm{d}\vec{a}}{\vec{a}\,^2}.$$

2.35. Suppose $\vec{r}\,(t)$ and $l(t)$ are three times continuously differentiable functions on an interval (a, b) and $l(t)$ in addition is a monotonous function. Denote $\vec{r}\,' = \dfrac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ and $\dot{\vec{r}} = \dfrac{\mathrm{d}\vec{r}}{\mathrm{d}l}$. Change the variable t to l in the following expressions: $\vec{r}\,'$, $\vec{r}\,''$, $\vec{r}\,'''$, $\vec{r}\,'^2$, $\vec{r}\,' \times \vec{r}\,''$, $\vec{r}\,'\vec{r}\,''\vec{r}\,'''$.

2.36. Prove that continuity of a vector function is equivalent to continuity of its components.

2.37. Prove that continuity of a function $|\vec{r}(t)|$ follows from continuity of the vector function $\vec{r}(t)$. Is the inverse proposition correct?

2.38. Prove if for all $t \in [a, b]$ the norm of a vector function $\vec{r}(t)$ is constant, then $\forall t \in [a, b] : \ \vec{r} \perp \vec{r}\,'$. Is the inverse statement correct?

2.39. Prove if $|\vec{r}\,'(t)| = 0$ for all $t \in [a, b]$, then $\vec{r}(t) = \mathrm{const}$, $\forall t \in [a, b]$. Is the inverse statement correct?

2.40. Prove that for vectors $\vec{r}(t)$ and $\vec{r}\,'(t)$ to be collinear it is necessary and sufficient for the vector $\vec{r}(t)$ to have a constant direction.

2.41. Prove that a necessary condition for vectors $\vec{r}(t)$ to be parallel to a fixed plane for all $t \in (a, b)$ is that $(\vec{r} \times \vec{r}\,') \cdot \vec{r}\,'' = 0$ for all $t \in (a, b)$. Is this condition sufficient?

2.2. Curves

Let T be a connected set of $\mathbb{R}$ (segment, interval, etc). The equations

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad t \in T, \tag{2.1}$$

where $x(t), y(t), z(t)$ are continuous functions of the parameter t, define the locus $\mathcal{L} \in E^3$ that is called a *curve*. If

$$(x(t_1), y(t_1), z(t_1)) \neq (x(t_2), y(t_2), z(t_2))$$

for $t_1 \neq t_2$, then the curve $\mathcal{L}$ is called the *simple curve* and equations (2.1) are called *parametric equations* of the curve $\mathcal{L}$.

The equation

$$\vec{r} = \vec{r}(t) \quad \text{or} \quad \vec{r} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \quad t \in T,$$

is called the *vectorial parametric* equation of the curve $\mathcal{L}$.

For $T = [a, b]$ the points $\vec{r}(a)$ and $\vec{r}(b)$ are called the boundary points of the curve. If the boundary points are the same and all other points of the curve are different then the curve is called the *simple closed* curve. The single curve can be parameterized in various ways, if the parameter t is considered as a continuous strictly monotone function of another parameter l: $x = x(t(l)),\ y = y(t(l)),\ z = z(t(l))$ or $\vec{r} = \vec{r}(t(l))$.

The system of equations

$$\begin{cases} F_1(x, y, z) = 0, \\ F_2(x, y, z) = 0, \end{cases}$$

defines the curve as the locus of common points of surfaces $F_1(x, y, z) = 0$ and $F_2(x, y, z) = 0$ if vectors

$$\frac{\partial F_1}{\partial x}\vec{i} + \frac{\partial F_1}{\partial y}\vec{j} + \frac{\partial F_1}{\partial z}\vec{k} \quad \text{and} \quad \frac{\partial F_2}{\partial x}\vec{i} + \frac{\partial F_2}{\partial y}\vec{j} + \frac{\partial F_2}{\partial z}\vec{k}$$

are not collinear.

A curve $\mathcal{L}:\ \vec{r} = \vec{r}(t),\ t \in T$ is called the *smooth curve* at a point $t_0 \in T$ if $\vec{r}(t) \in C^1$ and $\vec{r}\,'(t_0) \neq \vec{0}$. If $\vec{r}(t) \in C^n, n \geqslant 2$, and $\vec{r}\,' \neq 0$, then the curve is called the *regular curve*.

The *length* of a smooth simple curve $\vec{r} = \vec{r}(t),\ t \in [a, b]$ is

$$l = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\mathrm{d}t = \int_a^b |\vec{r}\,'(t)|\mathrm{d}t.$$

A continuous strictly increasing and positive for all $t > a$ function

$$l(t) = \int_a^t |\vec{r}'(\xi)| \mathrm{d}\xi$$

is called the *natural parameter* of a curve. We denote the derivative of a vector function $\vec{r}(t(l))$ in respect of the natural parameter as

$$\dot{\vec{r}} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}l}.$$

Example 1. Write a parametric equation of the curve

$$\cos x - x + y^2 = 0, \quad y \leqslant 0.$$

Solution. We can solve the equation $\cos x - x + y^2 = 0$ in respect of the variable y: $y = \pm\sqrt{\cos x - x}$. The condition $y \leqslant 0$ allows us to choose an appropriate branch of the square root namely the negative one. In addition in order to the real function $y(x)$ to be defined, the radicand are to be nonnegative. It is possible for all $x \in (-\infty, x_0]$, where x_0 is the solution of the equation $\cos x = x$. Finally the parametric equation of the curve is:

$$x = x, \quad y = -\sqrt{\cos x - x}, \quad x \in (-\infty, x_0].$$

Example 2. Write a parametric equation of the curve

$$x^3 = axy - ay^2.$$

Solution. Functions x^3 and $axy - ay^2$ are homogeneous. In this case it can be convenient to use a parametrisation as $y = xt$. Substituting y in the curve equation we obtain

$$x^3 = ax^2(t - t^2),$$

and finally

$$x = at(1 - t), \quad y = at^2(1 - t), \quad t \in (-\infty, +\infty).$$

Example 3. Write a parametric equation of the curve

$$(x^2 + y^2)^3 = 2a^4(x^2 - y^2).$$

Solution. Method 1. Similarly to the Example 2 we can notice that functions $(x^2+y^2)^3$ and x^2-y^2 are homogeneous and it is possible to introduce a parameter t as $y = xt$. In this case we obtain

$$x = \pm a\sqrt[4]{\frac{2(1-t)^2}{(1+t^2)^3}}, \quad y = \pm at\sqrt[4]{\frac{2(1-t)^2}{(1+t^2)^3}}, \quad |t| \leqslant 1.$$

A certain pair of signs before variables x and y specify the part of the curve which belongs to a certain quadrant. Therefore the whole curve is a union of the following four simple curves:

$$x = +a\sqrt[4]{\frac{2(1-t)^2}{(1+t^2)^3}}, \quad y = +at\sqrt[4]{\frac{2(1-t)^2}{(1+t^2)^3}}, \quad |t| \leqslant 1,$$

$$x = -a\sqrt[4]{\frac{2(1-t)^2}{(1+t^2)^3}}, \quad y = +at\sqrt[4]{\frac{2(1-t)^2}{(1+t^2)^3}}, \quad |t| \leqslant 1,$$

$$x = -a\sqrt[4]{\frac{2(1-t)^2}{(1+t^2)^3}}, \quad y = -at\sqrt[4]{\frac{2(1-t)^2}{(1+t^2)^3}}, \quad |t| \leqslant 1,$$

$$x = +a\sqrt[4]{\frac{2(1-t)^2}{(1+t^2)^3}}, \quad y = -at\sqrt[4]{\frac{2(1-t)^2}{(1+t^2)^3}}, \quad |t| \leqslant 1.$$

Of course the parametrization $y = xt$ is not very convenient.

Method 2. In this example it is better to use polar coordinates:

$$x = a\rho(\varphi)\cos\varphi, \quad y = a\rho(\varphi)\sin\varphi.$$

Substituting these x and y into the curve implicit equation we obtain

$$(a^2\rho^2(\cos^2\varphi + \sin\_^2\varphi))^3 = 2a^6\rho^2(\cos^2\varphi - \sin^2\varphi),$$

and $\rho^4 = 2\cos 2\varphi$. Thus the parametric equation is,

$$x = a\sqrt[4]{2\cos 2\varphi}\cos\varphi, \quad y = a\sqrt[4]{2\cos 2\varphi}\sin\varphi.$$

The functions $x(\varphi)$ and $y(\varphi)$ are 2π-periodical and defined at

$$\varphi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \bigcup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right].$$

Example 4. Write a parametric equation of the curve

$$x^2 + y^2 = cz, \quad \frac{y}{x} = \tan\frac{z}{c}$$

between points $A\left(\frac{c}{2}\sqrt{\frac{\pi}{2}}, \frac{c}{2}\sqrt{\frac{\pi}{6}}, \frac{c\pi}{6}\right)$ and $B(x_0, y_0, z_0)$, $z_0 > 0$.

Solution. Introduce a parameter t as $z = ct$. Then substitution of the second equation $y = x\tan t$ into the first yields

$$x^2(1 + \tan^2 t) = c^2 t.$$

Hence $x = \pm c\sqrt{t}\cos t$, $y = \pm c\sqrt{t}\sin t$. Since we are to find the part of the curve between points A and B, the coordinates x and y are positive. The point A corresponds to the parameter value $t = \frac{\pi}{6}$, and B to $t = \frac{z_0}{c}$. Finally the parametric equation is

$$x = c\sqrt{t}\cos t, \quad y = c\sqrt{t}\sin t, \quad z = t, \quad t \in \left[\frac{\pi}{6}, \frac{z_0}{c}\right].$$

Example 5. Write a parametric equation of the curve

$$x^2 + y^2 = R^2, \quad 2xy = z.$$

Solution. The curve consists of common points of the cylinder $x^2 + y^2 = R^2$ and hyperbolic paraboloid $2xy = z$. Since the first equation does not contain the coordinate z it is convenient to use a parametrization as $x = R\cos t$, $y = R\sin t$. Substituting x and y to the second equation we find $z = 2R^2 \sin t\cos t$. Finally we have

$$x = R\cos t, \quad y = R\sin t, \quad z = R^2 \sin 2t, \quad t \in [0, 2\pi].$$

Example 6. Write a parametric equation of the curve

$$x = y^2 + z^2, \quad x - 2y + 4z = 4.$$

Solution. Elimination of the variable x results in

$$y^2 + 2y + z^2 - 4z = 4.$$

This is the equation of the projection of the cure onto the plane yOz. Completing squares in both variables we obtain the equation of a circle in the form $(y + 1)^2 + (z - 2)^2 = 9$, that can be parameterized as

$$y + 1 = 3\cos t, \quad z - 2 = 3\sin t.$$

Then finding the x from, for example, the equation of plane we obtain

$x = 6(\cos t - 2\sin t - 1), \quad y = -1 + 3\cos t, \quad z = 2 + 3\sin t, \quad t \in [0, 2\pi].$

Example 7. Show that the curve

$$x = at\cos t, \quad y = at\sin t, \quad z = \frac{a^2t^2}{2p}$$

belongs to the paraboloid of revolution and the projection of the curve onto the plane xOy is the Archimedean spiral.

Solution. Raising the first two equation to the second power and then adding them we obtain

$$x^2 + y^2 = a^2t^2(\cos^2 t + \sin^2 t) = a^2t^2 = 2pz.$$

Therefore coordinates x, y, and z are related by the expression

$$x^2 + y^2 = 2pz,$$

that is the equation of a paraboloid of revolution.

The projection of the curve onto the plane xOy is

$$x = at\cos t, \quad y = at\sin t.$$

In polar coordinates this equation looks as $\rho = a\varphi$, that is the equation of a Archimedean spiral.

Example 8. Find natural parametrization of the curve

$$x = t, \quad y = \sqrt{2}\ln t, \quad z = \frac{1}{t}.$$

Solution. A natural parameter l is the length of a curve arc between an arbitrary fixed point corresponding $t = a$ and variable point $t = t$. The dependence $l(t)$ is found by the formula

$$l(t) = \int_a^t |\vec{r}\,'(\xi)| \mathrm{d}\xi.$$

Calculation gives us

$$x' = 1, \quad y' = \frac{\sqrt{2}}{t}, \quad z' = -\frac{1}{t^2},$$

hence

$$|\vec{r}\,'(t)| = \sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}} = 1 + \frac{1}{t^2}.$$

Suppose $a = 1$. Then

$$l(t) = \int_1^t \left(1 + \frac{1}{\xi^2}\right) \mathrm{d}\xi = t - \frac{1}{t}.$$

Find parameter t as a function of l: $t = \dfrac{l \pm \sqrt{l^2+4}}{2}$. Since $t > 0$, then we choose the plus sign.

Finally the natural parametrization can be written as

$$x = \frac{l + \sqrt{l^2+4}}{2}, \quad y = \sqrt{2} \ln \frac{l + \sqrt{l^2+4}}{2}, \quad z = \frac{2}{l + \sqrt{l^2+4}}.$$

Exercises

2.42. A point M rotates around a fixed straight line with constant angular velocity and simultaneously moves parallel to the line with constant velocity. The trajectory of the point is called cylindrical helix. Write a parametric equation of the curve and find projections onto coordinate planes.

2.43. A point M moves along the generatrix of a circular cylinder with a speed proportional to the path traveled. The cylinder rotates around its axis with a constant angular velocity. Write a parametric equation of the trajectory of the point M.

2.44. A line OT is located at an acute angle to the Oz axis and rotates around it with a constant angular velocity ω. A point M moves along the line OT with a constant speed. The trajectory of a point is called a conical helix. Write a parametric equation for this curve.

2.45. A line OT is located at an acute angle to the Oz axis and rotates around it with a constant angular velocity ω. A point M moves along the line OT with a speed proportional to the distance OM between the moving point M and the fixed point O. The point M describes the conical spiral. Write a parametric equation of the curve.

2.46. A circle of radius a rolls along a straight line without slipping. A point M is rigidly connected to the circle and is at a distance d from its center. For $d = a$ the trajectory of the point is called cycloid, for $d < a$ — *curtailed cycloid*, for $d > a$ — *elongated cycloid* . Write a parametric equation of the trajectory of the point M.

2.47. A circle of radius r rolls without slipping along a circle of radius R, remaining outside it. The trajectory of a point M, of the first circle is called epicycloid. Write a parametric equation for this curve.

2.48. A circle of radius r rolls without slipping along a circle of radius R, remaining inside it. The trajectory of a point M, of the first circle is called hypocycloid. Write a parametric equation for this curve.

2.49. The axes of two circular cylinders of radii a and b intersect at right angle. Two closed curves are formed at the intersection of the cylinders, the set of which is called *bicylindrics.* Write a parametric equation of the curve. What happens when $a = b$?

2.50. A sphere of radius $2a$ is intersected by a circular cylinder of diameter a so that one of the generators of the cylinder passes through the center of the sphere. The line obtained in the section is called the *Viviani's curve.* Write its parametric equation.

Find a parametric equation of the following plane curves.

2.51. The segment AB, connecting points $A(1, -2)$ and $B(4, -3)$.

2.52. A part of the parabola $y = 2x^2$, connecting points $A(-1, 2)$ and $B(2, 8)$.

2.53. $x^3 + 2x^2 + y^2 = 3,\ y \geqslant 0.$

2.54. $\ln x - y + \sin x = 0.$

2.55. Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

2.56. Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

2.57. $\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} = 1$ from the point $A(a, 0)$ to $B(0, b)$.

2.58. Astroid $x^{2/3} + y^{2/3} = a^{2/3}.$

2.59. $(x + y)^{2/3} - (x - y)^{2/3} = a^{2/3}.$

2.60. $x(x - y)^2 + y = 0,\ x \geqslant 0.$

2.61. $2(x + y) = (x - y)^2.$

2.62. $(y - x)^2 = a^2 - x^2.$

2.63. $x^4 = axy^2 + ay^3.$

2.64. $a^4y^4 = x^4(a^2 - x^2).$

2.65. $x^6 + y^6 = a^2x^4 + b^2y^4.$

2.66. $y^2(a-x) = x^2(a+x)$.

2.67. $(x^2+y^2)^2 = 2a^2xy$.

2.68. $x^4 - y^4 + xy = 0$ from the point $A\left(\sqrt{2/15}, 2\sqrt{2/15}\right)$ to $B(0,0)$.

Find a parametric equation of the following curves.

2.69. The segment AB, connecting points $A(1,3,-1)$ and $B(2,3,0)$.

2.70. $x^2+y^2 = R^2,\ z = h$.

2.71. $x^2+y^2 = R^2,\ x+y = z$.

2.72. $y^2 = 2px,\ x+y-z = 0$.

2.73. $x^2+y^2+z^2 = R^2,\ x^2+y^2 = \dfrac{R^2}{2},\ z > 0$.

2.74. $x^2+y^2+z^2 = 2ax,\ x^2+y^2 = z^2$.

2.75. $x^2+y^2+z^2 = R^2,\ x+y+z = 0$.

2.76. $x^2-y^2+z^2 = 1,\ y^2+z^2-x^2 = 1$.

2.77. $z = x^2-y^2,\ x+y-z = 0$.

2.78. $x^2-y^2 = \dfrac{9}{8}z^2,\ (x-y)^2 = a(x+y)$ from the point $A(0,0,0)$ to $B(x_0,y_0,z_0)$.

2.79. $x^2+y^2+z^2 = a^2,\ \sqrt{x^2+y^2}\cosh\left(\arctan\dfrac{y}{x}\right) = a,\ z \geqslant 0$, from the point $A(a,0,0)$ to $B(x_0,y_0,z_0)$.

2.80. Show that the curve $x = a\cos t,\ y = b\sin t,\ z = ct$ belongs to the elliptic cylinder.

2.81. Show that the curve $x = a\cosh t,\ y = b\sinh t,\ z = ct$ belongs to the hyperbolic cylinder.

2.82. Show that the curve $x = e^t\cos t,\ y = e^t\sin t,\ z = 2t$ belongs to the surface $x^2+y^2 = e^z$.

2.83. Show that the curve $x = a\cos^3 t,\ y = a\sin^3 t,\ z = a\cos 2t$ belongs to the cylinder whose directrix is an astroid and the generatrix is parallel to the Oz axis.

2.84. Show that the curve $x = \sin 2t,\ y = 1-\cos 2t,\ z = 2\cos t$ belongs to the sphere and is the intersection of the parabolic and circular cylinders.

2.85. Show that the curve $x = a\sin^2 t,\ y = b\sin t\cos t,\ z = c\cos t$ belongs to the ellipsoid.

2.86. To what class of regularity belongs the curve

$$x = \begin{cases} e^t, & t < 0, \\ 0, & t \geqslant 0; \end{cases} \qquad y = t; \qquad z = \begin{cases} 0, & t \leqslant 0, \\ e^{-1/t}, & t > 0. \end{cases}$$

2.87. Determine whether the following parameterizations are equivalent:

$$\begin{cases} x = a\cos t, \\ y = a\sin t, \\ z = ht \end{cases} \quad \text{and} \quad \begin{cases} x = a\cos(\tau^3 + 1), \\ y = a\sin(\tau^3 + 1), \\ z = h(\tau^3 + 1), \end{cases}$$

where $t \in [0, 2\pi]$, $\tau \in [-1, \sqrt[3]{2\pi - 1}]$.

2.88. Prove that parameterizations

$$x = a\cos t, \quad y = a\sin t, \quad z = ht, \quad t \in [0, 4\pi]$$

and

$$x = a\cos\tau^2, \quad y = a\sin\tau^2, \quad z = 2h\tau, \quad \tau \in [0, 2\pi]$$

are not equivalent, i. e. these are different curves.

2.89. Show that the length of a curve $\rho = \rho(\varphi)$, $\varphi_1 \leqslant \varphi \leqslant \varphi_2$ in polar coordinates is calculated as $l = \int\limits_{\varphi_1}^{\varphi_2} \sqrt{\rho^2 + (\rho')^2} d\varphi$.

Find the length of the following curves.

2.90. $y = a\cosh\dfrac{x}{a}$, $x \in [-a, a]$.

2.91. $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$, $t \in [0, 2\pi]$.

2.92. $x = a(2\cos t + \cos 2t)$, $y = a(2\sin t + \sin 2t)$, $t \in [0, 2\pi]$.

2.93. $x = a\left(\ln\cot\dfrac{t}{2} - \cos t\right)$, $y = a\sin t$, $t \in \left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$.

2.94. $\rho = a\cos\varphi$, $\varphi \in [0, 2\pi]$.

2.95. $\rho = a(1 + \cos\varphi)$, $\varphi \in [0, 2\pi]$.

2.96. $\rho = a\sin^3\dfrac{\varphi}{3}$, $\varphi \in [0, \pi]$.

2.97. $\rho = a\varphi$, $\varphi \in [0, 2\pi]$.

2.98. $\rho = a\tanh\dfrac{\varphi}{2}$, $\varphi \in [0, \pi]$.

2.99. $\rho = \dfrac{a}{1 + \cos\varphi}$, $\varphi \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

2.100. $x = \sqrt{\dfrac{3}{2}}t^2$, $y = 2 - t$, $z = t^3$, $t \in [0, 1]$.

2.101. $x = t$, $y = \dfrac{t^3}{3}$, $z = -\dfrac{1}{2t}$, $t \in [1, 2]$.

Write the natural parametrization of the following curves.

2.102. $x = a\cos t,\ y = a\sin t,\ z = ht.$

2.103. $x = \cos^3 t,\ y = \sin^3 t,\ z = \cos 2t.$

2.104. $x = a\cosh t,\ y = a\sinh t,\ z = at.$

2.105. $x = t - \sin t,\ y = 1 - \cos t,\ z = 4\sin\dfrac{t}{2}.$

2.106. $x = e^t,\ y = e^{-t},\ z = \sqrt{2}t.$

2.3. Frenet trihedron. Curvature and torsion

Let $\mathcal{L} \in C^2$ have a natural equation $\vec{r} = \vec{r}(l),\ l \in [0, l_0]$. Then at any point M of the curve $\mathcal{L}$ it is possible to build the *moving trihedron* of the curve or *Frenet trihedron* comprising tangent, principal normal, binormal, normal plane, osculating plane and rectifying plane (fig. 2.1).

The line through the point M of the curve $\mathcal{L}$ parallel to the vector $\dot{\vec{r}}$ is called the *tangent to the curve* at M. The unit directing vector of the tangent $\vec{\tau}$ is

$$\vec{\tau} = \dot{\vec{r}}.$$

The plane through the point M of the curve $\mathcal{L}$ perpendicular to the tangent is called the *normal plane*.

The line through the point M of the curve $\mathcal{L}$ parallel to the vector $\ddot{\vec{r}}$ is called the *principal normal* of the curve at the point M. The unit directing vector of the principal normal $\vec{n}$ is

$$\vec{n} = \frac{\ddot{\vec{r}}}{|\ddot{\vec{r}}|}.$$

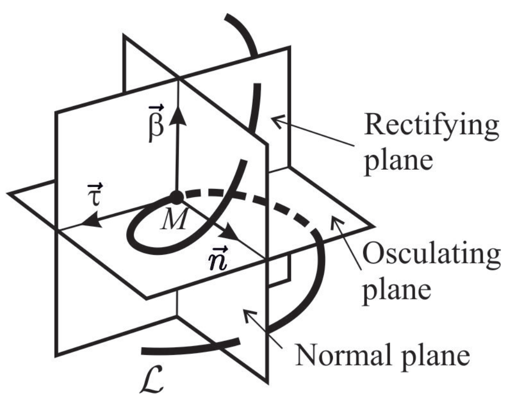

Fig. 2.1

The plane through the point M of the curve $\mathcal{L}$ spanned over the vectors $\dot{\vec{r}}$ and $\ddot{\vec{r}}$ or $\vec{\tau}$ and $\vec{n}$ is called the *osculating plane* of the curve. A plane curve belongs to its osculating plane.

The line through the point M of the curve $\mathcal{L}$ perpendicular to the osculating plane is called *binormal*. The unit directing vector of the binormal is chosen in such a way that the triple $(\vec{\tau}, \vec{n}, \vec{\beta})$ is right-handed:

$$\vec{\tau} \times \vec{n} = \vec{\beta}, \quad \vec{n} \times \vec{\beta} = \vec{\tau}, \quad \vec{\beta} \times \vec{\tau} = \vec{n}.$$

Thus the vector $\vec{\beta}$ can be calculated as

$$\vec{\beta} = \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{|\ddot{\vec{r}}|}.$$

The plane through the point M of the curve $\mathcal{L}$ spanned over the tangent and binormal is called the *rectifying plane*.

Let M_0 and M be to points of the curve $\mathcal{L}$, Δl the length of the arc between these points and $\Delta\theta$ the angle of the tangents at M_0 and M and $\Delta\vartheta$ the angle of the binormal at M_0 and M. The *curvature* k and *torsion* κ of the curve $\mathcal{L}$ at the point M_0 are the rate of change of direction of the tangent and binormal at M_0 respectively:

$$k = \lim_{M \to M_0} \left| \frac{\Delta\theta}{\Delta l} \right|, \quad \kappa = \lim_{M \to M_0} \frac{\Delta\vartheta}{\Delta l}.$$

The curvature of a curve in space is always positive and the torsion is considered negative if vectors $\vec{n}$ and $\dot{\vec{\beta}}$ are codirectional. The formulas for calculation of curvature and torsion differs in dependence on whether the parametrisation is natural or not:

$$k = |\ddot{\vec{r}}| \quad \text{or} \quad k = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3};$$

$$\kappa = \frac{\dot{\vec{r}}\ddot{\vec{r}}\dddot{\vec{r}}}{|\ddot{\vec{r}}|^2} \quad \text{or} \quad \kappa = \frac{\vec{r}\,'\vec{r}\,''\vec{r}\,'''}{|\vec{r}\,' \times \vec{r}\,''|^2}.$$

The value $R = \dfrac{1}{k}$ is called the *radius of curvature*. The circle in the osculating plane with center at the point $\vec{r} + \dfrac{1}{k}\vec{n}$ is called the *circle of curvature* or *osculating circle* and its center the *center if curvature*. The circle of curvature of a curve at a point of the curve has the contact of the second order ($\dot{\vec{r}} = \dot{\vec{r}}_C$, $\ddot{\vec{r}} = \ddot{\vec{r}}_C$) with the curve.

The equations

$$\begin{cases} \dot{\vec{\tau}} = k\vec{n}, \\ \dot{\vec{n}} = \varkappa\vec{\beta} - k\vec{\tau}, \\ \dot{\vec{\beta}} = -\varkappa\vec{n}, \end{cases}$$

are called *Frenet formulas* and are fundamental in the theory of curves.

Example 1. Prove that for the curve $\vec{r} = \vec{r}(t)$, $t \in T$, the second derivative $\vec{r}''$ belongs to the osculating plane, and the vector $\vec{r}' \times \vec{r}''$ is co-directional with the binormal vector $\vec{\beta}$.

Solution. It is known that (see problem 2.35)

$$\vec{r}' = \dot{\vec{r}}l' = l'\vec{\tau},$$
$$\vec{r}'' = \ddot{\vec{r}}l'^2 + \dot{\vec{r}}l'' = kl'^2\vec{n} + l''\vec{\tau}.$$

Thus the vector $\vec{r}''$ is a linear combination of the vectors $\vec{n}$ and $\vec{\tau}$ and belongs to the osculating plane.

Then calculation of $\vec{r}' \times \vec{r}''$ yields

$$\vec{r}' \times \vec{r}'' = kl'^3(\vec{\tau} \times \vec{n}).$$

Since the curvature is $k > 0$ and $l' > 0$, hence $\vec{r}' \times \vec{r}'' \upuparrows \vec{\beta}$.

Example 2. Write equations of elements of Frenet trihedron for the curve

$$r = \vec{r}(t) = \vec{i}a\cos t + \vec{j}a\sin t + \vec{k}\,bt$$

at the point $M_0 :\ t_0 = \pi/4$.

Solution. Recall in the beginning basic equations of a line and plane. A straight line is uniquely defined by a point (x_0, y_0, z_0) and a directing vector $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$. The equation

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} = \frac{z - z_0}{a_z}. \tag{2.2}$$

is the canonical equation of the line. The general equation of a plane is written as

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0, \tag{2.3}$$

where (x_0, y_0, z_0) is a point of the plane and $\vec{N} = A\vec{i} + B\vec{j} + C\vec{k}$ is its normal vector.

In this example the vector equation of the curve is equivalent to the system of equations

$$x = a\cos t, \quad y = a\sin t, \quad z = bt.$$

Substituting the value of the parameter $t_0 = \dfrac{\pi}{4}$ in this system we obtain the cartesian coordinates of the point M_0:

$$x_0 = \frac{a}{\sqrt{2}}, \quad y_0 = \frac{a}{\sqrt{2}}, \quad z_0 = \frac{b\pi}{4}.$$

Then the directing vector of the tangent of the curve at an arbitrary point is the vector $\vec{r}\,'(t)$:

$$\vec{r}\,'(t) = -a\sin t\vec{i} + a\cos t\vec{j} + b\vec{k},$$

and its value at the point t_0 is

$$\vec{r}\,'(t_0) = -\frac{a}{\sqrt{2}}\vec{i} + \frac{a}{\sqrt{2}}\vec{j} + b\vec{k}.$$

Using formula (2.2) we can write the equation of the tangent at the point M_0:

$$\frac{x - \dfrac{a}{\sqrt{2}}}{-\dfrac{a}{\sqrt{2}}} = \frac{y - \dfrac{a}{\sqrt{2}}}{\dfrac{a}{\sqrt{2}}} = \frac{z - \dfrac{b\pi}{4}}{b}.$$

The vector $\vec{r}\,'(t_0)$ is the perpendicular of the normal plane at the point t_0. Then rewriting the equation (2.3) for our case we obtain the equation of the normal plane in the form

$$-\frac{a}{\sqrt{2}}\left(x - \frac{a}{\sqrt{2}}\right) + \frac{a}{\sqrt{2}}\left(y - \frac{a}{\sqrt{2}}\right) + b\left(z - \frac{b\pi}{4}\right) = 0.$$

In order to simplify further calculations we find the norm of the vector $\vec{r}\,'(t)$. If $|\vec{r}\,'(t)|$ is constant for all t, then $\vec{r}\,'(t) \perp \vec{r}\,''(t)$, and the vector $\vec{r}\,''(t_0)$ in this case is directing vector of the principal normal. Thus

$$|\vec{r}\,'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2} = \text{const}, \quad \forall t;$$

$$\vec{r}\,''(t) = -\vec{i}a\cos t - \vec{j}a\sin t,$$

and

$$\vec{r}\,''(t_0) = -\frac{a}{\sqrt{2}}\vec{i} - \frac{a}{\sqrt{2}}\vec{j} \quad \| \quad \vec{i} + \vec{j}.$$

Finally the equation of the principal normal at the point t_0 is

$$\frac{x-\dfrac{a}{\sqrt{2}}}{1}=\frac{y-\dfrac{a}{\sqrt{2}}}{1}=\frac{z-\dfrac{b\pi}{4}}{0}$$

or after simplifying

$$x=y,\quad z=\frac{b\pi}{4}.$$

Since the principal normal coincides with the normal of the rectifying plane, the equation of the latter is

$$\left(x-\frac{a}{\sqrt{2}}\right)+\left(y-\frac{a}{\sqrt{2}}\right)=0,$$

or

$$x+y=\sqrt{2}a.$$

It was shown in Example 1 that vectors $\vec{r}'(t)$ and $\vec{r}''(t)$ belong to the osculating plane. Thus to write the equation of this plane we are to expand the following determinant which is equal to zero:

$$(\vec{r}-\vec{r}_0)\vec{r}'(t_0)\vec{r}''(t_0)=\begin{vmatrix} x-\dfrac{a}{\sqrt{2}} & y-\dfrac{a}{\sqrt{2}} & z-\dfrac{b\pi}{4} \\ -\dfrac{a}{\sqrt{2}} & \dfrac{a}{\sqrt{2}} & b \\ 1 & 1 & 0 \end{vmatrix}=0.$$

Finally the equation of the osculating plane is

$$-b\left(x-\frac{a}{\sqrt{2}}\right)+b\left(y-\frac{a}{\sqrt{2}}\right)-a\sqrt{2}\left(z-\frac{b\pi}{4}\right)=0.$$

Coefficients before parentheses are coordinates of the normal vector to the osculating plane and this vector $\vec{N}=-b\vec{i}+b\vec{j}-a\sqrt{2}\vec{k}$ is collinear with the binormal. Thus the equation of the binormal is

$$\frac{x-\dfrac{a}{\sqrt{2}}}{-b}=\frac{y-\dfrac{a}{\sqrt{2}}}{b}=\frac{z-\dfrac{b\pi}{4}}{-a\sqrt{2}}.$$

Example 3. Write equations of elements of Frenet trihedron for the curve

$$r = \vec{r}(t) = e^t\vec{i} + e^{-t}\vec{j} + \sqrt{2}t\,\vec{k}$$

at the point $M_0 : t_0 = 0$.

Solution. As it was done in Example 2 we find firstly the coordinates of the point M_0

$$x_0 = 1, \quad y_0 = 1, \quad z_0 = 0.$$

The directing vector of the tangent is

$$\vec{r}\,'(t) = e^t\vec{i} - e^{-t}\vec{j} + \sqrt{2}\,\vec{k}$$

and

$$\vec{r}\,'(t_0) = \vec{i} - \vec{j} + \sqrt{2}\vec{k}.$$

Therefore we easily can write the equations of the tangent at the point t_0:

$$\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z}{\sqrt{2}},$$

and the normal plane:

$$(x-1) - (y-1) + \sqrt{2}z = 0 \quad \text{or} \quad x - y + \sqrt{2}z = 0.$$

Then, calculation of $|\vec{r}\,'(t)|$ yields:

$$|\vec{r}\,'(t)| = \sqrt{e^{2t} + e^{-2t} + 2} = 2\cosh t \neq \text{const},$$

that means that $\vec{r}\,''(t)$ is not collinear to the principal normal.

In that case we find the equations of the osculating plane and binormal. For that purpose we calculate $\vec{r}\,''(t)$:

$$\vec{r}\,''(t) = e^t\vec{i} + e^{-t}\vec{j}, \quad \text{and} \quad \vec{r}\,''(t_0) = \vec{i} + \vec{j}.$$

And then the equation of the osculating plane is

$$\begin{vmatrix} x-1 & y-1 & z \\ 1 & -1 & \sqrt{2} \\ 1 & 1 & 0 \end{vmatrix} = -\sqrt{2}(x-1) + \sqrt{2}(y-1) + 2z = 0$$

or finally

$$-x + y + \sqrt{2}z = 0.$$

The normal vector to the osculating plane $\vec{N}_1 = -\vec{i} + \vec{j} + \sqrt{2}\vec{k}$ is the directing vector of the binormal that readily give us the equation of the binormal

$$\frac{x-1}{-1} = \frac{y-1}{1} = \frac{z}{\sqrt{2}}.$$

The binormal and the tangent belongs to the rectifying plane therefore the equation of the latter can be written in the form

$$\begin{vmatrix} x-1 & y-1 & z \\ 1 & -1 & \sqrt{2} \\ -1 & 1 & \sqrt{2} \end{vmatrix} = -2\sqrt{2}(x-1) - 2\sqrt{2}(y-1) = 0,$$

and finally

$$x + y = 2.$$

Finding the normal vector to the rectifying plane $\vec{N}_2 = \vec{i} + \vec{j}$ we write the equation of the principal normal

$$\frac{x-1}{1} = \frac{y-1}{1} = \frac{z}{0}.$$

Example 4. Write equations of elements of Frenet trihedron for the line of intersection of the sphere $x^2 + y^2 + z^2 = 9$ and hyperbolic cylinder $x^2 - y^2 = 3$ at the point $M(2, 1, 2)$.

Solution. The given curve is defined by a system

$$\begin{cases} F_1(x, y, z) = 0, \\ F_2(x, y, z) = 0. \end{cases} \tag{2.4}$$

Taking as a parameter for example x we can find from the system (2.4) functions $y(x)$ and z and write a parametric equation of the curve in the form

$$\vec{r} = x\vec{i} + y(x)\vec{j} + z(x)\vec{k}.$$

Differentiating the system (2.4) with respect to x, we can find functions $y'(x)$, $z'(x)$ and $y''(x)$, $z''(x)$ and finally obtain

$$\vec{r}'_x = \vec{i} + y'_x\vec{j} + z'_x\vec{k}, \tag{2.5}$$

$$\vec{r}''_{xx} = y''_{xx}\vec{j} + z''_{xx}\vec{k}. \tag{2.6}$$

The system (2.4) in our case is

$$\begin{cases} x^2 + y^2 + z^2 = 9, \\ x^2 - y^2 = 3. \end{cases} \tag{2.7}$$

Differentiating each of equations in (2.7) with respect to x and simplifying yields

$$\begin{cases} x + yy' + zz' = 0, \\ x - yy' = 0. \end{cases} \tag{2.8}$$

At the point $M(2, 1, 2)$ we have

$$\begin{cases} 2 + y' + 2z' = 0, \\ 2 - y' = 0, \end{cases}$$

and solfing this system we find $y' = 2,\ z' = -2$. Then according to (2.5)

$$\vec{r}\,'(M) = \vec{i} + 2\vec{j} - 2\vec{k}.$$

Similarly we find the second derivative:

$$\begin{cases} 1 + (y')^2 + yy'' + (z')^2 + zz'' = 0, \\ 1 - (y')^2 - yy'' = 0. \end{cases} \tag{2.9}$$

Substituting into the system (2.9) the point M and the previously found vector $\vec{r}\,'(M)$, we obtain

$$\begin{cases} 1 + 4 + y'' + 4 + 2z'' = 0, \\ 1 - 4 - y'' = 0, \end{cases}$$

and finally $y'' = z'' = -3$. Hence

$$\vec{r}\,''(M) = -3\vec{j} - 3\vec{k} \parallel \vec{j} + \vec{k}.$$

Then we act as in Example 2 or Example 3. The equation of the tangent:

$$\frac{x-2}{1} = \frac{y-1}{2} = \frac{z-2}{-2};$$

the normal plane

$$(x-2) + 2(y-1) - 2(z-2) = 0, \quad \text{or} \quad x + 2y - 2z = 0;$$

the osculating plane

$$\begin{vmatrix} x-2 & y-1 & z-2 \\ 1 & 2 & -2 \\ 0 & 1 & 1 \end{vmatrix} = 4(x-2) - (y-1) + (z-2) = 0,$$

or

$$4x - y + z - 9 = 0;$$

the binormal

$$\frac{x-2}{4} = \frac{y-1}{-1} = \frac{z-2}{1};$$

the rectifying plane

$$\begin{vmatrix} x-2 & y-1 & z-2 \\ 1 & 2 & -2 \\ 4 & -1 & 1 \end{vmatrix} = -9(y-1) - 9(z-2) = 0,$$

or

$$y + z - 3 = 0.$$

the principal normal

$$\frac{x-2}{0} = \frac{y-1}{1} = \frac{z-2}{1}.$$

Example 5. Calculate the curvature and the torsion of the curve

$$x = a\cosh t, \quad y = a\sinh t, \quad z = bt.$$

at an arbitrary point. For what values of a and b does the torsion of the curve equal its curvature at all points?

Solution. The formulas for calculating the curvature k and the torsion κ are

$$k = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}, \qquad \kappa = \frac{\vec{r}\,'\vec{r}\,''\vec{r}\,'''}{|\vec{r}\,' \times \vec{r}\,''|^2}. \tag{2.10}$$

Since

$$\vec{r}(t) = \vec{i}a\cosh t + \vec{j}a\sinh t + \vec{k}bt,$$

we find

$$\vec{r}\,'(t) = \vec{i}a\sinh t + \vec{j}a\cosh t + \vec{k}b,$$
$$\vec{r}\,''(t) = \vec{i}a\cosh t + \vec{j}a\sinh t,$$
$$\vec{r}\,'''(t) = \vec{i}a\sinh t + \vec{j}a\cosh t.$$

Then calculate the cross product $\vec{r}\,' \times \vec{r}\,''$:

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\sinh t & a\cosh t & b \\ a\cosh t & a\sinh t & 0 \end{vmatrix} = -ab\sinh t\vec{i} + ab\cosh t\vec{j} - a^2\vec{k}.$$

Here for simplification we used the property $\cosh^2 t - \sinh^2 t = 1$. Find the norms of required vectors:

$$|\vec{r}\,'(t)| = \sqrt{a^2\sinh^2 t + a^2\cosh^2 t + b^2} = \sqrt{a^2\cosh 2t + b^2},$$

$$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{a^2b^2\sinh^2 t + a^2b^2\cosh^2 t + a^4} = a\sqrt{b^2\cosh 2t + a^2}.$$

Substituting the found quantities into the first formula (2.10) we obtain the curvature:

$$k = \frac{a\sqrt{b^2\cosh 2t + a^2}}{(\sqrt{a^2\cosh 2t + b^2})^3}.$$

To find the torsion calculate firstly scalar triple product $\vec{r}\,'\vec{r}\,''\vec{r}\,'''$:

$$\vec{r}\,'\vec{r}\,''\vec{r}\,''' = (\vec{r}\,' \times \vec{r}\,'') \cdot \vec{r}\,''' =$$

$$= \left(-\vec{i}ab\sinh t + \vec{j}ab\cosh t - a^2\vec{k}\right) \cdot \left(\vec{i}a\sinh t + \vec{j}a\cosh t\right) = a^2b.$$

Therefore the torsion is

$$\varkappa = \frac{b}{b^2\cosh 2t + a^2}.$$

To answer the second question of the problem, we equate the curvature of the curve with its torsion:

$$b\left(\sqrt{a^2\cosh 2t + b^2}\right)^3 = a\left(\sqrt{b^2\cosh 2t + a^2}\right)^3,$$

or

$$b^{2/3}\left(a^2\cosh 2t + b^2\right) = a^{2/3}\left(b^2\cosh 2t + a^2\right).$$

Since this equality have to be true for all t, we write the system

$$\begin{cases} a^2b^{2/3} = b^2a^{2/3}, \\ a^{2/3} = b^{2/3}, \end{cases}$$

from which it immediately follows that $a = b$.

Exercises

Write equations of elements of Frenet trihedron, curvature and torsion of the following curves at the specified points.

2.107. $x = t,\ y = t^3,\ z = t^2 + 4,\ t_0 = 1.$

2.108. $x = t,\ y = \dfrac{t^3}{3},\ z = -\dfrac{1}{2t},\quad t_0 = 1.$

2.109. $x = \sqrt{\dfrac{3}{2}}t^2,\ y = 2 - t,\ z = t^3,\ t_0 = 1.$

2.110. $x = t,\ y = \sqrt{at - t^2},\ z = \sqrt{a^2 - at},\ t_0 = \dfrac{a}{2}.$

2.111. $x = \cos^3 t,\ y = \sin^3 t,\ z = \cos 2t,\ t_0 = \dfrac{\pi}{4}.$

2.112. $x = a\cosh t,\ y = a\sinh t,\ z = ht,\ t_0 = 0.$

2.113. $x = \ln(\cos t),\ y = \ln(\sin t),\ z = \sqrt{2}t,\ t_0 = \dfrac{\pi}{4}.$

2.114. $x = t - \sin t,\ y = 1 - \cos t,\ z = 4\sin\dfrac{t}{2},\ t_0 = \dfrac{\pi}{2}.$

2.115. $x = e^t\cos t,\ y = e^t\sin t,\ z = e^t,\ t_0 = 0.$

2.116. $x = \sin t,\ y = \cos t,\ z = tgt,\ t_0 = \dfrac{\pi}{4}.$

2.117. $x = t\cos t,\ y = t\sin t,\ z = at,\ t_0 = 0.$

2.118. $\begin{cases} x^2 + y^2 + z^2 = 3, \\ x^2 + y^2 = 2, \end{cases} \quad M_0(1, 1, 1).$

2.119. $\begin{cases} y^2 + z^2 = 25, \\ x^2 + y^2 = 10, \end{cases} \quad M_0(1, 3, 4).$

2.120. $\begin{cases} x^2 + y^2 + z^2 = 1, \\ x^2 + y^2 = x, \end{cases} \quad M_0\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{\sqrt{2}}\right).$

2.121. $\begin{cases} x^2 + y^2 = z^2, \\ 1 + x = y, \end{cases} \quad M_0(3, 4, 5).$

2.122. $\begin{cases} y^2 = x, \\ x^2 = z, \end{cases} \quad M_0(1, 1, 1).$

2.123. $\begin{cases} x^2 + y^2 + z^2 = 3, \\ xy = z, \end{cases} \quad M_0(1, 1, 1).$

2.124. $\begin{cases} x^2 + z^2 - y^2 = 1, \\ y^2 - 2x + z = 0, \end{cases} \quad M_0(1, 1, 1).$

2.125. $\begin{cases} x + \sinh x = \sin y + y, \\ z + e^z = x + \ln(1 + x) + 1, \end{cases} \quad M_0(0, 0, 0).$

2.126. Prove that all normal planes of the curve $x = a\sin^2 t$, $y = a\sin t\cos t$, $z = a\cos t$ pass through the origin of coordinates.

2.127. Prove that all normal planes of the curve $x = a\cos t$, $y = a\sin\alpha\sin t$, $z = a\cos\alpha\sin t$ pass through the straight line $x = 0$, $z + y\tan\alpha = 0$.

2.128. Find tangents to the curve $x = 3t - t^3$, $y = 3t^2$, $z = 3t + t^3$ that are perpendicular to the vector $\vec{a}(3, 1, 1)$.

2.129. Find tangents to the curve $x = t^2$, $y = t$, $z = e^t$, that are parallel to the plane $x - 2y - 5 = 0$.

2.130. Find osculating planes of the curve $x = t$, $y = t^2$, $z = t^3$ passing through the point $M_0(2, -1/3, -6)$.

2.131. Show that the straight line passing through an arbitrary point M of the curve $x = t$, $y = t^2$, $z = t^3$ and the axis Oz parallel to the plane $z = 0$ belongs to the osculating plane of the curve at the point M.

2.132. Find points of the curve $x = 2/t$, $y = \ln t$, $z = -t^2$ at which the binormal is parallel to the plane $x - y + 8z + 2 = 0$.

2.133. Suppose N is a point on the binormal of the helix at the Point M, the length of the segment NM being equal for each point M of the curve. Prove that the locus formed by point N is another helix.

2.134. Let $\mathcal{L}$ be a smooth closed curve. Prove that for all vectors $\vec{a}$ there is a point $M \in \mathcal{L}$ that the tangent to the curve at this point is perpendicular to the vector $\vec{a}$.

2.135. Prove that if all normal planes of the spatial curve pass through a fixed point, then the curve belongs to a sphere (such curves are called spherical).

2.136. Prove that if all osculating planes of the spatial curve pass through a fixed point, then this curve is flat.

2.137. Prove that if all osculating planes of the spatial curve are perpendicular to some fixed straight line, the curve is flat.

2.138. Show that for the curve $\vec{r} = \vec{r}(l)$ the following relations hold:

$$|\dddot{\vec{r}}| = k^4 + k^2\varkappa^2 + \dot{k}^2,$$

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} = 0, \quad \dot{\vec{r}} \cdot \dddot{\vec{r}} = -k^2, \quad \ddot{\vec{r}} \cdot \dddot{\vec{r}} = k\dot{k}.$$

2.139. Prove that Frenet formulas can be written in the form

$$\dot{\vec{\tau}} = \vec{\omega} \times \vec{\tau}, \quad \dot{\vec{n}} = \vec{\omega} \times \vec{n}, \quad \dot{\vec{\beta}} = \vec{\omega} \times \vec{\beta}.$$

Find Darboux vector $\vec{\omega}$ and explain its kinematic meaning.

2.140. Prove that
a) $\vec{\tau}\,\vec{\beta}\,\dot{\vec{\beta}} = \kappa$; b) $\dot{\vec{\beta}}\,\ddot{\vec{\beta}}\,\dddot{\vec{\beta}} = \kappa^5 \dfrac{\mathrm{d}}{\mathrm{d}l}\left(\dfrac{k}{\kappa}\right)$; c) $\dot{\vec{\tau}}\,\ddot{\vec{\tau}}\,\dddot{\vec{\tau}} = k^5 \dfrac{\mathrm{d}}{\mathrm{d}l}\left(\dfrac{\kappa}{k}\right)$.

2.141. Prove that if the principal normals of a curve form a constant angle with some fixed straight line, then

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{k^2+\kappa^2}{k\dfrac{\mathrm{d}}{\mathrm{d}l}\left(\dfrac{\kappa}{k}\right)}\right) + \kappa = 0.$$

2.142. Prove that if the curvature and torsion of a curve $\vec{r} = \vec{r}(l)$ are not zero then the curve is spherical then and only then

$$\frac{k}{\kappa} = \frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{\dot{k}}{\kappa k^2}\right).$$

2.143. Find the formula for the curvature of a flat curve with the equation $y = f(x)$.

2.144. Find the formula for the curvature of a flat curve with the equation in polar coordinates $\rho = \rho(\varphi)$.

2.145. Prove that a curve is the straight line if and only if its curvature is zero.

2.146. Prove that a curve is flat if and only if its torsion is zero.

2.147. Prove that the curvature of a curve $\mathcal{L}$ at a point M is equal to the curvature of the projection of the curve onto the osculating plane at the point M.

2.148. Prove that for all closed curve lying on the sphere there is a point at which the torsion of the curve is zero.

2.149. Prove that the radius of curvature of the conical spiral

$$x = ae^{kt}\cos t, \quad y = ae^{kt}\sin t, \quad z = be^{kt}$$

is proportional to the distance from the point of the spiral to the cone axis.

2.150. Prove that the curve $x = a_1t^2 + b_1t + c_1$, $y = a_2t^2 + b_2t + c_2$, $z = a_3t^2 + b_3t + c_3$ is flat and find the plane which contains this curve.

2.151. Find all functions $f(t)$ for which the curve $x = a\cos t$, $y = a\sin t$, $z = f(t)$ is flat.

2.152. Find all functions $f(t)$ for which the curve $x = e^t$, $y = 2e^{-t}$, $z = tf(t)$ is flat.

2.153. Prove that the torsion of the curve $\vec{r}(t) = a\int\limits_0^t \vec{e}(\xi) \times \vec{e}\,'(\xi)d\xi$, where $\vec{e}(t)$ is a vector function satisfying $|\vec{e}(t)| = 1$ and $\vec{e}\,'(t) \neq 0$, is constant.

2.4. Smooth surfaces

Let $D \subset E^2$ be a closed finite domain of a plane, (u, v) be a cartesian coordinate system of the plane and (x, y, z) be cartesian system of Euclidean space E^3. If $F : D \to E^3$ is a mapping that

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in D, \tag{2.11}$$

where functions $x(u, v), y(u, v), z(u, v)$ are continuously differentiable in the domain D and

$$\text{rang} \begin{pmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{pmatrix} = 2,$$

then this mapping F is called the *smooth mapping*, and the locus $\Omega = F(D)$ is called the *smooth surface*. If the mapping F is biunique, then the surface is called the *simple* surface, and equations (2.11) is called *parametric equations* of the surface or *parametrization*.

The image of the boundary of the domain D at mapping F is called the *surface boundary*. A surface composed of a finite number of smooth surfaces is called a *piecewise smooth* surface.

Equation

$$\vec{r} = \vec{r}(u, v) \quad \text{or} \quad \vec{r} = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k}, \quad (u, v) \in D$$

is called *vectorial parametric* equation of a smooth surface.

Equations

$$\Phi(x, y, z) = 0 \quad \text{and} \quad z = f(x, y)$$

are called *implicit* and *explicit* equations of a surface correspondingly.

Coordinates (u, v) are called *curvilinear coordinates* on a surface. The curves

$$\vec{r} = \vec{r}(u, v_0) \quad \text{and} \quad \vec{r} = \vec{r}(u_0, v),$$

where $(u, v_0) \in D$, $(u_0, v) \in D$ are called the *coordinate curves* in the surface (u-curve and v-curve). The tangent vectors to the coordinate curves at a point M_0 are

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = x'_u \vec{i} + y'_u \vec{j} + z'_u \vec{k},$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v} = x'_v \vec{i} + y'_v \vec{j} + z'_v \vec{k}.$$

From the definition of a smooth surface it follows that $\vec{r}_u \nparallel \vec{r}_v$.

A plane through a point M of a surface Ω parallel to vectors $\vec{r}_u(M)$ and $\vec{r}_v(M)$, is called the *tangent plane* to the surface Ω at the point M. A straight line perpendicular to the tangent plane is called the *normal* to the surface. The directing vector of the normal can be calculated as

$$\vec{N} = \vec{r}_u \times \vec{r}_v.$$

If a surface is defined implicitly by $\Phi(x, y, z) = 0$, then the normal is

$$\vec{N} = \frac{\partial \Phi}{\partial x}\vec{i} + \frac{\partial \Phi}{\partial y}\vec{j} + \frac{\partial \Phi}{\partial z}\vec{k}.$$

Example 1. Write the parametric equation of a surface of revolution that is result of rotation of a curve $\mathcal{L}: \ x = \varphi(u), \ z = \psi(u)$, around the axis Oz.

Solution. Suppose the point $M_0 \in \mathcal{L}$ has coordinates $\big(\varphi(u), 0, \psi(u)\big)$ (fig. 2.2). Let v be an angle of revolution around the axis Oz. Then the point M_0 moves to the point $M(x, y, z)$ with coordinates

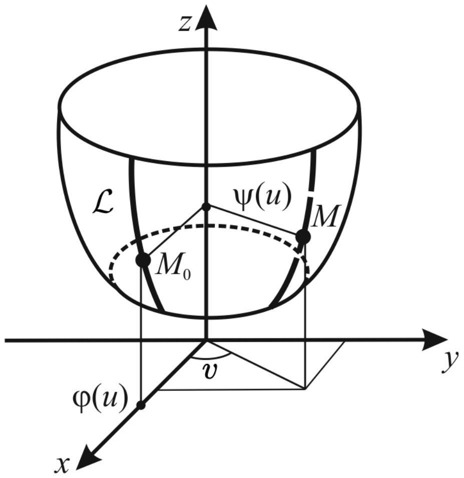

Fig. 2.2

$M\big(\varphi(u)\cos v, \varphi(u)\sin v, \psi(u)\big)$. Then the equation of the surface of revolution is

$$x = \varphi(u)\cos v, y = \varphi(u)\sin v, \quad z = \psi(u).$$

Example 2. Find the tangent plane and the normal to the pseudosphere

$$x = a\cos v\sin u, \quad y = a\sin v\sin u, \quad z = a\cos u + a\ln\tan\frac{u}{2}$$

at the point $M_0: \ u_0 = \dfrac{\pi}{4}, \ v_0 = \dfrac{\pi}{4}$.

Solution. To start we write the vectorial parametric equation of the surface:

$$\vec{r} = \vec{r}(u,v) = a\cos v\sin u\vec{i} + a\sin v\sin u\vec{j} + a\left(\cos u + \ln\tan\frac{u}{2}\right)\vec{k}.$$

Then we find two tangent vectors at the point M:

$$\vec{r}_u(u,v) = a\cos v\cos u\,\vec{i} + a\sin v\cos u\vec{j} + a\frac{\cos^2 u}{\sin u}\vec{k},$$

$$\vec{r}_v(u,v) = -a\sin v\sin u\,\vec{i} + a\cos v\sin u\vec{j}.$$

Since the point $M_0(u = u_0, v = v_0)$ and vectors $\vec{r}_u(u_0, v_0)$ and $\vec{r}_v(u_0, v_0)$ belong to the tangent plane to find the equation of the plane is

$$(\vec{r} - \vec{r}_0)\vec{r}_u(u_0, v_0)\vec{r}_v(u_0, v_0) = 0.$$

Therefore performing calculations we obtain

$$\begin{vmatrix} x - \frac{a}{2} & y - \frac{a}{2} & z - a\left(\ln\tan\frac{\pi}{8} + \frac{1}{\sqrt{2}}\right) \\ \frac{a}{2} & \frac{a}{2} & \frac{a}{\sqrt{2}} \\ -\frac{a}{2} & \frac{a}{2} & 0 \end{vmatrix} =$$

$$= -\frac{a^2}{2\sqrt{2}}\left(x - \frac{a}{2}\right) - \frac{a^2}{2\sqrt{2}}\left(y - \frac{a}{2}\right) + \frac{a^2}{2}\left[z - a\left(\ln\tan\frac{\pi}{8} + \frac{1}{\sqrt{2}}\right)\right] = 0,$$

and finally

$$\sqrt{2}z - x - y = \sqrt{2}a\ln\tan\frac{\pi}{8}.$$

The normal vector to the tangent plane is $\vec{N} = -\vec{i} - \vec{j} + \sqrt{2}\vec{k}$, thus the equation of the normal can be written as

$$\frac{x - \dfrac{a}{2}}{-1} = \frac{y - \dfrac{a}{2}}{-1} = \frac{z - a\left(\ln\tan\dfrac{\pi}{8} + \dfrac{1}{\sqrt{2}}\right)}{\sqrt{2}}.$$

Example 3. Find the tangent plane and the normal to the surface $x^2 - 2y^2 - 3z^2 - 4 = 0$ at the point $M_0(3, 1, -1)$.

Solution. The surface is given by an implicit equation, therefore it is more convenient to calculate the direction vector of the normal by the formula

$$\vec{N} = \frac{\partial \Phi}{\partial x}\vec{i} + \frac{\partial \Phi}{\partial y}\vec{j} + \frac{\partial \Phi}{\partial z}\vec{k},$$

where $\Phi(x, y, z) = x^2 - 2y^2 - 3z^2 - 4$. Then

$$\vec{N}(x, y, z) = 2x\vec{i} - 4y\vec{j} - 6z\vec{k} \quad \text{and} \quad \vec{N}(M_0) = 6\vec{i} - 4\vec{j} + 6\vec{k}.$$

Therefore the equation of the normal is

$$\frac{x-3}{3} = \frac{y-1}{-2} = \frac{z+1}{3},$$

and we can readily write the equation of the tangent plane:

$$3(x-3) - 2(y-1) + 3(z+1) = 0, \quad \text{or} \quad 3x - 2y + 3z - 4 = 0.$$

Exercises

2.154. Write a parametric equation of the torus, obtained by rotating the circle $(x-a)^2 + z^2 = b^2$, $a > b$, around the Oz axis.

2.155. Write a parametric equation of the catenoid, obtained by rotating the chain line $x = a\cosh(u/a)$, $y = 0$, $z = u$ around Oz axis.

2.156. Write a parametric equation of the pseudosphere, obtained by rotating the tractrix $x = a\sin u$, $y = 0$, $z = a\cos u + a\ln\tan\dfrac{u}{2}$ around Oz axis.

2.157. Write a parametric equation of the hyperbolic paraboloid $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2z$, if its coordinate lines are its rectilinear generators. How will these equations be written if the equation of the surface is taken in the form $z = pxy$?

2.158. Write a parametric equation of a cylindrical surface whose generatrices are parallel to the Oz axis, and the directrix is given by equation $x = \varphi(u),\ y = \psi(u),\ z = 0$.

2.159. Write a parametric equation of a cylindrical surface whose directrix has polar equation $\vec{\rho} = \vec{\rho}(u)$ and generatrices are parallel to a vector $\vec{e}$.

2.160. Write a parametric equation of a cylindrical surface whose directrix is $x = u,\ y = u^2,\ z = u^3$ and generatrices are parallel to the vector $\vec{a}(1, 2, 3)$.

2.161. Prove that the equation of a cylindrical surface whose generatrices are parallel to a vector $\vec{a}(l, m, n)$ is $f(nx - lz, ny - mz) = 0$.

2.162. Write the implicit equation of a cylindrical surface whose directrix is $x = \cos u,\ y = \sin u,\ z = 0$ and generatrices are parallel to the vector $\vec{a}(-1, 3, -2)$.

2.163. Write a parametric equation of a cylindrical surface whose directrix is $x^2 + y^2 = ay,\ z = 0$ and generatrices are parallel to a vector $\vec{a}(l, m, n)$.

2.164. Write a parametric equation of a cone whose vertex is $M(a, b, c)$ and directrix is a curve $x = \varphi(u),\ y = \psi(u),\ z = \chi(u)$.

2.165. Write an equation of a cone formed by straight lines passing through a point $M(a, b, c)$ and intersecting the parabola $y^2 = 2px,\ z = 0$.

2.166. Write an equation of a cone with the vertex $M(-1, 0, 0)$ circumscribed around the paraboloid $2y^2 + z^2 = 4x$.

2.167. Write a parametric equations of a circular cylinder in such a way that the coordinate lines are: a) helices and circles; b) helices and rectilinear generators; c) two families of helices.

2.168. Write a parametric equations of a figure formed by the tangents to the curve $\vec{\rho} = \vec{\rho}(u)$.

2.169. Write a parametric equations of a figure formed by the tangents to the helix $x = a\cos u,\ y = a\sin u,\ z = bu$. Is this figure a surface?

2.170. A helicoid of general form is a figure formed by a curve rotating around the axis and simultaneously moving in the direction of this axis, the speeds of these movements being proportional. Write the equations of a helicoid of general form.

2.171. A helicoid whose profile is a straight line intersecting the axis is called right if the line is perpendicular to the axis, and oblique otherwise.

Write down the equations of these helicoids, taking the Oz axis as the axis of rotation.

2.172. Find the equation of a surface formed by the principal normals of a helix.

2.173. A right conoid is a figure obtained by rotating a straight line around an axis orthogonal to it and simultaneously translating this line along the axis. Write the equation of the conoid whose axis coincides with the axis Oz.

2.174. Prove that the coordinates x and y of the point of an arbitrary second-order surface can always be expressed by rational functions of two parameters u and v.

Find the unit direction vector of the normal at an arbitrary point to the following surfaces.

2.175. $x = a\cos v \sin u,\ y = b\sin v \sin u,\ z = c\cos u.$

2.176. $x = (a+b\cos u)\cos v,\ y = (a+b\cos u)\sin v,\ z = b\sin u, a > b.$

2.177. $x = \sqrt{u^2+a^2}\cos v,\ y = \sqrt{u^2+a^2}\sin v,\ z = au.$

2.178. $x = \sqrt{u^2+a^2}\cos v,\ y = \sqrt{u^2+a^2}\sin v,\ z = \ln\left(u + \sqrt{u^2+a^2}\right).$

2.179. $x = \cosh u \cos v,\ y = \cosh u \sin v,\ z = u.$

2.180. $x = u^2+v^2,\ y = u^2-v^2,\ z = uv,\ |u|+|v| \neq 0.$

2.181. $x = u\cos v,\ y = u\sin v,\ z = v.$

2.182. $x = u\cos v,\ y = u\sin v,\ z = u+v.$

Write the equation of the tangent plane to the surface at the point M_0.

2.183. $x = u\cos v,\ y = u\sin v,\ z = u,\ M_0(0,1,1).$

2.184. $x = 2u - v,\ y = u^2+v^2,\ z = u^3-v^3,\ M_0(3,5,7).$

2.185. $x = u+v,\ y = u-v,\ z = uv,\ M_0(3,1,2).$

2.186. $x = u,\ y = u^2 - 2uv,\ z = u^3 - 3u^2v,\ M_0(1,3,4).$

2.187. $x = \dfrac{u}{u^2+v^2},\ y = \dfrac{v}{u^2+v^2},\ z = \dfrac{1}{u^2+v^2},\ M_0(1,1,2).$

2.188. $z = x^3 + y^3,\ M_0(1,2,9).$

2.189. $x^2+y^2+z^2 = 169,\ M_0(3,4,12).$

2.190. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1,\ M_0(x_0,y_0,z_0).$

2.191. Find the tangent plane to the surface $xyz = 1$ parallel to the plane $x + y + z = 3$.

2.192. Prove that the volume of a tetrahedron formed by any tangent plane to the surface $xyz = a^3$ and planes $x = 0,\ y = 0,\ z = 0$ is constant.

2.193. Show that the tangent plane to a cone at an arbitrary point pass through its vertex.

2.194. Show that all tangent planes to the surface $z = x^3 + y^3$ at points $M(\alpha, -\alpha, 0)$ form a sheaf of planes.

2.195. Find the points of the torus

$$x = (a + b\cos u)\cos v, \quad y = (a + b\cos u)\sin v, \quad z = b\sin u, a > b,$$

at which the normal is perpendicular to the plane $Ax + By + Cz + D = 0$.

2.196. Find the points of the torus

$$x = (3 + 2\cos u)\cos v, \quad y = (3 + 2\cos u)\sin v, \quad z = 2\sin u,$$

at which the normal is parallel to the plane $x + y + \sqrt{2}z + 5 = 0$.

2.197. Investigate the sign change of the function $\vec{n}\cdot\vec{r}$ for a torus, where $\vec{r}$ is the radius vector of the torus point, $\vec{n}$ directing vector of the normal to the torus. Find the locus of the torus satisfying $\vec{n} \cdot \vec{r} = 0$ (choose the torus center sa the origin of coordinates).

2.198. Show that the tangent plane at an arbitrary point of the surface $f(x - az, y - bz) = 0$ is parallel to a specific direction.

2.199. Show that the tangent planes of the surface $z = x\varphi(y/x)$ pass through the origin of coordinates.

2.200. Show that surfaces $z = \tan(xy)$ and $x^2 - y^2 = a$ are perpendicular each other at intersection points.

Prove that the following families of surfaces are pairwise orthogonal (λ, μ, ν are parameters of families).

2.201. $4x + y^2 + z^2 = \lambda, \quad y = \mu z, \quad y^2 + z^2 = \nu e^x.$

2.202. $x^2 + y^2 + z^2 = \lambda, \quad x^2 + y^2 + z^2 = \mu y, \quad x^2 + y^2 + z^2 = \nu z.$

2.203. $xy = \lambda z^2, \quad x^2 + y^2 + z^2 = \mu, \quad x^2 + y^2 + z^2 = \nu(x^2 - y^2).$

2.204. Show that the tangent plane to the surface $x = u\cos v,\ y = u\sin v,\ z = f(v) + au$, at any point of the line $v = c$ pass through a fixed straight line.

2.205. Prove that if all normals to a surface pass through one point, then this surface is a sphere or a region of the sphere.

2.206. Prove that the normal to the surface of revolution coincides with the principal normal of the meridian and intersects the axis of rotation.

2.207. Prove that if all normals to a surface intersect the same straight line, then the surface is a surface of revolution.

2.5. The first quadratic form of a surface

Let Ω be a smooth surface with an equation $\vec{r} = \vec{r}(u, v)$. The positive definite quadratic form of differentials $\mathrm{d}u$ and $\mathrm{d}v$

$$\mathrm{d}\vec{r}^{\,2} = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2,$$

where

$$E = \vec{r}_u^{\,2}, \quad F = \vec{r}_u \cdot \vec{r}_v, \quad G = \vec{r}_v^{\,2}, \quad EG - F^2 > 0,$$

is called *the first quadratic form of a surface*. Functions E, G, F of u and v are called coefficients of the first quadratic form of a surface.

The length of a curve $u = u(t),\ v = v(t),\ t \in [a, b]$, in a smooth surface $\vec{r} = \vec{r}(u, v)$ (or $\vec{r} = \vec{r}(u(t), v(t)),\ t \in [a, b]$) is calculated by the formula

$$l = \int\limits_a^b \sqrt{E(u(t), v(t))u'^2 + 2F(u(t), v(t))u'v' + G(u(t), v(t))v'^2}\mathrm{d}t.$$

The angle between two intersecting curves $\mathcal{L}_1 :\ u = u_1(t_1),\ v = v_1(t_1)$ and $\mathcal{L}_2 :\ u = u_2(t_2),\ v = v_2(t_2)$ in a surface $\vec{r} = \vec{r}(u, v)$ is the angle between their tangents at the intersection point and can be calculated as

$$\cos\varphi = \frac{E\mathrm{d}u_1\mathrm{d}u_2 + F(\mathrm{d}u_1\mathrm{d}v_2 + \mathrm{d}u_2\mathrm{d}v_1) + G\mathrm{d}v_1\mathrm{d}v_2}{\sqrt{E\mathrm{d}u_1^2 + 2F\mathrm{d}u_1\mathrm{d}v_1 + G\mathrm{d}v_1^2}\sqrt{E\mathrm{d}u_2^2 + 2F\mathrm{d}u_2\mathrm{d}v_2 + G\mathrm{d}v_2^2}}.$$

The coefficients of the first quadratic form are taken in the intersection point.

The area of a smooth surface $\vec{r} = \vec{r}(u, v), \quad (u, v) \in D$, where D is squarable domain is

$$S = \iint_D |\vec{r}_u \times \vec{r}_v|\mathrm{d}u\mathrm{d}v = \iint_D \sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v.$$

Example 1. Find the first quadratic form of a surface of revolution.

Solution. The parametric equation of a surface of revolution is (example 1 in section 2.4)

$$\vec{r} = \vec{r}(u, v) = \varphi(u) \cos v\vec{i} + \varphi(u) \sin v\vec{j} + \psi(u)\vec{k}.$$

To find the coefficients of the first quadratic form we calculate partial derivatives:

$$\vec{r}_u(u, v) = \varphi'(u) \cos v\vec{i} + \varphi(u)' \sin v\vec{j} + \psi(u)'\vec{k},$$

$$\vec{r}_v(u, v) = -\varphi(u) \sin v\vec{i} + \varphi(u) \cos v\vec{j}.$$

Then

$$E = \vec{r}_u^{\,2} = \varphi'(u)^2 + \psi'(u)^2, \quad F = \vec{r}_u \cdot \vec{r}_v = 0, \quad G = \vec{r}_v^{\,2} = \varphi(u)^2,$$

and finally

$$\mathrm{d}\vec{r}^{\,2} = \big(\varphi'(u)^2 + \psi'(u)^2\big)\mathrm{d}u^2 + \varphi(u)^2\mathrm{d}v^2.$$

Example 2. Express the angle between coordinate curves u and v of a surface $\vec{r} = \vec{r}(u, v)$ in terms of coefficients of the first quadratic form.

Solution. The parametric equation of the coordinate curve u is $u_1(t_1) = t_1$, $v_1(t_1) = const$, and $\mathrm{d}u_1 = \mathrm{d}t_1$, $\mathrm{d}v_1 = 0$. The similar is for the coordinate curve v: $u_2(t_2) = const$, $v_2(t_2) = t_2$, and $\mathrm{d}u_2 = 0$, $\mathrm{d}v_2 = \mathrm{d}t_2$. then cosine of the angle between coordinate curves is

$$\cos\varphi = \frac{F\mathrm{d}u_1\mathrm{d}v_2}{\sqrt{E\mathrm{d}u_1^2}\sqrt{G\mathrm{d}v_2^2}} = \frac{F}{\sqrt{EG}}.$$

It follows from this formula that a necessary and sufficient condition that at each point of a surface the coordinate curves meet at right angles, is that $F = 0$.

Example 3. Find the perimeter and angles of the triangle formed by lines

$$u = \frac{av^2}{2}, \quad u = -\frac{av^2}{2}, \quad v = 1,$$

in the surface with the first quadratic form

$$\mathrm{d}\vec{r}^{\,2} = \mathrm{d}u^2 + (u^2 + a^2)\mathrm{d}v^2.$$

Solution. Fist we find that coefficients of the first quadratic form $d\vec{r}^2 = du^2 + (u^2 + a^2)dv^2$ are

$$E = 1, \quad F = 0, \quad G = u^2 + a^2.$$

The triangle in the plane (u, v) is shown on fig. 2.3. The coordinates of vertexes can be easily found and they are

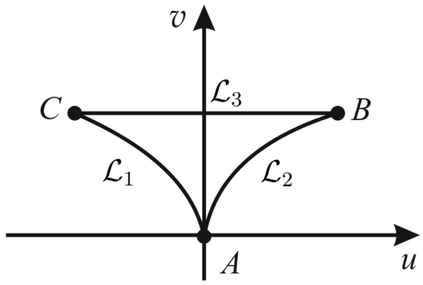

Fig. 2.3

$$A(u = 0, v = 0), \quad B\left(u = -\frac{a}{2}, v = 1\right), \quad C\left(u = \frac{a}{2}, v = 1\right).$$

Write parametric equation of sides of the triangle

$$\mathcal{L}_1: \quad u = -\frac{at_1^2}{2}, \quad v = t_1, \quad t_1 \in [0, 1], \quad u' = -at_1, \quad v' = 1;$$

$$\mathcal{L}_2: \quad u = \frac{at_2^2}{2}, \quad v = t_2, \quad t_2 \in [0, 1], \quad u' = at_2, \quad v' = 1;$$

$$\mathcal{L}_3: \quad u = t_3, \quad v = 1, \quad t_3 \in \left[-\frac{a}{2}, \frac{a}{2}\right], \quad u' = 1, \quad v' = 0.$$

Then the length of image of the curve $\mathcal{L}_1$ is

$$l_1 = \int_0^1 \sqrt{(-at_1)^2 + \left(\left(-\frac{at_1^2}{2}\right)^2 + a^2\right)} dt_1 = a \int_0^1 \sqrt{1 + t_1^2 + \frac{t_1^4}{4}} dt_1 =$$

$$= a \int_0^1 \left(1 + \frac{t_1^2}{2}\right) dt_1 = \frac{7a}{6}.$$

Similarly we find that $l_2 = \dfrac{7a}{6}$, $l_3 = a$. Thus the perimeter is $p = \dfrac{10a}{3}$.

Then find $\angle C$ between curves $\mathcal{L}_1$ and $\mathcal{L}_3$. From the curves parametrization we obtain $du_1 = -at_1 dt_1$, $dv_1 = dt_1$, $du_3 = dt_3$, $dv_3 = 0$. At the intersection point $t_1 = 1$, $E = 1$, $F = 0$, $G = \left(\dfrac{a}{2}\right)^2 + a^2 = \dfrac{5a^2}{4}$. Then

$$\cos \angle C = \frac{-a dt_1 dt_3}{\sqrt{(-a)^2 dt_1^2 + \frac{5a^2}{4} dt_1^2} \sqrt{dt_3^2}} = -\frac{2}{3}.$$

Similarly $\cos \angle B = \dfrac{2}{3}$, $\cos \angle A = 1$.

Example 4. Find the area of the curvilinear triangle formed by the curves

$$u = av, \quad u = -av, \quad v = 1,$$

in the surface with the first quadratic form

$$\mathrm{d}\vec{r}^{\,2} = \mathrm{d}u^2 + (u^2 + a^2)\mathrm{d}v^2.$$

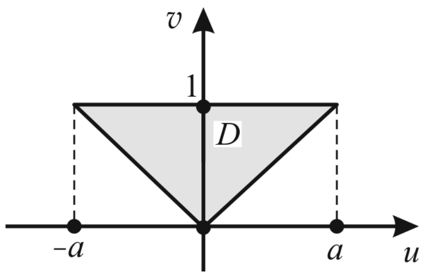

Fig. 2.4

Solution. Firstly we find that $E = 1$, $F = 0$, $G = u^2 + a^2$. The triangle in the plane (u, v) (fig. 2.4) is the integration domain D.

Then the area is calculated as follows:

$$S = \iint\limits_D \sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v =$$

$$= \iint_D \sqrt{u^2 + a^2}\mathrm{d}u\mathrm{d}v =$$

$$= 2\int\limits_0^a \sqrt{u^2 + a^2}\mathrm{d}u \int\limits_{u/a}^1 \mathrm{d}v = 2\int\limits_0^a \sqrt{u^2 + a^2}\mathrm{d}u - 2\int\limits_0^a \frac{u}{a}\sqrt{u^2 + a^2}\mathrm{d}u =$$

$$= \left[u\sqrt{u^2 + a^2} + a^2 \ln(2(u + \sqrt{u^2 + a^2})) - 2\frac{(u^2 + a^2)^{3/2}}{3a} \right]\Bigg|_0^a =$$

$$= a^2(\sqrt{2} + \ln(1 + \sqrt{2}) - \frac{2}{3}(2\sqrt{2} - 1)).$$

Exercises

Find the first quadratic form of the following surfaces.

2.208. $x = a\cos v \sin u,\ y = b\sin v \sin u,\ z = c\cos u.$

2.209. $x = \sqrt{u^2 + a^2}\cos v,\ y = \sqrt{u^2 + a^2}\sin v,\ z = au.$

2.210. $x = \sqrt{u^2 + a^2}\cos v,\ y = \sqrt{u^2 + a^2}\sin v,\ z = \ln(u + \sqrt{u^2 + a^2}).$

2.211. $x = u^2 + v^2,\ y = u^2 - v^2,\ z = uv,\ |u| + |v| \neq 0.$

2.212. $x = \sinh u \cos v,\ y = \sinh u \sin v,\ z = \cosh u.$

2.213. $x = u\cos v,\ y = u\sin v,\ z = f(u) + av.$

2.214. $z = z(x, y).$

2.215. Indicate which of the below quadratic forms can not be the first quadratic form of a surface:

a) $\mathrm{d}\vec{r}^{\,2} = \mathrm{d}u^2 + 4\mathrm{d}u\mathrm{d}v + \mathrm{d}v^2$;

b) $\mathrm{d}\vec{r}^{\,2} = \mathrm{d}u^2 + 4\mathrm{d}u\mathrm{d}v + 4\mathrm{d}v^2$;

c) $\mathrm{d}\vec{r}^{\,2} = \mathrm{d}u^2 - \mathrm{d}u\mathrm{d}v + 6\mathrm{d}v^2$;

d) $\mathrm{d}\vec{r}^{\,2} = \mathrm{d}u^2 + 4\mathrm{d}u\mathrm{d}v - 2\mathrm{d}v^2$.

2.216. Find the formulas connecting the coefficients of the first quadratic form and the expression $H = \sqrt{EG - F^2}$ in different curvilinear coordinate systems.

2.217. Show that for an appropriate choice of curvilinear coordinates in the surface of revolution, its first quadratic form can be of the form $\mathrm{d}\vec{r}^{\,2} = \mathrm{d}u^2 + G(u)\mathrm{d}v^2$.

2.218. Transform the first quadratic form of a sphere, torus, catenoid, and pseudo-sphere to the form $\mathrm{d}\vec{r}^{\,2} = \mathrm{d}\tilde{u}^2 + G(\tilde{u})\mathrm{d}\tilde{v}^2$.

2.219. A curvilinear coordinates are called isothermal if the first quadratic form of the surface in these coordinates has the form $\mathrm{d}\vec{r}^{\,2} = A(u, v)(\mathrm{d}u^2 + \mathrm{d}v^2)$. Find the isothermal coordinates of a pseudosphere.

2.220. Find the length of the curve $u = a$ in the right helicoid $x = u\cos v,\ y = u\sin v,\ z = av$ between points $M_1(u = a, v_1)$ and $M_2(u = a, v_2)$.

2.221. Find the length of the curve $v = \ln(u \pm \sqrt{u^2 + a^2})$ in the right helicoid $x = u\cos v,\ y = u\sin v,\ z = av$ between points $M_1(u_1, v_1)$ and $M_2(u_2, v_2)$.

2.222. Find the length of the curve $u = v$ in the oblique helicoid $x = u\cos v,\ y = u\sin v,\ z = u + v$ between points $M_1(u = 0, v = 0)$ and $M_2(u = 1, v = 1)$.

2.223. Find the length of the curve $u + v = 0$ in the catenoid $x = \cosh u\cos v,\ y = \cosh u\sin v,\ z = u$ between points $M_1(u_1, v_1)$ and $M_2(u_2, v_2)$.

2.224. Find the length of the curve $v^2 = u$ in the cone $x = u\cos v$, $y = u\sin v$, $z = u$ between points $M_1(u = 0, v = 0)$ and $M_2(u = 1, v = 1)$.

2.225. Find the length of the curve $v = 2u$ in the surface with the first quadratic form $\mathrm{d}\vec{r}^{\,2} = \mathrm{d}u^2 + \dfrac{1}{4}\sinh^2 u\mathrm{d}v^2$ between points $M_1(u_1, v_1)$ and $M_2(u_2, v_2)$.

2.226. Suppose two families of curves $v = \pm a \ln \tan \frac{u}{2} + C$ are given in the pseudosphere

$$x = a \cos v \sin u, \quad y = a \sin v \sin u, \quad z = a \cos u + a \ln \tan \frac{u}{2}.$$

Prove that the length of arc of curves belonging to the one family between two fixed curves of another family is constant.

2.227. Prove that the coordinate curves of the surface

$$x = u\left(3v^2 - u^2 - \frac{1}{3}\right), \quad y = v\left(3u^2 - v^2 - \frac{1}{3}\right), \quad z = 2uv$$

are orthogonal.

2.228. Find the angle between curves $u + v = 0$ and $u - v = 0$ in the oblique helicoid $x = u \cos v$, $y = u \sin v$, $z = u + v$ at the point of intersection.

2.229. Find the angle between curves $u + v = 0$ and $u - v = 0$ in the catenoid $x = \cosh u \cos v$, $y = \cosh u \sin v$, $z = u$ at the point of intersection.

2.230. Find the angle between curves $v = u + 1$ and $v = 3 - u$ in the surface $x = u \cos v$, $y = u \sin v$, $z = u^2$ at the point of intersection.

2.231. Find the angle between curves $2u = v$ and $2u = -v$ in the surface with the first quadratic form $\mathrm{d}l^2 = \mathrm{d}u^2 + \mathrm{d}v^2$ at the point of intersection.

2.232. Find the angle between rectilinear generatrices of hyperbolic paraboloid $z = axy$.

2.233. Find the perimeter, angles and the area of curvilinear triangle formed by the lines $u = \pm v^2$ and $v = 2$ in the surface $x = u \cos v$, $y = u \sin v$, $z = 2v$.

2.234. Find the perimeter, angles and the area of curvilinear triangle formed by the lines $u = \pm v$ and $v = 1$ in the surface with the first quadratic form $\mathrm{d}l^2 = \mathrm{d}u^2 + (u^2 + 1)\mathrm{d}v^2$.

2.235. Find the area of the torus

$$x = (a + b \cos u) \cos v, \quad y = (a + b \cos u) \sin v, \quad z = b \sin u, \quad a > b.$$

2.236. Find the area of the region on a sphere limited by Viviani's curve.

2.337. Prove that the area of regions of paraboloids $z = a(x^2 + y^2)/2$ and $z = axy$, having the same projection onto the plane xOy are equal.

2.238. Prove that the curves, which at each points divide the angles between the coordinate curves in half, are defined by differential equations

$$\sqrt{E}\mathrm{d}u \pm \sqrt{G}\mathrm{d}v = 0.$$

2.239. Find the equation of curves in the right helix $x = u\cos v$, $y = u\sin v$, $z = av$,bisecting the angles between the coordinate curves.

2.240. Find the equations of curves intersecting the meridians of the surface of revolution at a constant angle α (loxodromes).

2.241. Find loxodromes in a sphere.

2.242. Write a differential equation of the orthogonal trajectories of the family of curves $\varphi(u, v) = \text{const}$ in the surface.

2.243. Find orthogonal trajectories of the family of curves $u+v = \text{const}$ in the sphere $x = R\cos v \sin u$, $y = R\sin v \sin u$, $z = R\cos u$.

2.244. Find orthogonal trajectories of the family of curves $u = Ce^v$, in the oblique helicoid $x = u\cos v$, $y = u\sin v$, $z = u + v$.

2.245. Write the equation of the oblique trajectory $x = u\cos v$, $y = u\sin v$, $z = u + v$, with curves $v = \text{const}$ and their orthogonal trajectories as coordinate curves.

2.246. Find orthogonal trajectories of the rectilinear generatrices of the surface $z = axy$.

2.6. The second quadratic form of a surface

The quadratic form of differentials $\mathrm{d}u$ and $\mathrm{d}v$

$$\mathrm{d}^2\vec{r} \cdot \vec{n} = L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2,$$

where $\vec{n}$ in a unit normal vector to a surface, is called the *second quadratic form* of the surface.

Coefficients of the second quadratic form are defined by

$$L = \frac{(\vec{r}_u \times \vec{r}_v) \cdot \vec{r}_{uu}}{\sqrt{EG - F^2}}, \quad M = \frac{(\vec{r}_u \times \vec{r}_v) \cdot \vec{r}_{uv}}{\sqrt{EG - F^2}}, \quad N = \frac{(\vec{r}_u \times \vec{r}_v) \cdot \vec{r}_{vv}}{\sqrt{EG - F^2}}.$$

The *normal section* of a surface Ω at a point M for the direction $\mathrm{d}u : \mathrm{d}v$ is called the locus of intersection of the surface and a plane through the normal to the surface at this point and the tangent in the direction du : dv. The curvature of the normal section is called the *normal curvature* and is calculated by the formula

$$k_n = \frac{L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2}{E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2}.$$

The direction du : dv is called a principle direction if the normal curvature achieve its extremum (maximum or minimum) in this direction. The extremal values of the normal curvature are called *principal curvatures* of the surface and can be found from the equation

$$\begin{vmatrix} L - kE & M - kF \\ M - kF & N - kG \end{vmatrix} = 0. \tag{2.12}$$

If the equation (2.12) has two different solutions k_1 and k_2, then there are two orthogonal principal directions (ξ_1, η_1) and (ξ_2, η_2), defined by the system

$$\begin{array}{l} (L - k_iE)\xi_i + (M - k_iF)\eta_i = 0, \\ (M - k_iF)\xi_i + (N - k_iG)\eta_i = 0, \end{array} \quad i = 1, 2.$$

If the equation (2.12) has two equal solutions $k_1 = k_2$, then each direction is considered principal.

If a normal section forms an angle φ with the first principal direction, then the normal curvature k_n of this section obeys *Euler formula*

$$k_n = k_1 \cos^2 \varphi + k_2 \sin^2 \varphi.$$

Gaussian curvature of the surface is called the quantity

$$K = k_1 k_2 = \frac{LN - M^2}{EG - F^2}.$$

Mean curvature of the surface is called the quantity

$$H = \frac{k_1 + k_2}{2} = \frac{EN + GL - 2FM}{2(EG - F^2)}.$$

Example 1. Find principal directions and principal curvatures of the right helicoid $x = u\cos v,\ y = u\sin v,\ z = av$.

Solution. The vectorial parametric equation of the surface is:

$$\vec{r} = \vec{r}(u, v) = u\cos v\vec{i} + u\sin v\vec{j} + av\vec{k}.$$

Find first partial derivatives of $\vec{r}(u,v)$ and then coefficients of the first quadratic form and the normal vector $\vec{r}_u \times \vec{r}_v$ to the surface and its norm:

$$\vec{r}_u(u,v) = \cos v\vec{i} + \sin v\vec{j},$$

$$\vec{r}_v(u,v) = -u\sin v\vec{i} + u\cos v\vec{j} + a\vec{k},$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & a \end{vmatrix} = a\sin v\vec{i} - a\cos v\vec{j} + u\vec{k},$$

$$E = \vec{r}_u^{\,2} = 1, \quad F = \vec{r}_u \cdot \vec{r}_v = 0, \quad G = \vec{r}_v^{\,2} = u^2 + a^2.$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{EG - F^2} = \sqrt{u^2 + a^2}.$$

To calculate coefficients of the second quadratic form we find the second partial derivatives of $\vec{r}(u,v)$:

$$\vec{r}_{uu}(u,v) = \vec{0},$$

$$\vec{r}_{uv}(u,v) = -\sin v\vec{i} + \cos v\vec{j},$$

$$\vec{r}_{vv}(u,v) = -u\cos v\vec{i} - u\sin v\vec{j}.$$

Thus

$$L = \frac{(\vec{r}_u \times \vec{r}_v)\cdot \vec{r}_{uu}}{\sqrt{EG - F^2}} = 0,$$

$$M = \frac{(\vec{r}_u \times \vec{r}_v)\cdot \vec{r}_{uv}}{\sqrt{EG - F^2}} = \frac{-a\sin^2 v - a\cos^2 v}{\sqrt{u^2 + a^2}} = -\frac{a}{\sqrt{u^2 + a^2}},$$

$$N = \frac{(\vec{r}_u \times \vec{r}_v)\cdot \vec{r}_{vv}}{\sqrt{EG - F^2}} = \frac{-ua\sin v\cos v + ua\cos v\sin v}{\sqrt{u^2 + a^2}} = 0.$$

Constructing the corresponding determinant and equating it to zero

$$\begin{vmatrix} L - kE & M - kF \\ M - kF & N - kG \end{vmatrix} = \begin{vmatrix} -k & -\dfrac{a}{\sqrt{u^2 + a^2}} \\ -\dfrac{a}{\sqrt{u^2 + a^2}} & -k(u^2 + a^2) \end{vmatrix} =$$

$$= k^2(u^2 + a^2) - \frac{a^2}{u^2 + a^2} = 0,$$

we find

$$k_1 = -k_2 = \frac{a}{u^2 + a^2}.$$

The first principal direction is the solution of the equation

$$(L - k_1E)\xi_1 + (M - k_1F)\eta_1 = -\frac{a}{u^2 + a^2}\xi_1 - \frac{a}{\sqrt{u^2 + a^2}}\eta_1 = 0,$$

thus

$$\frac{\xi_1}{\eta_1} = \frac{\mathrm{d}u_1}{\mathrm{d}v_1} = -\sqrt{u^2 + a^2}.$$

Similarly we find the second principal direction:

$$\frac{\xi_2}{\eta_2} = \frac{\mathrm{d}u_2}{\mathrm{d}v_2} = \sqrt{u^2 + a^2}.$$

Example 2. Find gaussian and mean curvatures of the pseudosphere

$$x = a\cos v\sin u, \quad y = a\sin v\sin u, \quad z = a\cos u + a\ln\tan\frac{u}{2}$$

at an arbitrary point.

Solution. In the beginning we write the vectorial parametric equation:

$$\vec{r} = \vec{r}(u, v) = a\cos v\sin u\vec{i} + a\sin v\sin u\vec{j} + a\left(\cos u + \ln\tan\frac{u}{2}\right)\vec{k}.$$

Then we calculate the first and second partial derivatives of the function $\vec{r}(u, v)$:

$$\vec{r}_u(u, v) = a\cos v\cos u\vec{i} + a\sin v\cos u\vec{j} + a\frac{\cos^2 u}{\sin u}\vec{k},$$

$$\vec{r}_v(u, v) = -a\sin v\sin u\vec{i} + a\cos v\sin u\vec{j},$$

$$\vec{r}_{uu}(u, v) = -a\cos v\sin u\vec{i} + a\sin v\sin u\vec{j} - a\frac{\cos u}{\sin^2 u}(\sin^2 u + 1)\vec{k},$$

$$\vec{r}_{uv}(u, v) = -a\sin v\cos u\vec{i} + a\cos v\cos u\vec{j},$$

$$\vec{r}_{vv}(u, v) = -a\cos v\sin u\vec{i} - a\sin v\sin u\vec{j}.$$

Combining found derivatives as necessary we find coefficients of the first and second quadratic form:

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos v\cos u & a\sin v\cos u & a\dfrac{\cos^2 u}{\sin u} \\ -a\sin v\sin u & a\cos v\sin u & 0 \end{vmatrix} =$$

$$= -a^2\cos^2 u\cos v\vec{i} - a^2\cos^2 u\sin v\vec{j} + a^2\cos u\sin u\vec{k},$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{EG - F^2} = a^2|\cos u|,$$

$$E = \vec{r}_u^{\,2} = a^2\cot^2 u, \quad F = \vec{r}_u \cdot \vec{r}_v = 0, \quad G = \vec{r}_v^{\,2} = a^2\sin^2 u.$$

$$L = \frac{(\vec{r}_u \times \vec{r}_v)\cdot\vec{r}_{uu}}{\sqrt{EG - F^2}} = -a\,\mathrm{sgn}(\cos u)\cot u,$$

$$M = \frac{(\vec{r}_u \times \vec{r}_v)\cdot\vec{r}_{uv}}{\sqrt{EG - F^2}} = 0,$$

$$N = \frac{(\vec{r}_u \times \vec{r}_v)\cdot\vec{r}_{vv}}{\sqrt{EG - F^2}} = a\,\mathrm{sgn}(\cos u)\cos u\sin u.$$

Now we can find the gaussian curvature of the pseudosphere:

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{1}{a^2}.$$

As it is seen the pseudosphere is a surface with constant negative gaussian curvature.

And finally we calculate the mean curvature:

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} = \frac{\mathrm{sgn}(\cos u)}{2a}(\cot u - \tan u).$$

Exercises

2.247. Prove that the second quadratic form is identically zero then the surface is a plain or its part.

2.248. Express the normal curvatures of a surface in the direction of coordinate curves in terms of coefficients of the first and the second quadratic forms.

2.249. Find the normal curvature of coordinate curves of the catenoid

$$x = \sqrt{u^2 + a^2}\cos v, \quad y = \sqrt{u^2 + a^2}\sin v, \quad z = \ln(u + \sqrt{u^2 + a^2}).$$

2.250. Calculate the curvature of the normal section of the surface

$$x = u^2 + v^2, \quad y = u^2 - v^2, \quad z = uv$$

at the point $M(2, 0, 1)$, passing through the tangent to the curve $v = u^2$.

Find the principal curvatures of the surfaces.

2.251. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ at the vertexes of the two-sheeted hyperboloid.

2.252. $z = xy$ at the point $M(1,1,1)$.

2.253. $\dfrac{x^2}{p}-\dfrac{y^2}{q}=2z$ at the point $M(0,0,0)$.

2.254. $x = u^2+v^2,\ y = u^2-v^2,\ z = uv$, at the point $M(2,0,1)$.

2.255. $x = \cos v - u\sin v,\ y = \sin v + u\cos v,\ z = u+v$ at an arbitrary point.

2.256. Prove that the principal directions of a right helicoid bisect the angles between the directions of generatrix and helix.

Find the gaussian and mean curvatures.

2.257. $x = a\cos v\sin u,\ y = a\sin v\sin u,\ z = c\cos u$.

2.258. $x = (a+b\cos u)\cos v,\ y = (a+b\cos u)\sin v,\ z = b\sin u, a > b$.

2.259. $x = \cosh u\cos v,\ y = \cosh u\sin v,\ z = \sinh u$.

2.260. $x = \sinh u\cos v,\ y = \sinh u\sin v,\ z = \cosh u$.

2.261. $x = u\cos v,\ y = u\sin v,\ z = u^2$.

2.262. $x = \cosh u\cos v,\ y = \cosh u\sin v,\ z = u$.

2.263. $x = u\cos v,\ y = u\sin v,\ z = v$.

2.264. A surface of revolution $z = f(\sqrt{x^2+y^2})$.

2.265. Find the gaussian curvature of a surface: a) $F(x,y,z) = 0$; b) $z = z(x,y)$.

2.266. Choose the function $f(u)$ in the parametrization of the surface of revolution

$$x = u, \quad y = f(u)\cos v, \quad z = f(u)\sin v, \quad f(u) > 0.$$

in such a way that the mean curvature of the surface is equal zero.

2.267. Find the gaussian curvature of the surface with the first quadratic form $\mathrm{d}\vec{r}^2 = \mathrm{d}u^2 + e^{2u}\mathrm{d}v^2$.

2.268. Prove that the gaussian curvature of the surface with the first quadratic form $\mathrm{d}\vec{r}^2 = \dfrac{\mathrm{d}u^2+\mathrm{d}v^2}{(u^2+v^2+c^2)^2}$ is constant.

2.269. Find the expression for the Gaussian curvature of a surface referred to isothermal coordinates.

2.270. Show that if the first quadratic of a surface is as

$$\mathrm{d}l^2 = \mathrm{d}u^2 + 2\cos\omega(u,v)\mathrm{d}u\mathrm{d}v + \mathrm{d}v^2,$$

then its gaussian curvature is

$$K = \frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u\partial v}.$$

2.271. There are n equiangular straight lines in the tangent plane passing through a point M of a surface. Show that

$$\frac{1}{n}(k_1 + k_2 + ... + k_n) = H,$$

where k_i, $i = \overline{1,n}$, are the normal curvatures of the lines in the surface contacting these straight lines.

2.272. Suppose that surfaces Ω_1 and Ω_2 intersect along the curve $\mathcal{L}$. Let k be the curvature of this curve at the point M, and $\lambda_i, i = 1, 2$, are the normal curvatures of the surfaces Ω_i at the point M for the direction of the curve $\mathcal{L}$, θ is the angle between the normals to the surfaces Ω_i at the point M. Prove that

$$k^2\sin\theta = \lambda_1^2 + \lambda_2^2 - \lambda_1\lambda_2\cos\theta.$$

Chapter 3

Scalar and vector fields

3.1. Scalar fields

Let D be a domain in the three-dimensional Euclidean space E^3. If a scalar u is uniquely assigned to each point M of the domain D the function $u(M)$ is said to define the *scalar field* in D. Another notation for the scalar field can be $u(\vec{r})$ or $u(x_1, x_2, x_3)$, if any coordinate system is given.

A scalar field $u(M)$ is *differentiable* at M_0, if the increment of the field Δu in the neighborhood of the point M_0 can be written as

$$\Delta u = u(M) - u(M_0) = \vec{c} \cdot \Delta\vec{r} + o(|\Delta\vec{r}|), \quad \text{while} \quad M \to M_0 \ (\Delta\vec{r} \to 0),$$

where the vector $\vec{c}$ does not depend on $\Delta\vec{r}$. The vector $\vec{c}$ is called the *derivative of the scalar field* at the point M_0 or the *gradient* of the scalar field and is denoted $\vec{c} = \operatorname{grad} u$. The scalar field is differentiable in the domain G, if it is differentiable at each point of D.

In cartesian coordinates (x, y, z) the gradient is

$$\operatorname{grad} u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}.$$

Let $\vec{l}$ be a unit vector. The *derivative* of the scalar field $u(M)$ *in the direction* of $\vec{l}$ at M_0 is the limit

$$\frac{\partial u}{\partial l} = \lim_{\varepsilon \to 0} \frac{u(M) - u(M_0)}{\varepsilon}, \quad \overrightarrow{M_0M} = \vec{l}\varepsilon.$$

The directional derivative $\dfrac{\partial u}{\partial l}$ shows the rate of change of the scalar field u in the direction $\vec{l}$ and is the projection of the gradient onto this direction:

$$\frac{\partial u}{\partial l} = \vec{l} \cdot \operatorname{grad} u.$$

The gradient of a scalar field is directed along the direction in which the directional derivative is maximal. The derivative of a scalar field in the direction of a curve is equal to the derivative in the direction of its tangent.

A scalar field may be described graphically by a family of *level surfaces* which are the surfaces of constant value of the scalar field $u(M)$, i. e. $u(x, y, z) = \text{const}$. If a scalar field is defined on a plane, the equation $u(x, y) = \text{const}$ defines a family of lines which are called *level curves.*

The gradient of a scalar field is perpendicular to a level surface at an arbitrary point of this surface.

Example 1. Draw level surfaces of the scalar field

$$u(x, y, z) = x^2 + y^2 - z^2.$$

Solution. The equation of a family of level surfaces is

$$u(x, y, z) = x^2 + y^2 - z^2 = C,$$

where $C = \text{const}$. If $C = 0$, then the surface is the cone which axis is Oz. For $C > 0$ there is the family of coaxial one-sheet hyperboloids, and for $C < 0$ the level surfaces are the family of coaxial two-sheet hyperboloids. The axis of hyperboloids is Oz (fig. 3.1).

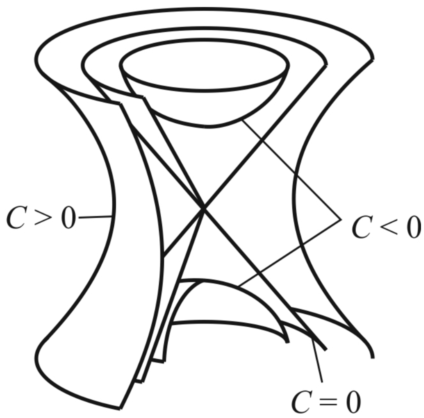

Fig. 3.1

Example 2. Compute the gradient of the scalar field $u = x^3y^2z$.
Solution. To calculate the gradient we us the formula

$$\operatorname{grad} u = \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}.$$

The partial derivatives of the field u are

$$\frac{\partial u}{\partial x} = 3x^2y^2z, \quad \frac{\partial u}{\partial y} = 2x^3yz, \quad \frac{\partial u}{\partial z} = x^3y^2.$$

Therefore the gradient is

$$\operatorname{grad} u = 3x^2y^2z\vec{i} + 2x^3yz\vec{j} + x^3y^2\vec{k}.$$

Example 3. Find the derivative of the scalar field $u = xe^y + ye^x - z^2$ at the point $M_0(3,0,2)$ in the direction to the point $M_1(4,1,3)$.

Solution. First we find the unit vector $\vec{l}$ specifying the direction of differentiation:

$$\vec{l} = \frac{\overrightarrow{M_0M_1}}{\left|\overrightarrow{M_0M_1}\right|} = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}.$$

Then we calculate the gradient of the scalar field at any point

$$\operatorname{grad} u = (e^y + ye^x)\vec{i} + (xe^y + e^x)\vec{j} - 2z\vec{k},$$

and at the point $M_0(3,0,2)$:

$$\operatorname{grad} u(M_0) = \vec{i} + (3 + e^3)\vec{j} - 4\vec{k}.$$

Therefore the directional derivative is

$$\frac{\partial u}{\partial l} = \vec{l} \cdot \operatorname{grad} u = \left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}\right) \cdot (\vec{i} + (3 + e^3)\vec{j} - 4\vec{k}) = \frac{e^3}{\sqrt{3}}.$$

Example 4. Find the derivative of the scalar field $u = \ln(x^2+y^2+z^2)$ at the point $M_0\left(0, R, \frac{\pi a}{2}\right)$ in the direction of the curve $\mathcal{L} : x = R\cos t$, $y = R\sin t,\ z = at$.

Solution. Since the derivative of a scalar field in the direction of a curve is the derivative in the direction of its tangent we first find the unit tangent vector to the curve at the point M_0. Analysing the parametric equation of the curve we conclude that the point M_0 corresponds to the parameter value $t_0 = \frac{\pi}{2}$. Therefore

$$\vec{\tau}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = -\vec{i}\frac{R}{\sqrt{R^2+a^2}}\sin t + \vec{j}\frac{R}{\sqrt{R^2+a^2}}\cos t + \frac{a}{\sqrt{R^2+a^2}}\vec{k}.$$

$$\vec{\tau}\left(\frac{\pi}{2}\right) = -\frac{R}{\sqrt{R^2+a^2}}\vec{i} + \frac{a}{\sqrt{R^2+a^2}}\vec{k}.$$

Then we find the gradient of the scalar field u:

$$\operatorname{grad} u = \frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z^2}\vec{k},$$

$$\operatorname{grad} u(M_0) = \frac{8R}{4R^2+\pi^2 a^2}\vec{j} + \frac{4\pi a}{4R^2+\pi^2 a^2}\vec{k}.$$

Finally the directional derivative is

$$\frac{\partial u}{\partial \tau} = \vec{\tau}\cdot \operatorname{grad} u = \frac{1}{\sqrt{R^2+a^2}}\frac{4\pi a}{4R^2+\pi^2 a^2}.$$

Exercises

Find level lines of the following scalar fields defined on a plane.

3.1. $u = 2x - y$.

3.2. $u = e^{x^2-y^2}$.

3.3. $u = \ln\sqrt{\dfrac{y}{2x}}$.

3.4. $u = \dfrac{2x}{x^2+y^2}$.

3.5. $u = \dfrac{y^2}{x}$.

3.6. $u = \dfrac{2x-y+1}{x^2}$.

3.7. $u = \dfrac{x^2}{y}$.

3.8. $u = \ln(x^2+y^2)$.

Find level surfaces of the following scalar fields.

3.9. $u = x^2 + y^2 - z$.

3.10. $u = \ln|\vec{r}|$.

3.11. $u = x - y^2 + z^2$.

3.12. $u = \dfrac{x^2+y^2}{z}$.

3.13. $u = 3^{x+2y-z}$.

3.14. $u = e^{\vec{a}\cdot\vec{r}}$, where $\vec{a}$ is a constant, $\vec{r}$ is a radius-vector.

3.15. $u = \dfrac{\vec{a}\cdot\vec{r}}{\vec{b}\cdot\vec{r}}$, where $\vec{a}, \vec{b}$ are constant vectors, $\vec{r}$ is a radius-vector.

3.16. $u = \sqrt{x^2+y^2+(z+8)^2} + \sqrt{x^2+y^2+(z-8)^2}$.

3.17. Find level lines of implicitly defined scalar field: $u + x\ln u + y = 0$.

Prove the following formulae.

3.18. $\operatorname{grad} c = \vec{0}, \quad c = \text{const}.$

3.19. $\operatorname{grad} cu = c \operatorname{grad} u, \quad c = \text{const}.$

3.20. $\operatorname{grad}(u + v) = \operatorname{grad} u + \operatorname{grad} v.$

3.21. $\operatorname{grad}(uv) = v \operatorname{grad} u + u \operatorname{grad} v.$

3.22. $\operatorname{grad} \dfrac{u}{v} = \dfrac{v \operatorname{grad} u - u \operatorname{grad} v}{v^2}.$

3.23. $\operatorname{grad} f(u) = f'(u) \operatorname{grad} u.$

3.24. $\operatorname{grad} f(u, v, w) = \dfrac{\partial f}{\partial u} \operatorname{grad} u + \dfrac{\partial f}{\partial v} \operatorname{grad} v + \dfrac{\partial f}{\partial w} \operatorname{grad} w.$

Find the gradient of the following scalar fields.

3.25. $u = \ln(x^2 + y^2 + z^2).$

3.26. $u = x^3 + y^3 + z^3 - 3xyz.$

3.27. $u = ze^{x^2+y^2+z^2}.$

3.28. $u = (x - y)(y - z)(z - x).$

3.29. $u = xyze^{x+y+z}.$

3.30. $u = (x - 1)(y - 2)(z - 3).$

3.31. $u = \arctan \dfrac{x + y + z - xyz}{1 - xy - yz - xz}.$

Find the gradient of the following implicitly defined scalar fields.

3.32. $u^3 - 3xyu = a^2.$

3.33. $x + y + u = e^u.$

3.34. $x + y + u = e^{-(x+y+u)}.$

Find the derivative of a scalar field u at a point M_0 in the direction to a point M_1.

3.35. $u = xyz,\ M_0(1, -1, 1),\ M_1(2, 3, 1).$

3.36. $u = x^2y + xz^2 - 2,\ M_0(1, 1, -1),\ M_1(2, -1, 3).$

3.37. $u = \sqrt{x^2 + y^2 + z^2},\ M_0(1, 1, 1),\ M_1(3, 2, 1).$

3.38. $u = \dfrac{x}{y} - \dfrac{y}{x},\ M_0(1, 1),\ M_1(4, 5).$

Find the derivative of a scalar field u at a point M_0 in the direction of a curve $\mathcal{L}$.

3.39. $u = x^2 + y^2,\ M_0(1, 2),\ \mathcal{L} : x^2 + y^2 = 5.$

3.40. $u = 2xy + y^2,\ M_0(\sqrt{2}, 1),\ \mathcal{L} : \dfrac{x^4}{4} + \dfrac{y^2}{2} = 1.$

3.41. $u = x^2 - y^2,\ M_0(5, 4),\ \mathcal{L} : x^2 - y^2 = 9.$

3.42. $u = \ln(x^2 + y^2),\ M_0(1, 2),\ \mathcal{L} : y^2 = 4x.$

3.43. $u = \arctan \dfrac{y}{x},\ M_0(2, -2),\ \mathcal{L} : x^2 + y^2 = 4x.$

3.44. $u = \ln(xy + yz + xz),\ M_0(0, 1, 1),\ \mathcal{L} : x = \cos t, y = \sin t, z = 1.$

Find the points where directional derivative of a scalar field u is equal to zero in any direction.

3.45. $u = x^3 + y^3 - 3xy.$

3.46. $u = x^2 + 2y^2 + 3z^2 + xy + 3x - 2y - 6z.$

3.47. $u = 2y^2 + z^2 - xy - yz + 2x.$

3.48. $u = x^2 + 2y^2 + 3z^2 - xz - yz - xy.$

3.49. Find the angle between the gradients of the field $u = (x + y)e^{x+y}$ at the points $M_1(0, 0)$ and $M_2(1, 1)$.

3.50. Find the angle between the gradients of the field $u = \dfrac{x}{x^2 + y^2 + z^2}$ at the points $M_1(1, 2, 2)$ and $M_2(-3, 1, 0)$.

3.51. Find the angle between the gradient of the fields $u = \sqrt{x^2 + y^2 + z^2}$ and $u = \ln(x^2 + y^2 + z^2)$ at the point $M_0(0, 0, 1)$.

3.52. Find the derivative of the scalar field

$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

at the point $M(x, y, z)$ in the direction of its radius-vector $\vec{r}$. In what direction the derivative is equal to the norm od the gradient?

3.53. Find the derivative of the scalar field $u = yze^x$ in the direction of its gradient.

3.54. Find the derivative of a scalar field $u = u(x, y, z)$ in the direction of the gradient of a function $v = v(x, y, z)$. In what direction is it zero?

3.2. Vector fields

If a vector $\vec{a}$ is uniquely assigned to each point M of a domain $D \subset E^3$ the vector function $\vec{(M)}$ is said to define the *vector field* in D. In Cartesian coordinate system (x, y, z) whose orthonormal basis is $(\vec{i}, \vec{j}, \vec{k})$ the vector field can be written as

$$\vec{a}(x, y, z) = a_x(x, y, z)\vec{i} + a_y(x, y, z)\vec{j} + a_z(x, y, z)\vec{k}.$$

A vector field may be described graphically by *vector lines* which are the lines at each point M of which the tangent is collinear to the vector $\vec{a}(M)$. Vector lines are the solution of the system of differential equations

$$\frac{\mathrm{d}x}{a_x} = \frac{\mathrm{d}y}{a_y} = \frac{\mathrm{d}z}{a_z}.$$

A vector field $\vec{a}(M)$ is *differentiable* at M_0, if the increment of the field $\Delta\vec{a}$ in the neighborhood of the point M_0 can be written as

$$\Delta\vec{a} = \vec{a}(M) - \vec{a}(M_0) = A\Delta\vec{r} + \vec{o}(|\Delta\vec{r}|), \quad M \to M_0 \ (\Delta\vec{r} \to 0),$$

where A is a linear operator that does not depend on $\Delta\vec{r}$. The linear operator a A is called the *derivative of the vector field.* The vector field is differentiable in the domain G, if it is differentiable at each point of D.

In Cartesian coordinates the operator A is

$$A_{ij} = \frac{\partial a_i}{\partial x_j}, \quad i, j = 1, 2, 3.$$

The *derivative* of a vector field $\vec{a}(M)$ *in the direction* of an unit vector $\vec{l}$ at a point M_0 is called the limit:

$$\frac{\partial\vec{a}}{\partial l} = \lim_{\varepsilon\to 0} \frac{\vec{a}(M) - \vec{a}(M_0)}{\varepsilon}, \quad \overrightarrow{M_0M} = \vec{l}\varepsilon.$$

In Cartesian coordinates the directional derivative is calculated as follows

$$\frac{\partial\vec{a}}{\partial l} = \left(l_x\frac{\partial}{\partial x} + l_y\frac{\partial}{\partial y} + l_z\frac{\partial}{\partial z}\right)\vec{a} = l_x\frac{\partial\vec{a}}{\partial x} + l_y\frac{\partial\vec{a}}{\partial y} + l_z\frac{\partial\vec{a}}{\partial z}.$$

The *divergence* of a vector field $\vec{a}$ is the scalar field $\operatorname{div}\vec{a}$ that in cartesian coordinates is given by the formula

$$\operatorname{div}\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}.$$

The *curl* of a vector field $\vec{a}$ is the vector field $\operatorname{curl}\vec{a}$ that in cartesian coordinates is given by the formula[1]

$$\operatorname{curl}\vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{k}.$$

It is more convenient to rewrite the above formula as a formal determinant

$$\operatorname{curl}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}.$$

A vector field is uniquely specified by giving its divergence and curl within a domain and its normal component over the boundary of the domain, a result known as Helmholtz's theorem.

Example 1. Find vector lines of the field $\vec{a} = (x^2+1)\vec{i} + (y^2+1)\vec{j}$.

Solution. Since the vector field is flat the system of differential equation for vector lines reduces to one equation

$$\frac{\mathrm{d}x}{x^2+1} = \frac{\mathrm{d}y}{y^2+1}.$$

The general solution gives a one-parameter family of vector lines. The given equation is a differential equation with separated variables, therefor the direct integration yields

$$\arctan x = \arctan y + C.$$

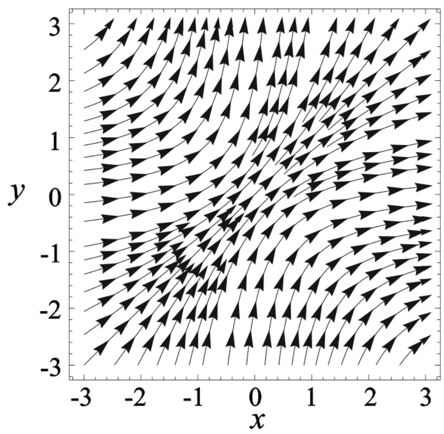

Fig. 3.2

Changing the value of the constant C we obtain the family of vector lines presented on fig. 3.2.

Example 2. Find the divergence of the vector field

$$\vec{a} = xyz\vec{i} + (2x+3y-z)\vec{j} + (x^2+z^2)\vec{k}.$$

Solution. The divergence in Cartesian coordinate is

$$\operatorname{div}\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}.$$

[1]There is another notation $\operatorname{rot}\vec{a} \equiv \operatorname{curl}\vec{a}$.

Then

$$\operatorname{div}\vec{a} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(2x + 3y - z) + \frac{\partial}{\partial z}(x^2 + z^2) = yz + 3 + 2z.$$

Example 3. Find the curl of the vector field

$$\vec{a} = (z - y)\,\vec{i} + (z - x)\,\vec{j} + (y - x)\,\vec{k}.$$

Solution. Expanding the formal determinant we obtain

$$\operatorname{curl}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z - y & z - x & y - x \end{vmatrix} =$$

$$= \vec{i}\,(1 - 1) - \vec{j}\,((-1) - 1) + \vec{k}((-1) - (-1)) = 2\vec{j}.$$

Exercises

Find vector lines of the following vector fields.

3.55. $\vec{a} = x\vec{i} + 2y\vec{j}$.

3.56. $\vec{a} = x^2\vec{i} + y^2\vec{j}$.

3.57. $\vec{a} = z\vec{j} - y\vec{k}$.

3.58. $\vec{a} = x\vec{i} + z\vec{k}$.

3.59. $\vec{a} = x\vec{i} + y\vec{j} + 2z\vec{k}$.

3.60. $\vec{a} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$.

3.61. $\vec{a} = (z - y)\vec{i} + (x - z)\vec{j} + (y - x)\vec{k}$.

3.62. $\vec{a} = x^2\vec{i} - y^3\vec{j} + z^2\vec{k}$.

Prove the following formulas.

3.63. $\operatorname{div}\vec{c} = 0, \quad \vec{c} = \text{const}$.

3.64. $\operatorname{div}(c\vec{a}) = c\operatorname{div}\vec{a}, \quad c = \text{const}$.

3.65. $\operatorname{div}(\vec{a} + \vec{b}) = \operatorname{div}\vec{a} + \operatorname{div}\vec{b}$.

3.66. $\operatorname{curl}\vec{c} = 0, \quad \vec{c} = \text{const}$.

3.67. $\operatorname{curl}(c\vec{a}) = c\operatorname{curl}\vec{a}, \quad c = \text{const}$.

3.68. $\operatorname{curl}(\vec{a} + \vec{b}) = \operatorname{curl}\vec{a} + \operatorname{curl}\vec{b}$.

Find the divergence of the following vector fields.

3.69. $\vec{a} = (x-y)(y-z)\vec{i} + (y-z)(z-x)\vec{j} + (z-x)(x-y)\vec{k}$.

3.70. $\vec{a} = (y^2+z^2)(x+y)\vec{i} + (z^2+x^2)(y+z)\vec{j} + (x^2+y^2)(z+x)\vec{k}$.

3.71. $\vec{a} = (x^2+y^2)(y-z)\vec{i} + (y^2+z^2)(z-x)\vec{j} + (z^2+x^2)(x-y)\vec{k}$.

3.72. $\vec{a} = f_1(y,z)\vec{i} + f_2(x,z)\vec{j} + f_3(x,y)\vec{k}$.

3.73. $\vec{a} = (x+f_1(y,z))\vec{i} + (y+f_2(x,z))\vec{j} + (z+f_3(x,y))\vec{k}$.

3.74. $\vec{a} = xf_1(y,z)\vec{i} + yf_2(x,z)\vec{j} + zf_3(x,y)\vec{k}$.

Find the curl of the following vector fields.

3.75. $\vec{a} = yz\vec{i} + zx\vec{j} + xy\vec{k}$.

3.76. $\vec{a} = yz\vec{i} + z(x+2y)\vec{j} + y(x+y)\vec{k}$.

3.77. $\vec{a} = \frac{y}{x^2}\vec{j} - \frac{1}{x}\vec{k}$.

3.78. $\vec{a} = y^2z^3\vec{i} + 2xzy^2\vec{j} + 3xy^2z^2\vec{k}$.

3.79. $\vec{a} = yz^2\vec{i} - x\vec{k}$.

3.80. $\vec{a} = \frac{y}{x}\vec{i} + \frac{z}{y}\vec{j} + \frac{x}{z}\vec{k}$.

3.81. $\vec{a} = y^2z\vec{i} + z^2x\vec{j} + x^2yk$.

3.82. $\vec{a} = xyz\vec{i} + (2x+3y-z)\vec{j} + (x^2+z^2)\vec{k}$.

3.83. Find the derivatives of the vector field $\vec{a} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ in the directions $\vec{l}_1 = \vec{i}$, $\vec{l}_2 = \frac{\vec{i}+\vec{j}}{\sqrt{2}}$, $\vec{l}_3 = \frac{\vec{j}+\vec{k}}{\sqrt{2}}$, $\vec{l}_4 = \frac{1}{\sqrt{3}}(\vec{i}+\vec{j}+\vec{k})$.

3.3. The operator $\vec{\nabla}$

The operator $\vec{\nabla}$, read "nabla" of "del", is the operator that combines properties of a vector and derivative and in cartesian coordinates takes the form

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}.$$

Using the $\vec{\nabla}$ operator the vector differential operations can be written as follows:

$$\operatorname{grad} u = \vec{\nabla} u, \quad \operatorname{div} \vec{a} = \vec{\nabla} \cdot \vec{a}, \quad \operatorname{curl} \vec{a} = \vec{\nabla} \times \vec{a},$$

$$\frac{\partial u}{\partial l} = (\vec{l} \cdot \vec{\nabla})u, \quad \frac{\partial \vec{a}}{\partial l} = (\vec{l} \cdot \vec{\nabla})\vec{a}.$$

There are two approaches to manipulate expressions with the operator $\vec{\nabla}$. The formal approach is based on the following steps.

1. If $\vec{\nabla}$ acts on a product of fields, then primarily it is considered as a differential operator.

2. Quantities under action of $\vec{\nabla}$ in complicated formula are designated by a downarrow, for example in the expression $\vec{\nabla}\cdot(\overset{\downarrow}{\vec{a}}\times\vec{b})$ the operator $\vec{\nabla}$ acts only on the vector field $\vec{a}$.

3. In final result those quantities on which $\vec{\nabla}$ does not act are placed to the left of $\vec{\nabla}$ using the rules of vector algebra:

   1) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$;
   2) $(\alpha\vec{a})\cdot\vec{b}=\vec{a}\cdot(\alpha\vec{b})$;
   3) $(\alpha\vec{a})\cdot\vec{b}=\alpha(\vec{a}\cdot\vec{b})$;
   4) $\alpha(\beta\vec{a})=(\alpha\beta)\vec{a}$;
   5) $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$;
   6) $(\alpha\vec{a})\times\vec{b}=\vec{a}\times(\alpha\vec{b})$;
   7) $(\alpha\vec{a})\times\vec{b}=\alpha(\vec{a}\times\vec{b})$;
   8) $(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c})$;
   9) $\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(a\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$;
   10) $(\vec{a}\times\vec{b})\times\vec{c}=\vec{b}(a\cdot\vec{c})-\vec{a}(\vec{b}\cdot\vec{c})$;
   11) $(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c})$

   for all vectors $\vec{a},\vec{b},\vec{c}$ and for all real numbers α and β.

The second approach of formulas manipulation with $\vec{\nabla}$ is to perform calculations in cartesian coordinates system. But they can be very cumbersome that overcomes by using index notation of vector algebraic operations and summation convention (section 1.7):

coordinates:	$x_1=x,\quad x_2=y,\quad x_3=z;$
basis:	$\vec{e}_1=\vec{i},\quad \vec{e}_2=\vec{j},\quad \vec{e}_3=\vec{k};$
decomposition of a vector:	$\vec{a}=a_i\vec{e}_i;$
dot product:	$\vec{a}\cdot\vec{b}=a_ib_i;$
cross product:	$\vec{a}\times\vec{b}=\vec{e}_i\varepsilon_{ijk}a_jb_k;$
scalar triple product:	$(\vec{a}\times\vec{b})\cdot\vec{c}=\varepsilon_{ijk}a_ib_jc_k;$
operator $\vec{\nabla}$:	$\vec{\nabla}=\vec{e}_i\dfrac{\partial}{\partial x_i}=\vec{e}_i\partial_i.$

Then the vector differential operations take the form:

$$\operatorname{grad} u = \vec{e}_i \frac{\partial u}{\partial x_i}, \quad \operatorname{div} \vec{a} = \frac{\partial a_i}{\partial x_i}, \quad \operatorname{curl} \vec{a} = \vec{e}_i \varepsilon_{ijk} \frac{\partial a_k}{\partial x_j},$$

$$(\vec{b} \cdot \vec{\nabla}) u = b_i \frac{\partial u}{\partial x_i}, \quad (\vec{b} \cdot \vec{\nabla}) \vec{a} = \vec{e}_k b_i \frac{\partial a_k}{\partial x_i}.$$

Example 1. Let $\vec{a}(\vec{r})$ and $u(\vec{r})$ be differentiable vector and scalar fields. Prove the equality

$$\operatorname{div}(u\vec{a}) = u \operatorname{div} \vec{a} + \vec{a} \cdot \operatorname{grad} u.$$

Solution. In this example we use the formal approach and taking primarily into account the differential character of $\vec{\nabla}$ we formally write out the derivative of a product:

$$\operatorname{div}(u\vec{a}) = \vec{\nabla} \cdot (u\vec{a}) = \vec{\nabla} \cdot (\overset{\downarrow}{u}\, \vec{a}) + \vec{\nabla} \cdot (u\, \overset{\downarrow}{\vec{a}}).$$

Then considering $\vec{\nabla}$ only as a vector we rearrange multipliers in such a way that the fields with the downarrow are placed to the right of the $\vec{\nabla}$, the rest to the left. Applying properties 1)–4) to the first term in the above expression we obtain

$$\vec{\nabla} \cdot (\overset{\downarrow}{u}\, \vec{a}) = (\vec{\nabla} \cdot \vec{a})\, \overset{\downarrow}{u} = (\vec{a} \cdot \vec{\nabla})\, \overset{\downarrow}{u} = \vec{a} \cdot (\vec{\nabla}\, \overset{\downarrow}{u}).$$

To rearrange the second term we use the property 2):

$$\vec{\nabla} \cdot (u\, \overset{\downarrow}{\vec{a}}) = u(\vec{\nabla} \cdot \overset{\downarrow}{\vec{a}}).$$

Finally we can omit the downarrows and write

$$\operatorname{div}(u\vec{a}) = \vec{a} \cdot (\vec{\nabla} u) + u(\vec{\nabla} \cdot \vec{a}) = \vec{a} \cdot \operatorname{grad} u + u \operatorname{div} \vec{a}.$$

Example 2. Suppose that $\vec{a}(\vec{r})$ and $\vec{b}(\vec{r})$ are differentiable vector fields. Prove the equality

$$\operatorname{curl}(\vec{a} \times \vec{b}) = \vec{a} \operatorname{div} \vec{b} - \vec{b} \operatorname{div} \vec{a} + (\vec{b} \cdot \vec{\nabla}) \vec{a} - (\vec{a} \cdot \vec{\nabla}) \vec{b}.$$

Solution. As in Example 1 we write

$$\operatorname{curl}(\vec{a} \times \vec{b}) = \vec{\nabla} \times (\vec{a} \times \vec{b}) = \vec{\nabla} \times (\overset{\downarrow}{\vec{a}} \times \vec{b}) + \vec{\nabla} \times (\vec{a} \times \overset{\downarrow}{\vec{b}}).$$

Then we expand the double cross product (property 9) in the first term:

$$\vec{\nabla} \times (\overset{\downarrow}{\vec{a}} \times \vec{b}) = \overset{\downarrow}{\vec{a}}(\vec{\nabla} \cdot \vec{b}) - \vec{b}(\vec{\nabla} \cdot \overset{\downarrow}{\vec{a}}).$$

Since the quantity $(\vec{\nabla} \cdot \vec{b})$ is a scalar we place it before the vector field $\vec{a}$ and change the order of vectors in the dot product:

$$\vec{\nabla} \times (\overset{\downarrow}{\vec{a}} \times \vec{b}) = (\vec{b} \cdot \vec{\nabla})\, \overset{\downarrow}{\vec{a}} - \vec{b}(\vec{\nabla} \cdot \overset{\downarrow}{\vec{a}}).$$

The fields in the second term are already in the right order. Similarly we process the second term

$$\vec{\nabla} \times (\vec{a} \times \overset{\downarrow}{\vec{b}}) = \vec{a}(\vec{\nabla} \cdot \overset{\downarrow}{\vec{b}}) - \overset{\downarrow}{\vec{b}}(\vec{\nabla} \cdot \vec{a}) = \vec{a}(\vec{\nabla} \cdot \overset{\downarrow}{\vec{b}}) - (\vec{a} \cdot \vec{\nabla})\, \overset{\downarrow}{\vec{b}}.$$

Thus we obtain

$$\operatorname{curl}(\vec{a} \times \vec{b}) = (\vec{b} \cdot \vec{\nabla})\vec{a} - \vec{b}(\vec{\nabla} \cdot \vec{a}) + \vec{a}(\vec{\nabla} \cdot \vec{b}) - (\vec{a} \cdot \vec{\nabla})\vec{b} =$$

$$= (\vec{b} \cdot \vec{\nabla})\vec{a} - \vec{b} \operatorname{div} \vec{a} + \vec{a} \operatorname{div} \vec{b} - (\vec{a} \cdot \vec{\nabla})\vec{b}.$$

Example 3. Let $\vec{a}(\vec{r})$ and $\vec{b}(\vec{r})$ be differentiable vector fields. Prove the equality

$$\operatorname{grad}(\vec{a} \cdot \vec{b}) = \vec{a} \times \operatorname{curl} \vec{b} + \vec{b} \times \operatorname{curl} \vec{a} + (\vec{b} \cdot \vec{\nabla})\vec{a} + (\vec{a} \cdot \vec{\nabla})\vec{b}.$$

Solution. We write as previously

$$\operatorname{grad}(\vec{a} \cdot \vec{b}) = \vec{\nabla}(\vec{a} \cdot \vec{b}) = \vec{\nabla}(\overset{\downarrow}{\vec{a}} \cdot \vec{b}) + \vec{\nabla}(\vec{a} \cdot \overset{\downarrow}{\vec{b}}).$$

There is no formula among 1) – 9) that is appropriate for the rearrangement of the above terms. To go further we consider relations

$$\vec{b} \times (\vec{\nabla} \times \overset{\downarrow}{\vec{a}}) = \vec{\nabla}(\overset{\downarrow}{\vec{a}} \cdot \vec{b}) - (\vec{b} \cdot \vec{\nabla})\, \overset{\downarrow}{\vec{a}},$$

$$\vec{a} \times (\vec{\nabla} \times \overset{\downarrow}{\vec{b}}) = \vec{\nabla}(\vec{a} \cdot \overset{\downarrow}{\vec{b}}) - (\vec{a} \cdot \vec{\nabla})\, \overset{\downarrow}{\vec{b}}.$$

Now we express the necessary terms and finally obtain

$$\operatorname{grad}(\vec{a} \cdot \vec{b}) = \vec{b} \times (\vec{\nabla} \times \vec{a}) + (\vec{b} \cdot \vec{\nabla})\vec{a} + \vec{a} \times (\vec{\nabla} \times \vec{b}) + (\vec{a} \cdot \vec{\nabla})\vec{b} =$$

$$= \vec{b} \times \operatorname{curl} \vec{a} + (\vec{b} \cdot \vec{\nabla})\vec{a} + \vec{a} \times \operatorname{curl} \vec{b} + (\vec{a} \cdot \vec{\nabla})\vec{b}.$$

Example 4. For arbitrary differentiable vector fields $\vec{a}(\vec{r})$ and $\vec{b}(\vec{r})$ prove the equality

$$(\vec{\nabla}\times\vec{a})\times\vec{b}=\vec{a}\,\mathrm{div}\,\vec{b}-(\vec{a}\cdot\vec{\nabla})\vec{b}-\vec{a}\times\mathrm{curl}\,\vec{b}-\vec{b}\times\mathrm{curl}\,\vec{a}.$$

Solution. In this example we use the second approach and perform calculations in cartesian coordinates. First we write the quantity $(\vec{\nabla}\times\vec{a})\times\vec{b}$ in index notation:

$$(\vec{\nabla}\times\vec{a})\times\vec{b}=\vec{e}_i\varepsilon_{ijk}(\vec{\nabla}\times\vec{a})_j b_k=\vec{e}_i\varepsilon_{ijk}\varepsilon_{jlm}\partial_l(a_m b_k).$$

Then we use the property of Levi-Civita's symbol:

$$\varepsilon_{ijk}\varepsilon_{jlm}=-\varepsilon_{ikj}\varepsilon_{jlm}=-(\delta_{il}\delta_{km}-\delta_{im}\delta_{kl}).$$

Thus

$$(\vec{\nabla}\times\vec{a})\times\vec{b}=\vec{e}_i(\delta_{im}\delta_{kl}-\delta_{il}\delta_{km})\partial_l(a_m b_k)=\vec{e}_i\partial_k(a_i b_k)-\vec{e}_i\partial_i(a_k b_k)=$$

$$=\vec{e}_i a_i\partial_k b_k+\vec{e}_i b_k\partial_k a_i-\vec{e}_i\partial_i(a_k b_k)=\vec{a}\,\mathrm{div}\,\vec{b}-(\vec{b}\cdot\vec{\nabla})\vec{a}-\mathrm{grad}(\vec{a}\cdot\vec{b}).$$

Substituting instead of $\mathrm{grad}(\vec{a}\cdot\vec{b})$ the result of the example 3 we prove the required.

Example 5. Find $\mathrm{grad}\,f(r)$, $\mathrm{div}\,f(r)\vec{r}$, $\mathrm{curl}\,f(r)\vec{r}$, where $\vec{r}$ is radius-vector, r is its norm, $f(r)$ is a differentiable function.

Solution. In the beginning we calculate the partial derivative $\partial_i r$ taking into account that $\vec{r}=\vec{e}_i x_i$ and $\partial_i x_k=\delta_{ik}$:

$$\partial_i r=\partial_i\sqrt{\vec{r}\cdot\vec{r}}=\frac{1}{2\sqrt{\vec{r}\cdot\vec{r}}}\partial_i(x_k x_k)=\frac{x_k\partial_i x_k}{r}=\frac{x_k\delta_{ik}}{r}=\frac{x_i}{r}.$$

Then

$$\mathrm{grad}\,f(r)=\vec{e}_i\partial_i f(r)=\vec{e}_i\frac{\partial f}{\partial r}\partial_i r=f'(r)\frac{\vec{e}_i x_i}{r}=f'(r)\frac{\vec{r}}{r}.$$

$$\mathrm{div}\,f(r)\vec{r}=\vec{\nabla}\cdot(f(r)\vec{r})=\partial_i(f(r)x_i)=x_i\partial_i f(r)+f(r)\partial_i x_i=$$
$$=x_i f'(r)\frac{x_i}{r}+f(r)\delta_{ii}=rf'(r)+3f(r).$$

$$\mathrm{curl}\,f(r)\vec{r}=\vec{\nabla}\times(f(r)\vec{r})=\vec{e}_i\varepsilon_{ijk}\partial_j(f(r)x_k)=\vec{e}_i\varepsilon_{ijk}x_k\partial_j f(r)+$$
$$+\vec{e}_i\varepsilon_{ijk}f(r)\partial_j x_k=\vec{e}_i\varepsilon_{ijk}x_k f'(r)\frac{x_j}{r}+\vec{e}_i\varepsilon_{ijk}f(r)\delta_{jk}=\vec{0}.$$

Exercises

For arbitrary differentiating vector fields $\vec{a}(\vec{r})$, $\vec{b}(\vec{r})$, $\vec{c}(\vec{r})$ and a differentiating scalar field $u(\vec{r})$ prove the following equalities.

3.84. $\operatorname{curl}(u\vec{a}) = u \operatorname{curl} \vec{a} - \vec{a} \times \operatorname{grad} u$.

3.85. $\operatorname{div}(\vec{a} \times \vec{b}) = \vec{b} \cdot \operatorname{curl} \vec{a} - \vec{a} \cdot \operatorname{curl} \vec{b}$.

3.86. $\vec{c} \cdot \operatorname{grad} \vec{a} \cdot \vec{b} = \vec{a} \cdot (\vec{c} \cdot \vec{\nabla})\vec{b} + \vec{b} \cdot (\vec{c} \cdot \vec{\nabla})\vec{a}$.

3.87. $(\vec{a} \cdot \vec{\nabla})u\vec{b} = \vec{b}(\vec{a} \cdot \operatorname{grad} u) + u(\vec{a} \cdot \vec{\nabla})\vec{b}$.

3.88. $(\vec{c} \cdot \vec{\nabla})(\vec{a} \times \vec{b}) = \vec{a} \times (\vec{c} \cdot \vec{\nabla})\vec{b} - \vec{b} \times (\vec{c} \cdot \vec{\nabla})\vec{a}$.

3.89. $(\vec{a} \times \vec{b}) \cdot \operatorname{curl} \vec{c} = \vec{b} \cdot (\vec{a} \cdot \vec{\nabla})\vec{c} - \vec{a} \cdot (\vec{b} \cdot \vec{\nabla})\vec{c}$.

3.90. $(\vec{a} \times \vec{\nabla}) \times \vec{b} = (\vec{a} \cdot \vec{\nabla})\vec{b} + \vec{a} \times \operatorname{curl} \vec{b} - \vec{a} \operatorname{div} \vec{b}$.

3.91. Prove that the vector $(\vec{b} \cdot \vec{\nabla})\vec{a}$ is the derivative of the vector field $\vec{a}$ in the direction of the vector $\vec{b}$ multiplied by the norm $\vec{b}$: $(\vec{b} \cdot \vec{\nabla})\vec{a} = |\vec{b}|\dfrac{\partial \vec{a}}{\partial b}$.

3.92. Find vectors $(\vec{b} \cdot \vec{\nabla})\vec{a}$ and $(\vec{a} \cdot \vec{\nabla})\vec{b}$ if $\vec{a} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and $\vec{b} = zx\vec{i} + xy\vec{j} + yz\vec{k}$.

3.93. Prove if $\vec{a}^{\,2} = \text{const}$ then

$$(\vec{a} \cdot \vec{\nabla})\vec{a} = -\vec{a} \times \operatorname{curl} \vec{a}.$$

Let $\vec{a}$ and $\vec{b}$ be constant vectors, $\vec{r}$ be a radius-vector and $r = |\vec{r}|$. Compute the following expression.

3.94. $\operatorname{grad} r$.

3.95. $\operatorname{grad} \dfrac{1}{r}$.

3.96. $(\vec{r} \cdot \vec{\nabla})r^n$.

3.97. $\operatorname{grad}(\vec{a} \cdot \vec{r})$.

3.98. $\operatorname{grad}(\vec{a} \cdot f(r)\vec{r})$.

3.99. $\operatorname{grad} \dfrac{\vec{a} \cdot \vec{r}}{r^3}$.

3.100. $\operatorname{grad}(\vec{a} \times \vec{r})^2$.

3.101. $\operatorname{grad} \dfrac{\vec{a} \cdot \vec{r}}{\vec{b} \cdot \vec{r}}$.

3.102. $\operatorname{div} \vec{r}$.

3.103. $\operatorname{div} \dfrac{\vec{r}}{r}$.

3.104. $\operatorname{div}(\vec{a} \cdot \vec{r})\vec{b}$.

3.105. $\operatorname{div}(\vec{a} \cdot \vec{r})\vec{r}$.

3.106. $\operatorname{div} f(r)\vec{a}$.

3.107. $\operatorname{div} r\vec{a}$.

3.108. $\operatorname{div} \vec{r}^{\,2}\vec{a}$.

3.109. $\operatorname{div}(\vec{r} \times \vec{a})$.

3.110. $\operatorname{div}((\vec{a} \times \vec{r}) \times \vec{b})$.

3.111. $\operatorname{div}((\vec{a} \times \vec{r}) \times \vec{r})$.

3.112. $\operatorname{curl} \vec{r}$.

3.113. $\operatorname{curl}(\vec{a} \cdot \vec{r})\vec{b}$.

3.114. $\operatorname{curl}(\vec{a} \cdot \vec{r})\vec{r}$.

3.115. $\operatorname{curl} r\vec{a}$.

3.116. $\operatorname{curl} f(r)\vec{a}$.

3.117. $\operatorname{curl} \dfrac{\vec{a} \times \vec{r}}{r^3}$.

3.118. $\operatorname{curl}(\vec{a} \times f(r)\,\vec{r})$.

3.119. $\operatorname{curl}(\vec{a} \times \vec{r})$.

3.120. $\operatorname{curl}((\vec{a} \times \vec{r}) \times \vec{b})$.

3.121. $\operatorname{curl}((\vec{a} \times \vec{r}) \times \vec{r})$.

Suppose that vector fields $\vec{a}$ and $\vec{b}$ depend only on the norm of radius vector. Compute the following expression.

3.122. $\operatorname{grad} \vec{a}(r) \cdot \vec{b}(r)$.

3.123. $\operatorname{grad} \vec{a}(r) \cdot \vec{r}$.

3.124. $(\vec{b} \cdot \vec{\nabla}) f(r)\vec{a}(r)$.

3.125. $\operatorname{div} f(r) \cdot \vec{a}(r)$.

3.126. $\operatorname{curl} f(r) \cdot \vec{a}(r)$.

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be constant vectors, $\vec{r}$ be a radius-vector and $r = |\vec{r}|$. Compute $\operatorname{div} \vec{p}$, $\operatorname{curl} \vec{p}$, $(\vec{c} \cdot \vec{\nabla})\vec{p}$.

3.127. $\vec{p} = r^n \vec{a}$.

3.128. $\vec{p} = r^n \vec{r}$.

3.129. $\vec{p} = \vec{a}(\vec{b} \cdot \vec{r}) r^n$.

3.130. $\vec{p} = \vec{r}(\vec{a} \cdot \vec{r}) r^n$.

3.131. $\vec{p} = \vec{r}(\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r}) r^n$.

3.132. $\vec{p} = (\vec{a} \times \vec{r}) r^n$.

3.133. $\vec{p} = (\vec{a} \times \vec{r})(\vec{b} \cdot \vec{r}) r^n$.

3.4. Differential operation of the second order

Suppose that a scalar field $u(M)$ and a vector field $\vec{a}(M)$ are twice continuously differentiable fields in a domain $D \subset E^3$. Then $\operatorname{grad} u$ is the differentiable in D vector field, $\operatorname{div} \vec{a}$ is the differentiable scalar field and $\operatorname{curl} \vec{a}$ is the differentiable vector field. Therefore from nine pair combinations of grad, div and curl there exist only five:

$$\operatorname{div} \operatorname{grad} u, \quad \operatorname{curl} \operatorname{curl} \vec{a}, \quad \operatorname{grad} \operatorname{div} \vec{a}, \quad \operatorname{curl} \operatorname{grad} u, \quad \operatorname{div} \operatorname{curl} \vec{a}.$$

Consider them step by step:

$$\operatorname{div}\ \operatorname{grad} u = \vec{\nabla} \cdot (\vec{\nabla} u) = (\vec{\nabla} \cdot \vec{\nabla}) u = \Delta u.$$

$$\operatorname{curl}\operatorname{curl}\vec{a} = \operatorname{grad}\operatorname{div}\vec{a} - \Delta\vec{a},$$
$$\operatorname{grad}\operatorname{div}\vec{a} = \operatorname{curl}\operatorname{curl}\vec{a} + \Delta\vec{a},$$
$$\operatorname{curl}\operatorname{grad} u = \vec{\nabla} \times (\vec{\nabla} u) = \vec{0},$$
$$\operatorname{div}\operatorname{curl}\vec{a} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{a}) = 0.$$

The operator Δ is called the *Laplace operator* and in cartesian coordinates takes the form

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

and acts as

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},$$
$$\Delta\vec{a} = \vec{i}\Delta a_x + \vec{j}\Delta a_y + \vec{k}\Delta a_z.$$

Example 1. Compute $\Delta(uv)$, where $u(M)$ and $v(M)$ are twice continuously differentiable scalar fields.

Solution. According to the result of the problem 3.21 we write using the formal approach the following:

$$\Delta(uv) = (\vec{\nabla} \cdot \vec{\nabla})(uv) = \vec{\nabla} \cdot (\vec{\nabla}(uv)) = \vec{\nabla} \cdot (v\vec{\nabla} u + u\vec{\nabla} v) =$$

$$= (\vec{\nabla} v) \cdot (\vec{\nabla} u) + v(\vec{\nabla} \cdot \vec{\nabla} u) + (\vec{\nabla} u) \cdot (\vec{\nabla} v) + u(\vec{\nabla} \cdot \vec{\nabla} v) =$$
$$= v\Delta u + 2(\vec{\nabla} u) \cdot (\vec{\nabla} v) + u\Delta v.$$

Example 2. Compute $\operatorname{div}(u \operatorname{grad} v)$, where $u(M)$ and $v(M)$ are twice continuously differentiable scalar fields.

Solution. Using formal approach we obtain the following chain of equalities:

$$\operatorname{div}(u \operatorname{grad} v) = \vec{\nabla} \cdot (u\vec{\nabla} v) = \vec{\nabla} \cdot (\overset{\downarrow}{u}\, \vec{\nabla} v) + \vec{\nabla} \cdot (u\, \overset{\downarrow}{\vec{\nabla}} v) =$$

$$= \vec{\nabla} v \cdot \vec{\nabla}\, \overset{\downarrow}{u} + u\vec{\nabla} \cdot \overset{\downarrow}{\vec{\nabla}} v = \vec{\nabla} v \cdot \vec{\nabla} u + u\Delta v.$$

Exercises

Suppose that $\vec{a}$, $\vec{b}$ u and v are arbitrary twice differentiable fields and $f(r)$ is a twice differentiable real-valued function of the norm of a radius vector. Compute the following expressions.

3.134. $\operatorname{curl}(u \operatorname{grad} v)$.

3.135. $\operatorname{div}(\operatorname{grad} f(r))$.

3.136. $\operatorname{curl}(\vec{a} \times \operatorname{curl} \vec{b})$.

3.137. $\operatorname{div}(\vec{\nabla} u \times \vec{\nabla} v)$.

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be constant vectors, $\vec{r}$ be a radius-vector and $r = |\vec{r}|$. Compute $\operatorname{grad} \operatorname{div} \vec{p}$, $\Delta\vec{p}$.

3.138. $\vec{p} = r^n \vec{a}$.

3.139. $\vec{p} = r^n \vec{r}$.

3.140. $\vec{p} = \vec{a}(\vec{b} \cdot \vec{r}) r^n$.

3.141. $\vec{p} = \vec{r}(\vec{a} \cdot \vec{r}) r^n$.

3.142. $\vec{p} = \vec{r}(\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r}) r^n$.

3.143. $\vec{p} = (\vec{a} \times \vec{r}) r^n$.

3.144. $\vec{p} = (\vec{a} \times \vec{r})(\vec{b} \cdot \vec{r}) r^n$.

3.145. Compute $\operatorname{curl} \operatorname{curl} \vec{a}$, $\operatorname{grad} \operatorname{div} \vec{a}$, $\Delta\vec{a}$ if $\vec{a} = x^2y^2\vec{i} + y^2z^2\vec{j} + z^2x^2\vec{k}$ and show and verify the formula $\operatorname{curl} \operatorname{curl} \vec{a} = \operatorname{grad} \operatorname{div} \vec{a} - \Delta\vec{a}$.

3.146. There given the following vector fields (any sign combination is possible):

$$\vec{a}_1 = \vec{i}e^{\pm x} + \vec{j}e^{\pm y} + \vec{k}e^{\pm z},$$
$$\vec{a}_2 = \vec{i}e^{\pm y} + \vec{j}e^{\pm z} + \vec{k}e^{\pm x},$$
$$\vec{a}_3 = \vec{i}e^{\pm z} + \vec{j}e^{\pm x} + \vec{k}e^{\pm y}.$$

Prove that:

a) $\operatorname{curl} \vec{a}_1 = 0$, and the fields $\vec{a}_2$ and $\vec{a}_3$ satisfy the equation $\vec{a} + \operatorname{curl} \operatorname{curl} \vec{a} = 0$;

b) $\operatorname{div} \vec{a}_2 = 0$, $\operatorname{div} \vec{a}_3 = 0$, and the field $\vec{a}_1$ satisfies the equation $\vec{a} - \operatorname{grad} \operatorname{div} \vec{a} = 0$.

3.147. Show that the function $\ln \dfrac{1}{r}$, where $r = \sqrt{x^2 + y^2}$, satisfies for $r \neq 0$ the Laplace equation on a plane $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

3.148. Show that the function $\dfrac{1}{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$, satisfies for $r \neq 0$ the Laplace equation $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.

3.149. Show that the electric field $\vec{E} = ke\frac{\vec{r}}{r^3}$ of a point charge e placed in the origin of coordinates satisfies for $r \neq 0$ the Laplace equation $\Delta\vec{E} = 0$.

3.150. Using Maxwell's equations for a homogeneous isotropic medium in the absence of charges and currents

$$\operatorname{curl}\vec{E} = -\frac{\mu}{c}\frac{\partial\vec{H}}{\partial t}, \quad \operatorname{curl}\vec{H} = \frac{\varepsilon}{c}\frac{\partial\vec{E}}{\partial t}, \quad \operatorname{div}\vec{E} = 0, \quad \operatorname{div}\vec{H} = 0.$$

show that the strengthes of electric $\vec{E}$ and magnetic $\vec{H}$ fields satisfy the equations:

$$\Delta\vec{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0, \qquad \Delta\vec{H} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0.$$

3.151. Show that the vectors $\vec{a}_1 = \vec{\nabla}u$, $\vec{a}_2 = \vec{\nabla}\times(\vec{c}u)$ and $\vec{a}_3 = \vec{\nabla}\times(\vec{\nabla}\times(\vec{c}u))$ satisfy the vectorial Helmholtz's equation $\Delta\vec{a} + k^2\vec{a} = \vec{0}$ if the scalar field is the solution of $\Delta u + k^2 u = 0$. Here the vector $\vec{c}$ is a constant vector. Prove that the vectors $\vec{a}_1$ and $\vec{a}_2$ are orthogonal. Find the divergence of the fields $\vec{a}_2$ and $\vec{a}_3$.

Chapter 4

Integral calculus of fields

4.1. Line integral of the first kind

Let $f(M)$ be a scalar field defined in the neighborhood of a simple smooth curve $\mathcal{L}$. Divide the curve by points M_i, $i = \overline{1, n}$, on arcs $M_{i-1}M_i$ of the length Δl_i and choose for each arc $M_{i-1}M_i$ a point $N_i \in M_{i-1}M_i$. If there exists the limit of integral sum $\lim\limits_{\max \Delta l_i \to 0} \sum\limits_{i=0}^{n} f(N_i)\,\Delta l_i$, which does not depend on the way of choosing point M_i and N_i, this limit is called the *line integral of the first kind* of the scalar field $f(M)$ along the curve $\mathcal{L}$ and denoted as

$$\int\limits_{\mathcal{L}} f(M)\mathrm{d}l \quad \text{or} \quad \oint\limits_{\mathcal{L}} f(M)\mathrm{d}l \quad \text{for a closed curve.}$$

Properties

1. *Linearity.* If there exist line integrals of fields $f(M)$ and $g(M)$ along a curve $\mathcal{L}$, then for all real numbers α and β

$$\int\limits_{\mathcal{L}} \big(\alpha f(M) + \beta g(M)\big)\mathrm{d}l = \alpha \int\limits_{\mathcal{L}} f(M)\mathrm{d}l + \beta \int\limits_{\mathcal{L}} g(M)\mathrm{d}l.$$

2. *Additivity.* If a curve $\mathcal{L}$ is the union of two curves $\mathcal{L}_1$ and $\mathcal{L}_2$ that may intersect only at a finite number of points, i. e. $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, and for a scalar field $f(M)$ there exists the line integral along the curve $\mathcal{L}$ then

$$\int\limits_{\mathcal{L}} f(M)\mathrm{d}l = \int\limits_{\mathcal{L}_1} f(M)\mathrm{d}l + \int\limits_{\mathcal{L}_2} f(M)\mathrm{d}l.$$

3. The line integral of a scalar field does not depend on the orientation of the curve.

4. The line integral of a scalar field does not depend on the parametrisation of the curve.

5. *Mean-value formula.* If a scalar field $f(M)$ is continuous along a curve $\mathcal{L}$, then there is a point $M^* \in \mathcal{L}$, that

$$\int\limits_{\mathcal{L}} f(M)\mathrm{d}l = f(M^*)l,$$

where l is the length of the curve $\mathcal{L}$.

6. $\int\limits_{\mathcal{L}} \mathrm{d}l = l(\mathcal{L})$.

To evaluate the line integral of a continuous scalar field along a smooth curve the curve is to be parameterised:

a) natural parametrization $\vec{r} = \vec{r}(l),\ l \in [0, l_0]$:

$$\int\limits_{\mathcal{L}} f(M)\mathrm{d}l = \int\limits_{0}^{l_0} f(x(l), y(l), z(l))\mathrm{d}l;$$

b) general parametrization $\vec{r} = \vec{r}(t),\ t \in [a, b]$:

$$\int\limits_{\mathcal{L}} f(M)\mathrm{d}l = \int\limits_{a}^{b} f(x(t), y(t), z(t))\sqrt{x_t'^2 + y_t'^2 + z_t'^2}\mathrm{d}t;$$

c) plane curve defined explicitly $y = y(x),\ x \in [a, b]$:

$$\int\limits_{\mathcal{L}} f(M)\mathrm{d}l = \int\limits_{a}^{b} f(x, y(x))\sqrt{1 + y_x'^2}\mathrm{d}x;$$

d) plane curve specified in the polar coordinates $\rho = \rho(\varphi),\ \varphi \in [\alpha, \beta]$:

$$\int\limits_{\mathcal{L}} f(M)\mathrm{d}l = \int\limits_{\alpha}^{\beta} f(\rho(\varphi)\cos\varphi, \rho(\varphi)\sin\varphi)\sqrt{\rho^2 + \rho_\varphi'^2}\mathrm{d}\varphi.$$

The line integral of the first kind can be defined for a vector field $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ as

$$\int\limits_{\mathcal{L}} \vec{a}(M)\mathrm{d}l = \vec{i}\int\limits_{\mathcal{L}} a_x(M)\mathrm{d}l + \vec{j}\int\limits_{\mathcal{L}} a_y(M)\mathrm{d}l + \vec{k}\int\limits_{\mathcal{L}} a_z(M)\mathrm{d}l.$$

Example 1. Evaluate the line integral of the first kind

$$\int\limits_{\mathcal{L}} (x + y)\mathrm{d}l,$$

where $\mathcal{L}$ is the triangle with vertexes at $A(1, 0)$, $B(0, 1)$ and $C(0, 0)$.

Solution. First we divide the piecewise smooth curve $\mathcal{L}$ into three smooth curves $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ as shown on fig. 4.1. Then we can parameterise each curve as follows:

$$\begin{aligned}
\mathcal{L}_1: &\quad x = 0,\ y = t, \quad t \in [0,1];\\
\mathcal{L}_2: &\quad x = t,\ y = 0, \quad t \in [0,1];\\
\mathcal{L}_3: &\quad x = t,\ y = 1 - t, \quad t \in [0,1].
\end{aligned}$$

Then taking into account that $\mathrm{d}l = \sqrt{x_t'^2 + y_t'^2}\mathrm{d}t$, we evaluate

$$\int\limits_{\mathcal{L}_1} (x+y)\mathrm{d}l = \int\limits_0^1 (0+t)\sqrt{0^2+1^2}\,\mathrm{d}t = \frac{1}{2};$$

$$\int\limits_{\mathcal{L}_2} (x+y)\mathrm{d}l = \int\limits_0^1 (t+0)\sqrt{1^2+0^2}\,\mathrm{d}t = \frac{1}{2};$$

$$\int\limits_{\mathcal{L}_3} (x+y)\mathrm{d}l = \int\limits_0^1 (t+1-t)\sqrt{1^2+(-1)^2}\,\mathrm{d}t = \sqrt{2}.$$

Thus using the property of additivity we obtain

$$\int\limits_{\mathcal{L}} (x+y)\mathrm{d}l = \sum_{k=1}^{3} \int\limits_{\mathcal{L}_k} (x+y)\mathrm{d}l = 1 + \sqrt{2}.$$

Example 2. Evaluate the line integral of the first kind

$$\int\limits_{\mathcal{L}} e^{\sqrt{x^2+y^2}}\mathrm{d}l,$$

where $\mathcal{L}$ is the convex contour formed by the curves $r = a$, $\varphi = 0$, $\varphi = \dfrac{\pi}{4}$ (r and φ are polar coordinates).

Solution. Given curve $\mathcal{L}$ is piecewise curve and we are to divide it into three simple smooth curves $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ as it is shown on fig. 4.2. The parametrisation of each smooth curve is

$$\begin{aligned}
\mathcal{L}_1: &\quad x = a\cos\varphi,\ y = a\sin\varphi, \quad \varphi \in \left[0, \frac{\pi}{4}\right];\\
\mathcal{L}_2: &\quad x = t,\ y = 0, \quad t \in [0,a];\\
\mathcal{L}_3: &\quad x = t,\ y = t, \quad t \in \left[0, \frac{a}{\sqrt{2}}\right].
\end{aligned}$$

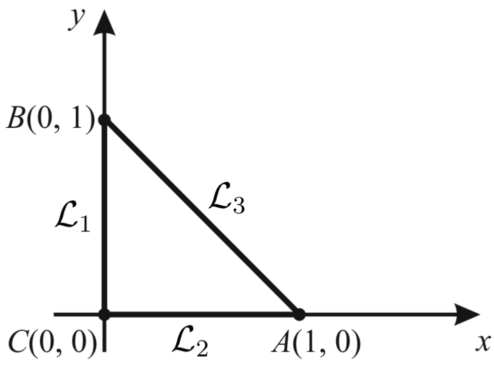

Fig. 4.1

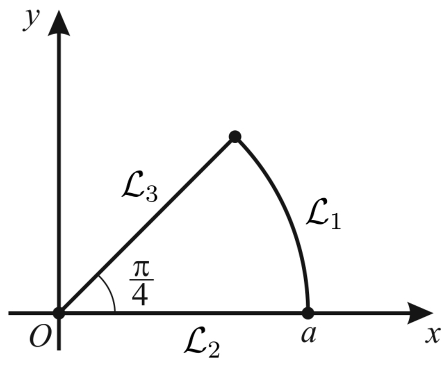

Fig. 4.2

Then

$$\int\limits_{\mathcal{L}_1} e^{\sqrt{x^2+y^2}} \mathrm{d}l = \int\limits_0^{\frac{\pi}{4}} e^{\sqrt{a^2\cos^2\varphi + a^2\sin^2\varphi}} \sqrt{a^2\sin^2\varphi + a^2\cos^2\varphi}\,\mathrm{d}\varphi =$$

$$= a\int\limits_0^{\frac{\pi}{4}} e^a \mathrm{d}\varphi = \frac{\pi a}{4} e^a;$$

$$\int\limits_{\mathcal{L}_2} e^{\sqrt{x^2+y^2}} \mathrm{d}l = \int\limits_0^a e^{\sqrt{t^2+0^2}} \sqrt{1^2+0^2}\mathrm{d}t = \int\limits_0^t e^a \mathrm{d}t = e^a - 1;$$

$$\int\limits_{\mathcal{L}_3} e^{\sqrt{x^2+y^2}} \mathrm{d}l = \int\limits_0^{a/\sqrt{2}} e^{\sqrt{t^2+t^2}} \sqrt{1^2+1^2}\mathrm{d}t = \sqrt{2}\int\limits_0^{a\sqrt{2}} e^{\sqrt{2}t}\mathrm{d}t = e^a - 1.$$

Finally we obtain

$$\int\limits_{\mathcal{L}} e^{\sqrt{x^2+y^2}} \mathrm{d}l = \sum_{k=1}^{3} \int\limits_{\mathcal{L}_k} e^{\sqrt{x^2+y^2}} \mathrm{d}l = 2(e^a - 1) + \frac{\pi a}{4} e^a.$$

Example 3. Evaluate the line integral of the first kind $\int\limits_{\mathcal{L}} xy\,\mathrm{d}l$ along the first arc of the cycloid $x = a(t - \sin t),\ y = a(1 - \cos t),\ 0 \leqslant t \leqslant 2\pi.$

Solution. The curve $\mathcal{L}$ is already parameterised. Therefore we can immediately find the element of the length $\mathrm{d}l$:

$$x' = a(1 - \cos t), \quad y' = a\sin t,$$

$$\mathrm{d}l = a\sqrt{(1-\cos t)^2 + \sin^2 t}\mathrm{d}t = a\sqrt{2(1-\cos t)}\mathrm{d}t = 2a\sin\frac{t}{2}\,\mathrm{d}t.$$

Substituting the parametrization into the integral we evaluate

$$\int\limits_{\mathcal{L}} xy\mathrm{d}l = 2a^3\int\limits_0^{2\pi}(t-\sin t)(1-\cos t)\sin\frac{t}{2}\mathrm{d}t = 4a^3\int\limits_0^{2\pi}(t-\sin t)\sin^3\frac{t}{2}\mathrm{d}t =$$

$$= 16a^3\int\limits_0^{\pi}\xi\sin^3\xi\,\mathrm{d}\xi - 8a^3\int\limits_0^{2\pi}\sin^4\frac{t}{2}\cos\frac{t}{2}\mathrm{d}t =$$

$$= 4a^3\int\limits_0^{\pi}\xi(3\sin\xi-\sin 3\xi)\mathrm{d}\xi - 16a^3\int\limits_0^{2\pi}\sin^4\frac{t}{2}\,\mathrm{d}\sin\frac{t}{2} =$$

$$= 12a^3(-\xi\cos\xi+\sin\xi)\Big|_0^{\pi} - 4a^3\left(-\frac{1}{3}\xi\cos 3\xi+\frac{1}{9}\sin 3\xi\right)\Big|_0^{\pi} -$$

$$- 16a^3\frac{1}{5}\sin^5\frac{t}{2}\Big|_0^{2\pi} = \frac{32\,\pi\,a^3}{3}.$$

Example 4. Evaluate the line integral of the first kind

$$\int\limits_{\mathcal{L}}\frac{(x^2-y^2)xy}{(x^2+y^2)^2}\mathrm{d}l,$$

where the curve $\mathcal{L}$ is defined by the polar equation $\rho = \sin 2\varphi,\ 0 \leqslant \varphi \leqslant \dfrac{\pi}{4}$.

Solution. The parametrization of the curve is

$$x = \rho(\varphi)\cos\varphi, \qquad y = \rho(\varphi)\sin\varphi, \quad 0 \leqslant \varphi \leqslant \frac{\pi}{4}.$$

Then

$$\mathrm{d}l = \sqrt{\rho^2+(\rho')^2}\,\mathrm{d}\varphi = \sqrt{1+3\cos^2 2\varphi}\,\mathrm{d}\varphi.$$

Substituting the parametrization into the integral we obtain

$$\int\limits_{\mathcal{L}}\frac{(x^2-y^2)xy}{(x^2+y^2)^2}\,\mathrm{d}l = \int\limits_0^{\frac{\pi}{4}}(\cos^2\varphi-\sin^2\varphi)\cos\varphi\sin\varphi\sqrt{1+3\cos^2 2\varphi}\mathrm{d}\varphi =$$

$$= \frac{1}{2}\int\limits_0^{\frac{\pi}{4}}\cos 2\varphi\sin 2\varphi\sqrt{1+3\cos^2 2\varphi}\mathrm{d}\varphi = \left|\begin{array}{c}\text{substitution}\\ \cos 2\varphi = t\end{array}\right| =$$

$$= \frac{1}{4}\int\limits_0^1 t\sqrt{1+3t^2}\mathrm{d}t = \frac{1}{8}\cdot\frac{1}{3}\cdot\frac{2}{3}\,(1+3t^2)^{\frac{3}{2}}\Big|_0^1 = \frac{7}{36}.$$

Example 5. Evaluate the line integral of the first kind

$$\int\limits_{\mathcal{L}} (x+y)\mathrm{d}l,$$

where $\mathcal{L}$ is a quarter of the circle $x^2+y^2+z^2=R^2,\ x=y$, being in the first octant (fig. 4.3).

Solution. We can parameterize the curve as

$$x=Rt, \quad y=Rt, \quad z=R\sqrt{1-2t^2}, \quad t\in\left[0,\frac{1}{\sqrt{2}}\right].$$

Then

$$\mathrm{d}l=R\sqrt{1+1+\frac{4t^2}{1-2t^2}}\mathrm{d}t=\frac{R\sqrt{2}}{\sqrt{1-2t^2}}\mathrm{d}t.$$

Substituting the parametrization into the integral we obtain

$$\int\limits_{\mathcal{L}} (x+y)\,\mathrm{d}l=\int\limits_0^{\frac{1}{\sqrt{2}}}\frac{2\sqrt{2}R^2t}{\sqrt{1-2t^2}}\mathrm{d}t=\left|\begin{array}{c}\text{substitution}\\ 2t^2=\xi\end{array}\right|=$$

$$=\int\limits_0^1\frac{R^2}{\sqrt{2}}\frac{d\xi}{\sqrt{1-\xi}}=\sqrt{2}R^2.$$

Example 6. Evaluate the line integral of the first kind

$$\int\limits_{\mathcal{L}} x^2\mathrm{d}l,$$

where the curve $\mathcal{L}$ is the circle $x^2+y^2+z^2=R^2,\ x+y+z=0$, (fig. 4.4).

Solution. Since the curve is symmetric with respect to permutations of coordinates x,y,z, then

$$\int\limits_{\mathcal{L}} x^2\mathrm{d}l=\int\limits_{\mathcal{L}} y^2\mathrm{d}l=\int\limits_{\mathcal{L}} z^2\mathrm{d}l.$$

Therefore

$$\int\limits_{\mathcal{L}} x^2\mathrm{d}l=\frac{1}{3}\int\limits_{\mathcal{L}}(x^2+y^2+z^2)\mathrm{d}l=\frac{1}{3}\int\limits_{\mathcal{L}} R^2\mathrm{d}l=\frac{2\pi R^3}{3}.$$

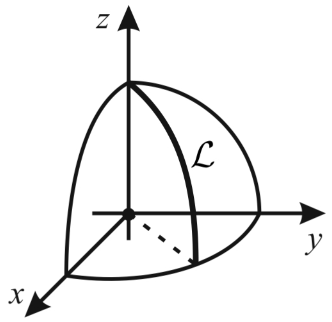

Fig. 4.3

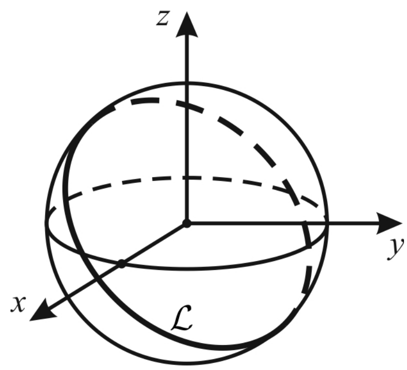

Fig. 4.4

Exercises

Evaluate the line integral of the first kind along the given plane curve. Consider all parameters like a, b, p etc. positive.

4.1. $\oint\limits_{\mathcal{L}} xy\,\mathrm{d}l$, $\mathcal{L}$: triangle with vertices at $(0,0)$, $(4,0)$, $(4,2)$, $(0,2)$.

4.2. $\int\limits_{\mathcal{L}} \dfrac{\mathrm{d}l}{\sqrt{x^2+y^2+1}}$, $\mathcal{L}$: segment of the straight line between points $(0,0)$ and $(1,2)$.

4.3. $\int\limits_{\mathcal{L}} (4\sqrt[3]{x} - 3\sqrt{y})\mathrm{d}l$, $\mathcal{L}$: segment of the straight line between points $(0,4)$ and $(4,0)$.

4.4. $\int\limits_{\mathcal{L}} \dfrac{\mathrm{d}l}{x-y}$, $\mathcal{L}$: segment of the straight line between points $(0,-2)$ and $(4,0)$.

4.5. $\int\limits_{\mathcal{L}} y\mathrm{d}l$, $\mathcal{L}$: arc of the parabola $y^2 = 2px$, cut off by the parabola $x^2 = 2py$.

4.6. $\int\limits_{\mathcal{L}} \dfrac{x}{y}\mathrm{d}l$, $\mathcal{L}$: parabola $y^2 = 2x$ between points $(1,\sqrt{2})$ and $(2,2)$.

4.7. $\int\limits_{\mathcal{L}} e^{2x}\mathrm{d}l$, $\mathcal{L}$: $x = \ln y,\ 1 \leqslant y \leqslant 4$.

4.8. $\int\limits_{\mathcal{L}} \dfrac{\mathrm{d}l}{y^2}$, $\mathcal{L}$: chain line $y = a\cosh\dfrac{x}{a}$.

4.9. $\int\limits_{\mathcal{L}} y\mathrm{d}l$, $\mathcal{L}$: $y = x^2 + |x^2 - x|,\ -1 \leqslant x \leqslant 2.$

4.10. $\int\limits_{\mathcal{L}} y^2\mathrm{d}l$, $\mathcal{L}$: $y = \max(2\sqrt{x}, 2x),\ 0 \leqslant x \leqslant 2.$

4.11. $\int\limits_{\mathcal{L}} 4xy\mathrm{d}l$, $\mathcal{L}$: $y = \min\left(\dfrac{x^2}{a}, \sqrt{2a^2 - x^2}\right),\ x \geqslant 0.$

4.12. $\int\limits_{\mathcal{L}} x^2\mathrm{d}l$, $\mathcal{L}$: arc of the circle $x^2 + y^2 = a^2,\ y \geqslant 0.$

4.13. $\int\limits_{\mathcal{L}} (x+y)\mathrm{d}l$, $\mathcal{L}$: arc of the circle $x = a\cos t,\ y = a\sin t,\ 0 \leqslant t \leqslant \dfrac{\pi}{2}.$

4.14. $\int\limits_{\mathcal{L}} (x^2 + y^2)^n\mathrm{d}l$, circle $\mathcal{L}$: $x = a\cos t, y = a\sin t.$

4.15. $\int\limits_{\mathcal{L}} (x - y)\,\mathrm{d}l$, circle $\mathcal{L}$: $x^2 + y^2 = ax.$

4.16. $\int\limits_{\mathcal{L}} xy\mathrm{d}l$, $\mathcal{L}$: arc of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ x \geqslant 0,\ y \geqslant 0.$

4.17. $\int\limits_{\mathcal{L}} xy\mathrm{d}l$, $\mathcal{L}$: arc of the hyperbola $x = a\cosh t,\ y = a\sinh t,$ $0 \leqslant t \leqslant t_0.$

4.18. $\int\limits_{\mathcal{L}} x^2y\mathrm{d}l$, $\mathcal{L}$: $x = 4\cos t,\ y = \sin 2t,\ x \geqslant 0,\ y \geqslant 0.$

4.19. $\int\limits_{\mathcal{L}} y^2\mathrm{d}l$, $\mathcal{L}$: arc of the cycloid $x = a(t - \sin t), y = a(1 - \cos t),$ $0 \leqslant t \leqslant 2\pi.$

4.20. $\int\limits_{\mathcal{L}} \sqrt{2y}\mathrm{d}l$, $\mathcal{L}$: arc of the cycloid $x = a(t - \sin t),\ y = a(1 - \cos t),$ $0 \leqslant t \leqslant 2\pi.$

4.21. $\int\limits_{\mathcal{L}} y\mathrm{d}l$, $\mathcal{L}$: arc of the astroid $x = \cos^3 t,\ y = \sin^3 t$ between points $(1, 0)$ and $(0, 1)$.

4.22. $\int\limits_{\mathcal{L}} (4x^2 - y^2)\mathrm{d}l$, $\mathcal{L}$: astroid $x = a\cos^3 t,\ y = a\sin^3 t,\ 0 \leqslant t \leqslant 2\pi.$

4.23. $\int\limits_{\mathcal{L}} \sqrt{x^2 + y^2}\mathrm{d}l$, $\mathcal{L}$: $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t),$ $0 \leqslant t \leqslant 2\pi.$

4.24. $\int\limits_{\mathcal{L}} \frac{\mathrm{d}l}{x^2+y^2}$, $\mathcal{L}$: $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$, $0 \leqslant t \leqslant 2\pi$.

4.25. $\int\limits_{\mathcal{L}} ye^{-x}\mathrm{d}l$, $\mathcal{L}$: $x = \ln(1+t^2)$, $y = 2\arctan t - t + 3$, $0 \leqslant t \leqslant 1$.

4.26. $\int\limits_{\mathcal{L}} \arctan\frac{y}{x}\mathrm{d}l$, $\mathcal{L}$: arc of the Archimedean spiral $\rho = 2\varphi$ in the interior of the circle $\rho = a$.

4.27. $\int\limits_{\mathcal{L}} x\mathrm{d}l$, $\mathcal{L}$: arc of the logarithmic spiral $\rho = e^{k\varphi}$, $k > 0$, in the interior of the circle $\rho = a$.

4.28. $\int\limits_{\mathcal{L}} x\mathrm{d}l$, $\mathcal{L}$: arc of cardioid $\rho = 1 + \cos\varphi$, $0 \leqslant \varphi \leqslant \pi$.

4.29. $\int\limits_{\mathcal{L}} \sqrt{x^2+y^2}\mathrm{d}l$, $\mathcal{L}$: loop of the curve $\rho = a\sin 3\varphi$, $x \geqslant 0$, $y \geqslant 0$.

4.30. $\int\limits_{\mathcal{L}} (x+y)\mathrm{d}l$, $\mathcal{L}$: loop of the curve $\rho^2 = \cos 2\varphi$, $-\frac{\pi}{4} \leqslant \varphi \leqslant \frac{\pi}{4}$.

4.31. $\int\limits_{\mathcal{L}} y^3\mathrm{d}l$, $\mathcal{L}$: loop of the curve $\rho = a\cos 4\varphi$, $-\frac{\pi}{8} \leqslant \varphi \leqslant \frac{\pi}{8}$.

4.32. $\int\limits_{\mathcal{L}} |y|\mathrm{d}l$, $\mathcal{L}$: loop of the curve $\rho = a(2+\cos\varphi)$, $0 \leqslant \varphi \leqslant 2\pi$.

4.33. $\int\limits_{\mathcal{L}} (x^3+y^3)\mathrm{d}l$, $\mathcal{L}$: $(x^2+y^2)^2 = 2a^2xy$, $x \geqslant 0$, $y \geqslant 0$.

4.34. $\int\limits_{\mathcal{L}} \sqrt{x^2+y^2}\mathrm{d}l$, $\mathcal{L}$: $(x^2+y^2)^2 = a^2(x^2-y^2)$.

4.35. $\int\limits_{\mathcal{L}} x\sqrt{x^2-y^2}\mathrm{d}l$, $\mathcal{L}$: $(x^2+y^2)^2 = a^2(x^2-y^2)$, $x \geqslant 0$.

Evaluate the line integral of the first kind along the given spatial curve. Consider all parameters like a, b, etc. positive.

4.36. $\int\limits_{\mathcal{L}} (x+z)\mathrm{d}l$, $\mathcal{L}$: $x = t$, $y = \frac{3t^2}{\sqrt{2}}$, $z = t^3$, $0 \leqslant t \leqslant 1$.

4.37. $\int\limits_{\mathcal{L}} \sqrt{2y}\mathrm{d}l$, $\mathcal{L}$: $x = t$, $y = \frac{t^2}{2}$, $z = \frac{t^3}{3}$, $0 \leqslant t \leqslant 1$.

4.38. $\int\limits_{\mathcal{L}} (2x - z^2y)\mathrm{d}l$, $\mathcal{L}$: $x = \frac{t^2}{2}$, $y = \frac{2\sqrt{2}}{3}t^{3/2}$, $z = t$, $0 \leqslant t \leqslant 1$.

4.39. $\int\limits_{\mathcal{L}} \frac{\mathrm{d}l}{x^2+y^2+z^2}$, $\mathcal{L}$: $x = a\cos t,\ y = a\sin t,\ z = bt,\ 0 \leqslant t \leqslant 2\pi$.

4.40. $\int\limits_{\mathcal{L}} \frac{z^2\mathrm{d}l}{x^2+y^2}$, $\mathcal{L}$: $x = a\cos t,\ y = a\sin t,\ z = at,\ 0 \leqslant t \leqslant 2\pi$.

4.41. $\int\limits_{\mathcal{L}} (x^2+y^2+z^2)\mathrm{d}l$, $\mathcal{L}$: $x = a\cos t,\ y = a\sin t,\ z = bt,\ 0 \leqslant t \leqslant 2\pi$.

4.42. $\int\limits_{\mathcal{L}} (2z - \sqrt{x^2+y^2})\mathrm{d}l$, $\mathcal{L}$: $x = t\cos t,\ y = t\sin t,\ z = t,$ $0 \leqslant t \leqslant 2\pi$.

4.43. $\int\limits_{\mathcal{L}} (x^2+y^2+z^2)\mathrm{d}l$, $\mathcal{L}$: $x = t\cos t - \sin t,\ y = t\sin t + \cos t,$ $z = t,\ 0 \leqslant t \leqslant 2\pi$.

4.44. $\int\limits_{\mathcal{L}} (x^2+y^2+z^2)\mathrm{d}l$, $\mathcal{L}$: $x = a(t - \sin t),\ y = a(1 - \cos t),$ $z = 4a\sin\frac{t}{2},\ 0 \leqslant t \leqslant 2\pi$.

4.45. $\int\limits_{\mathcal{L}} xz\mathrm{d}l$, $\mathcal{L}$: $\rho = a\left(1+\cos\varphi\right),\ z = 4a\left(1 - \cos\frac{\varphi}{2}\right),\ 0 \leqslant t \leqslant 4\pi$.

4.46. $\int\limits_{\mathcal{L}} \sqrt{x^2+y^2}\mathrm{d}l$, $\mathcal{L}$: $x = \sqrt{z}\cos\sqrt{z},\ y = \sqrt{z}\sin\sqrt{z}$ from the point (0,0,0) to $(-\pi, 0, \pi^2)$.

4.47. $\int\limits_{\mathcal{L}} z\mathrm{d}l$, $\mathcal{L}$: $x = t\cos t,\ y = t\sin t,\ z = t,\ 0 \leqslant t \leqslant 2\pi$.

4.48. $\int\limits_{\mathcal{L}} y^2\mathrm{d}l$, $\mathcal{L}$: circle $x^2+y^2+z^2 = a^2,\ x+y+z = a$.

4.49. $\int\limits_{\mathcal{L}} z\mathrm{d}l$, $\mathcal{L}$: arc of the curve $x^2+y^2 = z^2,\ y^2 = ax$ between points $(0,0,0)$ and $(a, a, a\sqrt{2})$.

4.50. $\int\limits_{\mathcal{L}} xyz\mathrm{d}l$, $\mathcal{L}$: a quarter of the circle $x^2+y^2+z^2 = a^2,\ x^2+y^2 = \frac{a^2}{4}$, being in the first octant.

4.51. $\int\limits_{\mathcal{L}} |y|\mathrm{d}l$, $\mathcal{L}$: $x^2+y^2 = z^2,\ x^2+y^2 = ax$.

4.2. Applications of the line integral of the first order

Suppose that a vector function $\vec{r}(t)$ specifies the position of a massive nonhomogeneous thread in E^3 and $\rho(\vec{r}(t))$ is its linear density function. Then

a) the total mass of the thread:

$$\mu = \int_{\mathcal{L}} \rho(\vec{r})\mathrm{d}l;$$

b) the center of gravity:

$$\vec{R}_c = \frac{1}{\mu}\int_{\mathcal{L}} \vec{r}\rho(\vec{r})\mathrm{d}l;$$

c) the moments of inertia with respect to coordinate axes:

$$I_x = \int_{\mathcal{L}} (y^2 + z^2)\rho(\vec{r})\mathrm{d}l, \quad I_y = \int_{\mathcal{L}} (x^2 + z^2)\rho(\vec{r})\mathrm{d}l,$$

$$I_z = \int_{\mathcal{L}} (x^2 + y^2)\rho(\vec{r})\mathrm{d}l;$$

d) the moments of inertia with respect to coordinate planes:

$$I_{yz} = \int_{\mathcal{L}} x^2\rho(\vec{r})\mathrm{d}l, \quad I_{xz} = \int_{\mathcal{L}} y^2\rho(\vec{r})\mathrm{d}l, \quad I_{xy} = \int_{\mathcal{L}} z^2\rho(\vec{r})\mathrm{d}l;$$

e) the moment of inertia with respect to the origin of coordinates:

$$I_0 = \int_{\mathcal{L}} (x^2 + y^2 + z^2)\rho(\vec{r})\mathrm{d}l.$$

The attraction force acting on a material point $P_0 = (x_0, y_0, z_0)$ of mass m by a massive thread:

$$\vec{F} = Gm\int_{\mathcal{L}} \frac{\vec{R}}{|\vec{R}|^3}\rho(x, y, z)\mathrm{d}l,$$

where $\vec{R} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$.

Example 1. Find the mass of the parabola $y^2 = 2px$, $0 \leqslant x \leqslant \dfrac{p}{2}$, if its linear density distributed as $\rho(x, y) = |y|$.

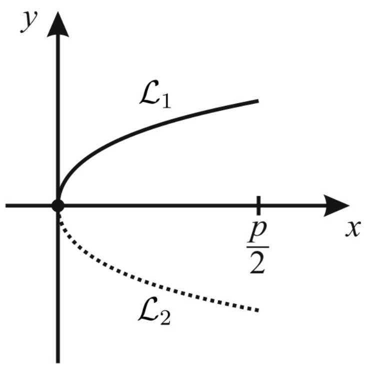

Fig. 4.5

Solution. At first we divide the curve $\mathcal{L}$ into two parts $\mathcal{L}_1$ and $\mathcal{L}_2$ symmetric in respect of the Ox axis as shown on fig. 4.5. Then the mass of the curve due to symmetry is

$$\mu = \int\limits_{\mathcal{L}} \rho(x, y)\mathrm{d}l = \int\limits_{\mathcal{L}} |y|\mathrm{d}l = 2\int\limits_{\mathcal{L}_1} y\mathrm{d}l.$$

Parameterizing the curve $\mathcal{L}_1$ as

$$x = x,\ y = \sqrt{2px}, \quad x \in \left[0, \frac{p}{2}\right],$$

we substitute it into the last integral and finally obtain

$$\mu = 2\int\limits_0^{p/2} \sqrt{2px}\sqrt{1 + \left(\frac{2p}{2\sqrt{2px}}\right)^2}\mathrm{d}x = 2\sqrt{2p}\int\limits_0^{p/2} \sqrt{x + \frac{p}{2}}\mathrm{d}x =$$

$$= \frac{4}{3}\sqrt{2p}\left(x + \frac{p}{2}\right)^{3/2}\Bigg|_0^{\frac{p}{2}} = \frac{4\sqrt{2}}{3}p^2\left(1 - \frac{1}{2\sqrt{2}}\right).$$

Example 2. Find the mass of the first coil of the helix $x = a\cos t$, $y = a\sin t$, $z = bt$, if its linear density $\rho(M)$ is proportional to the length of the radius-vector of the point M.

Solution. From the problem formulation we find out that $\rho(M) = \alpha|\vec{r}| = \alpha\sqrt{x^2 + y^2 + z^2}$, where α is a constant. Substituting parametric equation of the curve we obtain the density $\rho(t) = \alpha\sqrt{a^2 + b^2t^2}$ and the length element $\mathrm{d}l = |\vec{r}\,'(t)|\mathrm{d}t = \sqrt{a^2 + b^2}\mathrm{d}t$. Then the mass of the curve is

$$\mu = \int\limits_{\mathcal{L}} \rho(\vec{r})\mathrm{d}l = \int\limits_0^{2\pi} \alpha\sqrt{a^2 + b^2t^2}\sqrt{a^2 + b^2}\mathrm{d}t =$$

$$= \alpha\sqrt{a^2 + b^2}\left(\frac{t}{2}\sqrt{a^2 + b^2t^2} + \frac{a^2}{2b}\ln\left(bt + \sqrt{a^2 + b^2t^2}\right)\right)\Bigg|_0^{2\pi} =$$

$$= \alpha\sqrt{a^2 + b^2}\left[\pi\sqrt{a^2 + 4\pi^2b^2} + \frac{a^2}{2b}\ln\left(\frac{2\pi b + \sqrt{a^2 + 4\pi^2b^2}}{a}\right)\right].$$

Example 3. Find the center of gravity of the astroid arc

$$x^{3/2} + y^{3/2} = a^{3/2}, \quad x \geqslant 0,\ y \geqslant 0,$$

with constant linear density $\rho(x, y) = \rho_0$.

Solution. The parametric equation of the given part of the astroid is

$$x = a\cos^3\varphi, \quad y = \sin^3\varphi, \quad 0 \leqslant \varphi \leqslant \frac{\pi}{2}.$$

Then

$$x' = -3a\cos^2\varphi\sin\varphi, \quad y' = 3a\sin^2\varphi\cos\varphi,$$

$$\mathrm{d}l = 3a\sqrt{\cos^4\varphi\sin^2\varphi + \sin^4\varphi\cos^2\varphi}\mathrm{d}\varphi = 3a\sin\varphi\cos\varphi\mathrm{d}\varphi.$$

Primarily we find the curve total mass:

$$\mu = \int\limits_{\mathcal{L}} \rho\,(x, y)\,\mathrm{d}l = 3a\rho_0 \int\limits_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi\mathrm{d}\varphi = \frac{3a\rho_0}{2}.$$

Then we find the coordinates of the center of gravity:

$$x_c = \frac{1}{\mu}\int\limits_{\mathcal{L}} x\rho\,(x, y)\,\mathrm{d}l = \frac{1}{\mu}3a^2\rho_0 \int\limits_0^{\frac{\pi}{2}} \sin\varphi\cos^4\varphi\mathrm{d}\varphi = \frac{2}{3a\rho_0}\frac{3a^2\rho_0}{5} = \frac{2}{5}a,$$

$$y_c = \frac{1}{\mu}\int\limits_{\mathcal{L}} y\rho\,(x, y)\,\mathrm{d}l = \frac{1}{\mu}3a^2\rho_0 \int\limits_0^{\frac{\pi}{2}} \sin^4\varphi\cos\varphi\mathrm{d}\varphi = \frac{2}{3a\rho_0}\frac{3a^2\rho_0}{5} = \frac{2}{5}a.$$

Exercises

Consider all parameters like a, b etc. positive in this section.

Find the mass of a the given curve $\mathcal{L}$ with a linear density ρ.

4.52. $\mathcal{L}$: $y = \ln x,\ 0 < x_1 \leqslant x \leqslant x_2$; $\rho(x, y) = x^2$.

4.53. $\mathcal{L}$: triangle with vertices at $(0, 0)$, $(3, 0)$, $(0, 4)$; $\rho(x, y) = \dfrac{x}{3} + \dfrac{y}{4}$.

4.54. $\mathcal{L}$: arc of the chain line $y = a\cosh\dfrac{x}{a}$, $y \leqslant a\cosh 1$; $\rho(x, y) = b/y$.

4.55. $\mathcal{L}$: upper arc of the circle $x^2 + y^2 = a^2$; $\rho(x, y) = by^3$.

4.56. $\mathcal{L}$: lower arc of the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$; $\rho(x,y)=-y$.

4.57. $\mathcal{L}$: arc of the ellipse $x=a\cos t,\ y=b\sin t,\ a\geqslant b$; $\rho(x,y)=|y|$.

4.58. $\mathcal{L}$: arc of the parabola $y^2=2ax,\ 0\leqslant x\leqslant \dfrac{a}{2}$; $\rho(x,y)=|y|$.

4.59. $\mathcal{L}$: arc of the helix $x=a\cos t,\ y=a\sin t,\ z=bt,\ 0\leqslant t\leqslant 2\pi$; $\rho(x,y,z)=z^2$.

Find the center of gravity of the given homogeneous curve ($\rho(M)=1$).

4.60. $\mathcal{L}$: $x^2+y^2=a^2,\ x\geqslant 0,\ y\geqslant 0$.

4.61. $\mathcal{L}$: $y=a\cosh(x/a),\ -a\leqslant x\leqslant a$.

4.62. $\mathcal{L}$: $y=a\cosh(x/a),\ 0\leqslant x\leqslant a$.

4.63. $\mathcal{L}$: $y^2=x^2-x^4/a^2$.

4.64. $\mathcal{L}$: $x=(y^2-2\ln y)/4,\ 1\leqslant y\leqslant 2$.

4.65. $\mathcal{L}$: $x=a(t-\sin t),\ y=a(1-\cos t),\ 0\leqslant t\leqslant 2\pi$.

4.66. $\mathcal{L}$: $x=a(t-\sin t),\ y=a(1-\cos t),\ 0\leqslant t\leqslant \pi$.

4.67. $\mathcal{L}$: $x=a(\cos t+t\sin t),\ y=a(\sin t-t\cos t),\ 0\leqslant t\leqslant 2\pi$.

4.68. $\mathcal{L}$: $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

4.69. $\mathcal{L}$: $x^{2/3}+y^{2/3}=a^{2/3},\ y\geqslant 0$.

4.70. $\mathcal{L}$: $x=a\cos t,\ y=a\sin t,\ 0\leqslant t\leqslant \beta,\ 0<\beta<2\pi$.

4.71. $\mathcal{L}$: $x=a\cos t,\ y=a\sin t,\ z=bt,\ 0\leqslant t\leqslant 2\pi$.

4.72. $\mathcal{L}$: $x=e^t\cos t,\ y=e^t\sin t,\ z=e^t,\ t\leqslant 0$.

4.73. $\mathcal{L}$: $\rho=a\,(1+\cos\varphi)\,,\ 0\leqslant\varphi\leqslant\pi$.

4.74. $\mathcal{L}$: closed curve, which is intersection of the sphere $x^2+y^2+z^2=a^2$ and coordinate planes laying in the first octant.

4.75. $\mathcal{L}$: $x^2+y^2+z^2=a^2,\ |y|=x,\ z\geqslant 0$.

4.76. $\mathcal{L}$: $z=x^2-y^2,\ x+y-z=0,\ x\geqslant 0,\ y\leqslant 0$.

Find the center of gravity of the given nonhomogeneous curves.

4.77. $\mathcal{L}$: $x=a\cos t,\ y=a\sin t,\ z=bt,\ 0\leqslant t\leqslant 2\pi$; $\rho(x,y,z)=kz^2$.

4.78. $\mathcal{L}$: $x=a(t-\sin t),\ y=a(1-\cos t),\ z=4a\sin\dfrac{t}{2},\ 0\leqslant t\leqslant 2\pi$; $\rho(x,y,z)=kz^2$.

Find the moment of inertia of the given homogeneous curves.

4.79. $\mathcal{L}$: $x + 2y = 3,\ 1 \leqslant x \leqslant 2$, with respect to the Ox axis.

4.80. $\mathcal{L}$: $y = \sqrt{x},\ 1 \leqslant x \leqslant 2$, with respect to the Ox axis.

4.81. $\mathcal{L}$: $x^2 - y = 1,\ 0 \leqslant x \leqslant 1$, with respect to the coordinate axes.

4.82. $\mathcal{L}$: polygonal line ABC passing through the points $A(1, 1)$, $B(2, 3)$, $C(4, -1)$ with respect to the coordinate axes.

4.83. $\mathcal{L}$: triangle $\triangle ABC$ with vertices at the points $A(a, 0)$, $B\left(a,\ \dfrac{2\pi}{3}\right)$, $C\left(a,\ \dfrac{4\pi}{3}\right)$ given in polar coordinates with respect to the origin of coordinates.

4.84. $\mathcal{L}$: $x = a\cos t,\ y = a\sin t,\ 0 \leqslant t \leqslant \beta,\ 0 < \beta < 2\pi$ with respect to the coordinate axes.

4.85. $\mathcal{L}$: $\sqrt{x} + \sqrt{y} = \sqrt{a}$ with respect to the Ox axis and the origin.

4.86. $\mathcal{L}$: $x^{2/3} + y^{2/3} = a^{2/3},\ y \geqslant 0$, with respect to the coordinate axes.

4.87. $\mathcal{L}$: $x^{2/3} + y^{2/3} = a^{2/3},\ x \geqslant 0,\ y \geqslant 0$, with respect to the coordinate axes.

4.88. $\mathcal{L}$: $x = a\cos t,\ y = a\sin t,\ z = bt,\ 0 \leqslant t \leqslant 2\pi$, with respect to the coordinate axes.

4.89. Find the attraction force of a semicircle of radius R and mass M acting on a point mass m placed in the center of the semicircle.

4.90. Find the attraction force of an infinite homogeneous straight line directed along the Oz axis with linear density ρ acting on a point mass m placed at the distance h from the line.

4.91. Find the force exerted by the astroid arc $x = a\cos^3 t,\ y = a\sin^3 t$, $0 \leqslant t \leqslant \pi/2$ on a point mass m placed in the origin of coordinates, if the linear density of the astroid is proportional to the cubic power of the distance from the point of the astroid to the origin.

4.92. The circle $x^2 + y^2 = R^2$ is charged uniformly with linear charge density $\lambda(x,\ y) = \lambda_0$. Find electric field at the point $(0, 0, z_0)$.

4.93. The circle $x^2 + y^2 = R^2$ is charged with linear charge density $\lambda(x,\ y) = \lambda_0 \cos\varphi$ $(\tan\varphi = y/x)$. Find electric field at the center.

4.94. Let $\vec{v} = v_x(x, y)\vec{e}_x + v_y(x, y)\vec{e}_y$ be a velocity field of incompressible flow at a point $M(x, y)$. Find the flow rate through a closed smooth curve $\mathcal{L}$.

4.3. Line integral of the second kind

Let $\mathcal{L}$ be a simple curve with boundary points A and B ($A \neq B$). Ordering of boundary points defines the *orientation* of the curve. Expressions $\mathcal{L}_{AB}$ and $\mathcal{L}_{BA}$ are the notation of the curve $\mathcal{L}$ with opposite orientations. Another way to define the orientation of a curve is to choose the direction of unit tangent vector to the curve at fixed point. A closed simple curve can be oriented by cutting out one point and orienting the resulting simple open curve. A closed curve is said to be oriented positively or has the positive direction if when traversing, the region bounded by the curve remains on the left.

The line integral of the second kind along an oriented curve $\mathcal{L}$ of a vector field $\vec{a}(M)$ is called the integral

$$\int\limits_{\mathcal{L}} \vec{a}(M) \cdot \mathrm{d}\vec{r} = \int\limits_{\mathcal{L}} (\vec{a} \cdot \vec{\tau})\mathrm{d}l,$$

where $\vec{\tau}$ is the unit tangent vector of the curve.

In Cartesian coordinates

$$\vec{a}(M) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k},$$

$$\mathrm{d}\vec{r} = \mathrm{d}x\,\vec{i} + \mathrm{d}y\,\vec{j} + \mathrm{d}z\,\vec{k},$$

and

$$\int\limits_{\mathcal{L}} \vec{a} \cdot \mathrm{d}\vec{r} = \int\limits_{\mathcal{L}} P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z.$$

To compute the line integral of the second kind we are to parameterise the curve $\mathcal{L}: \ x = x(t),\ y = y(t),\ z = z(t),\ t \in [a, b]$, and put the parametrisation into the integral that transforms the line integral into the definite integral:

$$\int\limits_{\mathcal{L}} \vec{a} \cdot \mathrm{d}\vec{r} = \int\limits_{a}^{b} \vec{a}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\mathrm{d}t =$$

$$= \int\limits_{a}^{b} [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) +$$

$$+ R(x(t), y(t), z(t))z'(t)]\,\mathrm{d}t.$$

Properties

1. *Linearity.* If there exist line integrals of vector fields $\vec{a}(M)$ and $\vec{b}(M)$ along a curve $\mathcal{L}$ then for all real constant α and β

$$\int\limits_{\mathcal{L}} (\alpha\vec{a}(M) + \beta\vec{b}(M)) \cdot \mathrm{d}\vec{r} = \alpha \int\limits_{\mathcal{L}} \vec{a}(M) \cdot \mathrm{d}\vec{r} + \beta \int\limits_{\mathcal{L}} \vec{b}(M) \cdot \mathrm{d}\vec{r}.$$

2. *Additivity.* If a curve $\mathcal{L}$ is the union of two curves $\mathcal{L}_1$ and $\mathcal{L}_2$ that may intersect only at a finite number of points, i. e. $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, and for a vector field $\vec{a}(M)$ there exists the line integral along the curve $\mathcal{L}$ then

$$\int\limits_{\mathcal{L}} \vec{a}(M) \cdot \mathrm{d}\vec{r} = \int\limits_{\mathcal{L}_1} \vec{a}(M) \cdot \mathrm{d}\vec{r} + \int\limits_{\mathcal{L}_2} \vec{a}(M) \cdot \mathrm{d}\vec{r}.$$

3. Curve integrals of the second kind along curves with opposite orientation differ only in sign:

$$\int\limits_{\mathcal{L}_{AB}} \vec{a}(M) \cdot \mathrm{d}\vec{r} = -\int\limits_{\mathcal{L}_{BA}} \vec{a}(M) \cdot \mathrm{d}\vec{r}.$$

Example 1. Evaluate the line integral of the second kind

$$\oint\limits_{\mathcal{L}} x^2y\mathrm{d}x + x^3\mathrm{d}y,$$

where $\mathcal{L}$ is the closed piecewise curve composed by parabolas $y^2 = x$ and $x^2 = y$ taken in the clockwise direction.

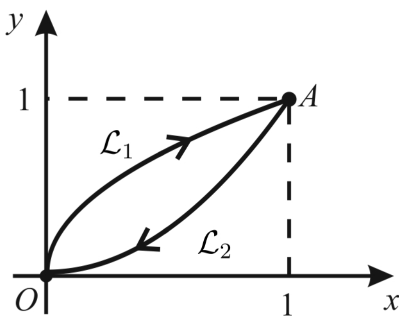

Fig. 4.6

Solution. First we need to divide the piecewise curve into two parabolas $\mathcal{L} = \mathcal{L}_1^- + \mathcal{L}_2^-$ oriented as shown in fig. 4.6. The parametrization of each curve is:

$$\mathcal{L}_1^- : \begin{cases} x = t^2, & x' = 2t, \\ y = t, & y' = 1, \end{cases} \quad t \in [0, 1];$$

$$\mathcal{L}_2^+ : \begin{cases} x = t, & x' = 1, \\ y = t^2, & y' = 2t, \end{cases} \quad t \in [0, 1].$$

Then we calculate the integral along the curve $\mathcal{L}_1^-$:

$$\int\limits_{\mathcal{L}_1^-} x^2y\mathrm{d}x + x^3\mathrm{d}y = \int\limits_0^1 (t^4 \cdot t \cdot 2t + t^6)\mathrm{d}t = \frac{3}{7}.$$

To evaluate the integral along the curve $\mathcal{L}_2^-$ we change the direction of the curve and use the property 3):

$$\int\limits_{\mathcal{L}_2^-} x^2y\mathrm{d}x + x^3\mathrm{d}y = -\int\limits_{\mathcal{L}_2^+} x^2y\mathrm{d}x + x^3\mathrm{d}y = -\int\limits_0^1 (t^2\cdot t^2 + t^3\cdot 2t)\mathrm{d}t = -\frac{3}{5}.$$

Then finally

$$\oint\limits_{\mathcal{L}} x^2y\mathrm{d}x + x^3\mathrm{d}y = \int\limits_{\mathcal{L}_1^+} x^2y\mathrm{d}x + x^3\mathrm{d}y + \int\limits_{\mathcal{L}_2^+} x^2y\mathrm{d}x + x^3\mathrm{d}y = -\frac{6}{35}.$$

Example 2. Evaluate the curve integral of the second kind

$$\int\limits_{\mathcal{L}} x^2\mathrm{d}y - xy\mathrm{d}x,$$

where $\mathcal{L}$ is the curve $x^4 - y^4 = 6x^2y$ from the point $(-4\sqrt{2}, 4)$ to $(0,0)$.

Solution. First we need to parameterise the curve. Since the curve implicit equation is uniform we introduce parameter t as $y = xt$. Substituting y in the curve equation we obtain

$$\mathcal{L}: \quad x = \frac{6t}{1-t^4}, \quad y = \frac{6t^2}{1-t^4}, \quad t\in\left[-\frac{1}{\sqrt{2}}, 0\right].$$

Then we find differentials

$$\mathrm{d}x = 6\frac{3t^4+1}{(1-t^4)^2}\mathrm{d}t, \quad \mathrm{d}y = 12\frac{t\left(t^4+1\right)}{(1-t^4)^2}\mathrm{d}t,$$

and finally evaluate the integral

$$\int\limits_{\mathcal{L}} x^2\mathrm{d}y - xy\mathrm{d}x = 6^3\int\limits_{-\frac{1}{\sqrt{2}}}^{0}\left(\frac{t^2}{(1-t^4)^2}\frac{2t(t^4+1)}{(1-t^4)^2} - \frac{t^3}{(1-t^4)^2}\frac{3t^4+1}{(1-t^4)^2}\right)\mathrm{d}t =$$

$$= 6^3\int\limits_{-\frac{1}{\sqrt{2}}}^{0}\frac{t^3}{(1-t^4)^3}\mathrm{d}t = \frac{6^3}{8}\frac{1}{(1-t^4)^2}\Bigg|_{-\frac{1}{\sqrt{2}}}^{0} = -21.$$

Example 3. Evaluate the line integral

$$\oint\limits_{\mathcal{L}} (y-z)\mathrm{d}x + (z-x)\mathrm{d}y + (x-y)\mathrm{d}z,$$

around the curve of intersection of the sphere $x^2+y^2+z^2=a^2$ and the plane $y = x\tan\alpha$, $0<\alpha<\pi/2$ taken in the counterclockwise direction as viewed from the point $(2a,-2a,0)$, $a>0$.

Solution. Since the curve is the intersection of the sphere and the plane we parameterize it as follows:

$$\mathcal{L}: \quad \begin{cases} x = a\cos\alpha\cos t, & x' = -a\cos\alpha\sin t, \\ y = a\sin\alpha\cos t, & y' = -a\sin\alpha\sin t, \\ z = a\sin t, & z' = a\cos t, \end{cases} \qquad t\in[0,2\pi].$$

The growth of the parameter generates the necessary direction of the curve. Substituting the parametrisation into the integral we obtain

$$\int\limits_{\mathcal{L}} (y-z)\mathrm{d}x + (z-x)\mathrm{d}y + (x-y)\mathrm{d}z =$$

$$= a^2\int\limits_0^{2\pi} [-(\sin\alpha\cos t - \sin t)\cos\alpha\sin t + (\sin t - \cos\alpha\cos t)\sin\alpha\sin t +$$

$$+ (\cos\alpha - \sin\alpha)\cos^2 t]\,\mathrm{d}t = a^2\int\limits_0^{2\pi}(\cos\alpha-\sin\alpha)\,\mathrm{d}t =$$

$$= 2\pi\alpha^2(\cos\alpha - \sin\alpha)\,.$$

Exercises

Evaluate the line integral of the second kind of a vector field $\vec{a}$ along the given curve $\mathcal{L}$. Consider all parameters like a, b, p etc. positive.

4.95. $\vec{a} = (x^2+y^2)\vec{i} + (x^2-y^2)\vec{j}$, $\mathcal{L}$: $y = |x|$ from $(-1,1)$ to $(2,2)$.

4.96. $\vec{a} = (x^2+y^2)\vec{i} + (x^2-y^2)\vec{j}$, $\mathcal{L}$: $y = 1-|x-1|$, $0\leqslant x\leqslant 2$.

4.97. $\vec{a} = (x^2-2xy)\vec{i} + (y^2-2xy)\vec{j}$, $\mathcal{L}$: $y = x^2$ from $(-1,1)$ to $(1,1)$.

4.98. $\vec{a} = 2xy\vec{i} + x^2\vec{j}$, $\mathcal{L}$: $x^2+y^2=1$ from $(1,0)$ to $(0,1)$.

4.99. $\vec{a} = \dfrac{x+y}{x^2+y^2}\vec{i} + \dfrac{x-y}{x^2+y^2}\vec{j}$, $\mathcal{L}$: $x^2+y^2=1$ oriented positively.

4.100. $\vec{a} = (x+y)\vec{i} + (x-y)\vec{j}$, $\mathcal{L}$: $(x-1)^2+(y-1)^2=1$ oriented negatively.

4.101. $\vec{a} = (2a-y)\vec{i} + (x-a)\vec{j}$, $\mathcal{L}$: $x = a(t-\sin t)$, $y = a(1-\cos t)$ from $(0,0)$ to $(2\pi a, 0)$.

4.102. $\vec{a} = x^2y\vec{i} - y^2x\vec{j}$, $\mathcal{L}$: $x = \sqrt{\cos t}$, $y = \sqrt{\sin t}$ from $(1,0)$ to $(0,1)$.

4.103. $\vec{a} = z\vec{i} + x\vec{j} + y\vec{k}$, $\mathcal{L}$: $x = a\cos t$, $y = a\sin t$, $z = bt$ from $(a,0,0)$ to $(a,0,2\pi b)$.

4.104. $\vec{a} = (y^2-z^2)\vec{i} + 2yz\vec{j} - x^2\vec{k}$, $\mathcal{L}$: $x = t$, $y = t^2$, $z = t^3$ from $(0,0,0)$ to $(1,1,1)$.

4.105. $\vec{a} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2+y^2+z^2-x-y+2z}}$, $\mathcal{L}$: straight line from $(1,1,1)$ to $(4,4,4)$.

4.106. $\vec{a} = z\vec{i} + 2x\vec{j} - y\vec{k}$, $\mathcal{L}$: $x^2+y^2 = 2ax$, $az = xy$, $z \geqslant 0$, from $(0,0,0)$ to (a,a,a).

Evaluate the line integral of the second kind along the given curve $\mathcal{L}$. Consider all parameters like a, b etc. positive.

4.107. $\oint\limits_{\mathcal{L}}(xy+x^2+y^2)\mathrm{d}x + (x^2-y^2)\mathrm{d}y$, $\mathcal{L}$: positively oriented triangle $\triangle ABC$ with vertexes at $A(1,2)$, B(0,2),C$(0,0)$.

4.108. $\int\limits_{\mathcal{L}} 4x\sin^2 y\mathrm{d}x + y\cos^2 2x\mathrm{d}y$, $\mathcal{L}$: straight line from $(0,0)$ to $(3,6)$.

4.109. $\int\limits_{\mathcal{L}} x\mathrm{d}y$, $\mathcal{L}$: straight line $bx+ay=ab$ from $(a,0)$ to $(0,b)$.

4.110. $\int\limits_{\mathcal{L}} \dfrac{y\mathrm{d}x + x\mathrm{d}y}{1+x^2y^2}$, $\mathcal{L}$: straight line from $(0,0)$ to $(1,1)$.

4.111. $\oint\limits_{\mathcal{L}} \dfrac{\mathrm{d}x + \mathrm{d}y}{|x|+|y|}$, $\mathcal{L}$: square with vertexes at $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$ traced in the counterclockwise direction.

4.112. $\int\limits_{\mathcal{L}} (y^2+2xy)\mathrm{d}x + (x^2-2xy)\mathrm{d}y$, $\mathcal{L}$: parabola $y = x^2$ from $(1,1)$ to $(2,4)$.

4.113. $\int\limits_{\mathcal{L}} y\mathrm{d}x - (y + x^2)\mathrm{d}y$, $\mathcal{L}$: parabola $y = 2x - x^2$ from $(2,0)$ to $(0,0)$.

4.114. $\int\limits_{\mathcal{L}} (4x + y)\mathrm{d}x + (x + 4y)\mathrm{d}y$, $\mathcal{L}:$ $y = x^4$ from $(1,1)$ to $(-1,1)$.

4.115. $\int\limits_{\mathcal{L}} (2a - y)\mathrm{d}x + x\mathrm{d}y$, $\mathcal{L}:$ cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ from $(0,0)$ to $(2\pi a, 0)$.

4.116. $\oint\limits_{\mathcal{L}} xy^2\mathrm{d}y - x^2y\mathrm{d}x$, $\mathcal{L}$: positively oriented circle $x^2 + y^2 = a^2$.

4.117. $\oint\limits_{\mathcal{L}} x\mathrm{d}y + 2y\mathrm{d}x$, $\mathcal{L}$: closed line composed by curves $y = 0$, $y = x$, $y = \sqrt{1 - x^2}$, traced in the counterclockwise direction.

4.118. $\oint\limits_{\mathcal{L}} y\mathrm{d}x - x\mathrm{d}y$, $\mathcal{L}$: astroid $x^{3/2} + y^{3/2} = a^{3/2}$, traced in the clockwise direction.

4.119. $\int\limits_{\mathcal{L}} a(x + y)\mathrm{d}x - xy\mathrm{d}y$, $\mathcal{L}$: $x^{1/4} + y^{1/4} = a^{1/4}$ from $(0, a)$ to $(a, 0)$.

4.120. $\int\limits_{\mathcal{L}} xy^2\mathrm{d}x - x^2y\mathrm{d}y$, $\mathcal{L}$: $2(x + y) = (x - y)^2$ from $(0,2)$ to $(2,0)$.

4.121. $\int\limits_{\mathcal{L}} xy\mathrm{d}x - x^2\mathrm{d}y$, $\mathcal{L}$: $x^4 - 2x^2y^2 + y^3 = 0$ from $(-1/4, -1/8)$ to the origin.

4.122. $\int\limits_{\mathcal{L}} y^2\mathrm{d}x - 2xy\mathrm{d}y$, $\mathcal{L}$: $x^3 + 2x^2 + y^2 = 3$ from $(-1, \sqrt{2})$ to $(1,0)$.

4.123. $\int\limits_{\mathcal{L}} (y + \pi)\,\mathrm{d}x + x\cos y\mathrm{d}y$, $\mathcal{L}$: $\pi \ln x - y + \sin y = 0$ from $(1,0)$ to (e, π).

4.124. $\oint\limits_{\mathcal{L}} x\mathrm{d}y - y\mathrm{d}x$, $\mathcal{L}:$ $x^4 + y^4 = a^2(x^2 + y^2)$, traced in the counterclockwise direction.

4.125. $\oint\limits_{\mathcal{L}} xy\mathrm{d}x - x^3y^3\mathrm{d}y$, $\mathcal{L}$: square $|x - y| + |x + y| = 1$ traced in the clockwise direction.

4.126. $\oint\limits_{\mathcal{L}} y\mathrm{d}x - x\mathrm{d}y$, $\mathcal{L}$: loop of the curve $\rho = a\cos 3\varphi,\ x \geqslant 0$, traced in the counterclockwise direction.

4.127. $\int\limits_{\mathcal{L}} y\mathrm{d}x + x\mathrm{d}y + (x+y+z)\mathrm{d}z$, $\mathcal{L}$: straight line from $(2,3,4)$ to $(3,4,5)$.

4.128. $\int\limits_{\mathcal{L}} x^2\mathrm{d}x + y^2\mathrm{d}y + z^2\mathrm{d}z$, $\mathcal{L}$: straight line from $(0,0,0)$ to $(1,\ 1,\ 1)$.

4.129. $\oint\limits_{\mathcal{L}} z\mathrm{d}x + x\mathrm{d}y + y\mathrm{d}z$, $\mathcal{L}:$ circle $x^2 + y^2 = 4,\ z = 0$, traced in the counterclockwise direction as viewed from the point $(0,0,1)$.

4.130. $\oint\limits_{\mathcal{L}} y\mathrm{d}x - x\mathrm{d}y + z\mathrm{d}z$, $\mathcal{L}:\ x^2 + y^2 + z^2 = 4,\ x^2 + y^2 = z^2,\ z \geqslant 0$, traced in the clockwise direction as viewed from the origin.

4.131. $\oint\limits_{\mathcal{L}} 2xz\mathrm{d}x - y\mathrm{d}y + z\mathrm{d}z$, $\mathcal{L}$: the curve of intersection of the plane $x + y + 2z = 2$ with the coordinate planes, traced in the counterclockwise direction as viewed from the origin.

4.132. $\oint\limits_{\mathcal{L}} (y^2 - z^2)\mathrm{d}x + (z^2 - x^2)\mathrm{d}y + (x^2 - y^2)\mathrm{d}z$, $\mathcal{L}$: the curve of intersection of the sphere $x^2+y^2+z^2 = 1$ with the coordinate planes laying in the first octant, traced in the counterclockwise direction as viewed from the point $(2,2,2)$.

4.133. $\oint\limits_{\mathcal{L}} y\mathrm{d}x - x\mathrm{d}y + (x+y)\mathrm{d}z$, $\mathcal{L}$: $x^2 + y^2 = z,\ z = 1$, traced in the clockwise direction as viewed from the point $(0,0,2)$.

4.134. $\oint\limits_{\mathcal{L}} zy^2\mathrm{d}x + xz^2\mathrm{d}y + yx^2\mathrm{d}z$, $\mathcal{L}:\ y^2 + z^2 = x,\ x = 9$, traced in the counterclockwise direction as viewed from the origin.

4.135. $\oint\limits_{\mathcal{L}} y^2\mathrm{d}x + z^2\mathrm{d}y$, $\mathcal{L}:\ x^2 + y^2 = 9,\ 3y + 4z = 5$, traced in the counterclockwise direction as viewed from the origin.

4.136. $\oint\limits_{\mathcal{L}} y\mathrm{d}x - x\mathrm{d}y + z\mathrm{d}z$, $\mathcal{L}:\ x^2 + y^2 + z^2 = 1,\ x = z$, traced in the counterclockwise direction as viewed from the point $(0,2,0)$.

4.137. $\int\limits_{\mathcal{L}} xz\mathrm{d}x + ax\mathrm{d}y - x^2\mathrm{d}z$, $\mathcal{L}$: $az = yx$, $x + y + z = a$, $x \geqslant 0$, $y \geqslant 0$, from the point $(0, a, 0)$ to $(a, 0, 0)$.

4.138. $\int\limits_{\mathcal{L}} yz\mathrm{d}x + ay\mathrm{d}y - az\mathrm{d}z$, $\mathcal{L}$: $x^2 + y^2 = z^2$, $x^2 + y^2 = ax$, $y \geqslant 0$, $z \geqslant 0$, from the point $(0, 0, 0)$ to $(a, 0, a)$.

4.139. $\int\limits_{\mathcal{L}} x^2y^3\mathrm{d}x + \mathrm{d}y + z\mathrm{d}z$, $\mathcal{L}$: $x^2 + y^2 = a$, $z = h$, from $(a, 0, h)$ to $(-a, 0, h)$ through $(0, a, h)$.

4.140. $\oint\limits_{\mathcal{L}} (y - z)\mathrm{d}x + (z - x)\mathrm{d}y + (x - y)\mathrm{d}z$, $\mathcal{L}$: $x^2 + y^2 + z^2 = a^2$, $y = x \tan \alpha$, $0 < \alpha < \dfrac{\pi}{2}$, traced in the counterclockwise direction as viewed from the point $(2a, -2a, 0)$.

4.141. $\int\limits_{\mathcal{L}} y^2\mathrm{d}x + z^2\mathrm{d}y + x^2\mathrm{d}z$, $\mathcal{L}$: upper arc of the Viviani's curve $x^2 + y^2 + z^2 = a^2$, $x^2 + y^2 = ax$, $z \geqslant 0$, traced in the counterclockwise direction as viewed from the point $(2a, 0, 0)$.

4.4. Applications of the line integral of the second order

Work done by a force field $\vec{F}$ along a curve $\mathcal{L}$:

$$A = \int\limits_{\mathcal{L}} \vec{F} \cdot \mathrm{d}\vec{r}.$$

If a curve $\mathcal{L}$ is closed then the line integral of the second kind is called *circulation* of a vector field.

Induction of magnetic field at a point (x_0, y_0, z_0), induced by a current I, passing through a closed conductor $\mathcal{L}$:

$$\vec{B} = \gamma I \oint\limits_{\mathcal{L}} \frac{\mathrm{d}\vec{r} \times \vec{R}}{|\vec{R}|^3},$$

where $\vec{R} = (x_0 - x)\vec{i} + (y_0 - y)\vec{j} + (z_0 - z)\vec{k}$.

Example 1. Find the induction of magnetic field at the Oz axis, induced by a current I, passing through the coil $x^2 + y^2 = a^2$, (fig. 4.7).

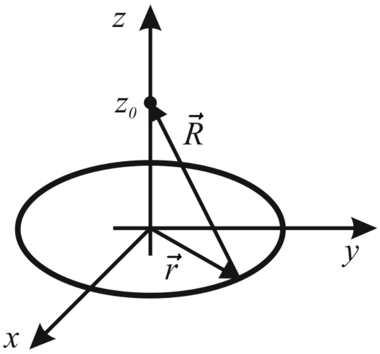

Fig. 4.7

Solution. To find the induction at the point $(0, 0, z_0)$ we parameterise the circle coil as

$$\vec{r} = \vec{i}a\cos t + \vec{j}a\sin t, \quad t \in [0, 2\pi].$$

Then

$$\mathrm{d}\vec{r} = (-\vec{i}a\sin t + \vec{j}a\cos t)\mathrm{d}t;$$

$$\vec{R} = -\vec{i}a\cos t - \vec{j}a\sin t + z_0\vec{k}, \quad |\vec{R}| = \sqrt{a^2 + z_0^2};$$

$$\mathrm{d}\vec{r} \times \vec{R} = (\vec{i}az_0\cos t\ + \vec{j}az_0\sin t\ + a^2\vec{k})\mathrm{d}t.$$

Finally aggregating preliminary calculations we obtain

$$\vec{B} = \gamma I \oint\limits_{\mathcal{L}} \frac{\mathrm{d}\vec{r} \times \vec{R}}{|\vec{R}|^3} = \gamma I \int\limits_0^{2\pi} \frac{az_0\cos t\vec{i} + az_0\sin t\vec{j} + a^2\vec{k}}{(a^2 + z_0^2)^{3/2}}\mathrm{d}t = \frac{2\pi\gamma I a^2}{(a^2 + z_0^2)^{3/2}}\vec{k}.$$

Exercises

4.142. Evaluate the work done by the force $\vec{F} = F\vec{i}$ along the arc of the circle $x^2 + y^2 = a^2$, laying in the first quadrant.

4.143. Evaluate the work done by the force $\vec{F} = xy\vec{i} + (x + y)\vec{j}$ while a particle moves from the origin to the point $(1, 1)$: a) along the straight line $y = x$; b) along the parabola $y = x^2$; c) along two-element broken line, elements of which are parallel to the coordinate axes (consider two cases).

4.144. Suppose that a particle moves along the ellipse $x = a\cos t$, $y = b\sin t$. Evaluate the work done by the force $\vec{F}$ directed to the origin, whose value is proportional to the distance between position of the particle M and the origin, a) along the arc of the ellipse in the first quadrant, b) along the whole ellipse.

4.145. The force $\vec{F}$ directed to the origin has the value inversely proportional to the distance from a point of appliance to the plane xOy. Evaluate the work done by this force along the straight line $x = at,\ y = bt,\ z = ct$ from the point (a, b, c) to $(2a, 2b, 2c)$.

4.146. The force $\vec{F}$ is directed to the Oz axis perpendicular to it and has the value inversely proportional to the distance from a point of appliance to the axis Oz. Evaluate the work done by this force along the circle $x = \cos t,\ y = 1,\ z = \sin t$ from the point $(1, 1, 0)$ to $(0, 1, 1)$.

4.147. Express the fluid flow rate obtained in the Exercise 4.94 in terms of the line integral of the second kind.

4.148. Let $\vec{v} = -xy\vec{i} + y^2/2\,\vec{j}$ be a stationary velocity field of an incompressible fluid. Evaluate the flow rate through the boundary of the domain $G = \{(x, y) : -1 \leqslant x \leqslant 1,\ x^4 \leqslant y \leqslant 1\}$.

4.149. Find the magnetic field induction value in the origin induced by a current I, passing through the coil $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

4.150. Find the magnetic field induction value induced by a current I, passing through the infinite rectilinear conductor at the distance h from the conductor.

4.5. Surface integral of the first kind

Let $f(M)$ be a scalar field defined and continuous in a domain D. Let Ω be a simple smooth surface in D. Suppose that the surface Ω is divided into nonintersecting parts Ω_{ik}[1] whose area is S_{ik}. Compose the integral sum

$$\sum_{i,k} f(N_{ik})S_{ik},$$

where $N_{ik} \in \Omega_{ik}$. If there exists the limit of the integral sum while $\max(S_{ik})$ approaches zero and it does not depend either on surface partitioning or position of the points N_{ik}, then the limit is called the surface integral of the first kind of a scalar field $f(M)$ on Ω and is denotes as

$$\iint\limits_{\Omega} f(M)\mathrm{d}S \quad \text{or} \quad \oiint\limits_{\Omega} f(M)\mathrm{d}S \quad \text{for a closed surface.}$$

[1]Surfaces Ω_{ik} may have only common boundaries.

Properties

1. *Linearity.* If there exist surface integrals of fields $f(M)$ and $g(M)$ on a surface Ω, then for all real numbers α and β

$$\iint\limits_{\Omega}(\alpha f(M)+\beta g(M))\mathrm{d}S=\alpha\iint\limits_{\Omega}f(M)\mathrm{d}S+\beta\iint\limits_{\Omega}g(M)\mathrm{d}S.$$

2. *Additivity.* If a surface Ω is the union of two nonintersecting surfaces Ω_1 and Ω_2, i. e. $\Omega=\Omega_1+\Omega_2$, and for a scalar field $f(M)$ there exists the surface integral on the surface Ω then

$$\iint\limits_{\Omega}f(M)\mathrm{d}S=\iint\limits_{\Omega_1}f(M)\mathrm{d}S+\iint\limits_{\Omega_2}f(M)\mathrm{d}S.$$

3. The surface integral of a scalar field does not depend on the orientation of the surface (see section 4.7).

4. The surface integral of a scalar field does not depend on the parametrisation of the surface.

5. *Mean-value formula.* If a scalar field $f(M)$ is continuous on a surface Ω, then there is a point $M^*\in\Omega$, that

$$\iint\limits_{\Omega}f(M)\mathrm{d}S=f(M^*)S(\Omega),$$

where S is the area of the surface Ω.

6. $\iint\limits_{\Omega}\mathrm{d}S=S(\Omega).$

The evaluation of the surface integral of a continuous scalar field on a smooth surface depends on the surface equation:

a) parametric equation $\vec{r}=\vec{r}\,(u,v)\,,\;(u,v)\in D_{uv}$,

$$\iint\limits_{\Omega}f(M)\mathrm{d}S=\iint\limits_{D_{uv}}f(x(u,v),y(u,v),z(u,v))|\vec{r}_u\times\vec{r}_v|\mathrm{d}u\mathrm{d}v;$$

b) explicit equation:

- $z=z(x,y),\;(x,y)\in D_{xy}$,

$$\iint\limits_{\Omega}f\,(M)\,\mathrm{d}S=\iint\limits_{D_{xy}}f\,(x,y,z\,(x,y))\,\sqrt{1+{z'_x}^2+{z'_y}^2}\mathrm{d}x\mathrm{d}y;$$

- $x = x(y, z),\ (y, z) \in D_{yz}$,

$$\iint\limits_{\Omega} f(M)\,\mathrm{d}S = \iint\limits_{D_{yz}} f(x(y,z), y, z)\sqrt{1 + {x'_y}^2 + {x'_z}^2}\mathrm{d}y\mathrm{d}z;$$

- $y = y(x, z),\ (x, z) \in D_{zy}$,

$$\iint\limits_{\Omega} f(M)\,\mathrm{d}S = \iint\limits_{D_{xz}} f(x, y(x,z), z)\sqrt{1 + {y'_x}^2 + {y'_z}^2}\mathrm{d}x\mathrm{d}z.$$

If a surface Ω is symmetric with respect to the coordinate plane xOy and a function $f(M) = f(x, y, z)$ is odd function with respect to z, then

$$\iint\limits_{\Omega} f(M)\,\mathrm{d}S = 0;$$

if $f(x, y, z)$ is even with respect to z, then

$$\iint\limits_{\Omega} f(M)\mathrm{d}S = 2\iint\limits_{\widetilde{\Omega}} f(M)\mathrm{d}S,$$

where $\widetilde{\Omega}$ is the upper (lower) half of the Ω.

Example 1. Evaluate the surface integral of the first kind

$$\iint\limits_{\Omega}(6x + 4y + 3z)\mathrm{d}S,$$

where Ω is the part of the plane $x+2y+3z = 6$ in the first octant (fig. 4.8).

Solution. First we need to find a parametric or explicit equation of the surface. In this example it is easier to write the explicit equation expressing for example x coordinate:

$$x = 6 - 2y - 3z, \quad (y, z) \in D_{yz},$$

where D_{yz} is the projection of the initial plane onto the plane yOz:

$$D_{yz} = \{(y, z) : y \geqslant 0, z \geqslant 0, 2y + 3z \leqslant 6\}.$$

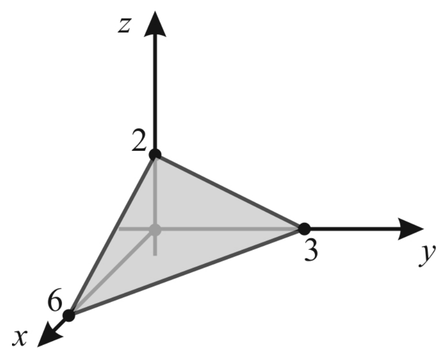

Fig. 4.8

Then we find the element of area

$$\mathrm{d}S = \sqrt{1 + {x'_y}^2 + {x'_z}^2}\mathrm{d}y\mathrm{d}z = \sqrt{1 + (-2)^2 + (-3)^2}\mathrm{d}y\mathrm{d}z = \sqrt{14}\mathrm{d}y\mathrm{d}z$$

and substitute all into the surface integral:

$$\iint\limits_{\Omega}(6x + 4y + 3z)\mathrm{d}S = \iint\limits_{D_{yz}}(6(6 - 2y - 3z) + 4y + 3z)\sqrt{14}\mathrm{d}y\mathrm{d}z =$$

$$= \sqrt{14}\int\limits_0^3 \mathrm{d}y \int\limits_0^{2-\frac{2}{3}y}(36 - 8y - 15z)\mathrm{d}z =$$

$$= 2\sqrt{14}\int\limits_0^3 \left(21 - 10y + y^2\right)\mathrm{d}y = 54\sqrt{14}.$$

Example 2. Evaluate the surface integral of the first kind

$$\iint\limits_{\Omega}(x^2 + y^2 + az)\mathrm{d}S,$$

on the upper hemisphere $x^2 + y^2 + z^2 = a^2,\ z \geqslant 0,\ a > 0$.

Solution. In this example it is more convenient to use parametric equation of the sphere:

$$x = a\cos v\sin u, \quad y = a\sin v\sin u, \quad z = a\cos u,$$

$$0 \leqslant v \leqslant 2\pi,\ 0 \leqslant u \leqslant \frac{\pi}{2}.$$

To find the element os area we compute coefficients of the first quadratic form of the surface:

$$\vec{r}_u = \frac{\partial\vec{r}}{\partial u} = a\cos v\cos u\vec{i} + a\sin v\cos u\,\vec{j} - a\sin u\vec{k},$$

$$\vec{r}_v = \frac{\partial\vec{r}}{\partial v} = -a\sin v\sin u\,\vec{i} + a\cos v\sin u\,\vec{j},$$

$$E = \vec{r}_u^{\,2} = a^2, \quad F = \vec{r}_u\cdot\vec{r}_v = 0, \quad G = \vec{r}_v^{\,2} = a^2\sin^2 u.$$

and thus

$$\mathrm{d}S = |\vec{r}_u \times \vec{r}_v|\,\mathrm{d}u\mathrm{d}v = \sqrt{EG - F^2}\,\mathrm{d}u\mathrm{d}v = a^2\sin u\,\mathrm{d}u\mathrm{d}v.$$

Substituting the parameterizations we evaluate the integral:

$$\iint\limits_{\Omega}(x^2+y^2+z)\mathrm{d}S=\int\limits_0^{\frac{\pi}{2}}\mathrm{d}u\int\limits_0^{2\pi}(a^2\cos^2 v\sin^2 u+a^2\sin^2 v\sin^2 u+$$

$$+a^2\cos u)a^2\sin u\mathrm{d}v=2\pi a^4\int\limits_0^{\frac{\pi}{2}}(\sin^3 u+\cos u\sin u)\mathrm{d}u=$$

$$=2\pi a^4\left(\frac{1}{12}\cos 3u-\frac{3}{4}\cos u-\frac{1}{2}\cos^2 u\right)\Bigg|_0^{\frac{\pi}{2}}=\frac{7}{3}\pi a^4.$$

Example 3. Evaluate the surface integral of the first kind

$$\iint\limits_{\Omega}(x^2-y^2+z^3)\mathrm{d}S,$$

on lateral surface of the cylinder $x^2+y^2=a^2$ enclosed by the planes $x+z=0$ and $x-z=0$.

Solution. The surface of integration is depicted by a dark gray color in fig. 4.9. The parametrisation of the surface is

$$x=a\cos\varphi,\quad y=a\sin\varphi,\quad z=z,$$

where parameters φ and z belong to the domain

$$D=\{(\varphi,z):\ 0\leqslant\varphi\leqslant 2\pi,\ |z|\leqslant a|\cos\varphi|\}.$$

The surface area element of the cylinder is $\mathrm{d}S=a\mathrm{d}\varphi\mathrm{d}z$. Aggregating previous calculations we obtain

$$\iint\limits_{\Omega}(x^2-y^2+z^3)\mathrm{d}S=a\int\limits_0^{2\pi}\mathrm{d}\varphi\int\limits_{-a|\cos\varphi|}^{a|\cos\varphi|}(a^2\cos^2\varphi-a^2\sin^2\varphi+z^3)\mathrm{d}z=$$

$$=2a^4\int\limits_0^{2\pi}(\cos^2\varphi-\sin^2\varphi)|\cos\varphi|\mathrm{d}\varphi=$$

$$=4a^4\int\limits_{-\pi/2}^{\pi/2}(1-2\sin^2\varphi)\cos\varphi\mathrm{d}\varphi=\frac{8a^2}{3}.$$

Example 4. Evaluate the surface integral of the first kind

$$\iint\limits_{\Omega} xyz\,\mathrm{d}S,$$

on the part of the hyperbolic paraboloid $z = xy$, enclosed by the cylinder $x^2 + y^2 = 4$.

Solution. The projection of the surface onto the plane xOy (fig. 4.10) is the disk $x^2 + y^2 \leqslant 4$. Then the explicit equation is

$$z = xy, \quad (x, y) \in D_{xy}, \quad D_{xy} = \{(x, y) \,:\, x^2 + y^2 \leqslant 4\}.$$

The area element is

$$\mathrm{d}S = \sqrt{1 + {z'_x}^2 + {z'_y}^2}\mathrm{d}x\mathrm{d}y = \sqrt{1 + x^2 + y^2}\mathrm{d}x\mathrm{d}y.$$

Substitution in integral results in

$$\iint\limits_{\Omega} xyz\mathrm{d}S = \iint\limits_{D_{xy}} x^2y^2\sqrt{1 + x^2 + y^2}\mathrm{d}x\mathrm{d}y.$$

Since the integration domain is a disk it is convenient to use polar coordinates $x = \rho\cos\varphi,\ y = \rho\sin\varphi$. The area element $\mathrm{d}x\mathrm{d}y$ transforms to $\rho\mathrm{d}\rho\mathrm{d}\varphi$, and the integration domain is $0 \leqslant \varphi \leqslant 2\pi,\ 0 \leqslant \rho \leqslant 2.$ Thus

$$\iint\limits_{\Omega} xyz\mathrm{d}S = \int\limits_0^{2\pi}\mathrm{d}\varphi\int\limits_0^{2}\rho^4\cos^2\varphi\sin^2\varphi\sqrt{1+\rho^2}\rho\mathrm{d}\rho =$$

$$= \left(\frac{\varphi}{8} - \frac{1}{32}\sin 2\varphi\right)\Bigg|_0^{2\pi}\frac{1}{105}\,\frac{8 - 12\rho^2 + 15\rho^4}{(1+\rho^2)^{3/2}}\Bigg|_0^{2} = \frac{2\pi}{105}(125\sqrt{5} - 1).$$

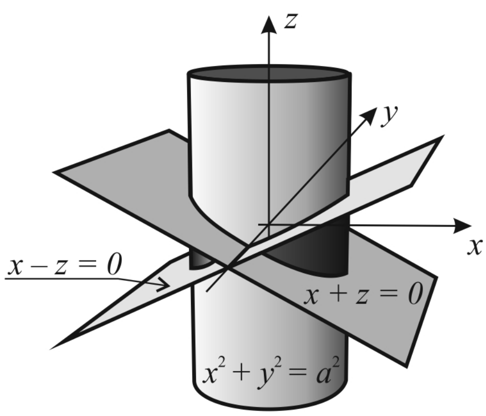

Fig. 4.9

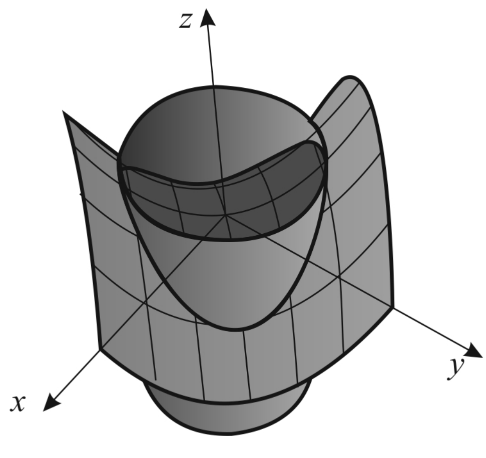

Fig. 4.10

Exercises

Evaluate the surface integral of the first kind on the given surface Ω. Consider all parameters like a, b, h etc. positive.

4.151. $\iint\limits_{\Omega} xyz\mathrm{d}S$, Ω: part of the plane $x+y+z=1$, laying in the first octant.

4.152. $\iint\limits_{\Omega} (4y-x+z)\mathrm{d}S$, Ω: part of the plane $y-x+z=2$, laying in the second octant.

4.153. $\iint\limits_{\Omega} (3x+2y+z)\mathrm{d}S$, Ω: part of the plane $x+2y+3z=6$, laying in the first octant.

4.154. $\iint\limits_{\Omega} \left(z+2x+\dfrac{4}{3}y\right)\mathrm{d}S$, Ω: part of the plane $\dfrac{x}{2}+\dfrac{y}{3}+\dfrac{z}{4}=1$, laying in the first octant.

4.155. $\oiint\limits_{\Omega} \dfrac{\mathrm{d}S}{(1+x+y)^2}$, Ω: faces of the tetrahedron $x+y+z\leqslant 1$, $x\geqslant 0$, $y\geqslant 0$, $z\geqslant 0$.

4.156. $\oiint\limits_{\Omega} (x^2+y^2+z^2)\mathrm{d}S$, Ω: faces of the cube $|x|\leqslant a$, $|y|\leqslant a$, $|z|\leqslant a$.

4.157. $\iint\limits_{\Omega} x\mathrm{d}S$, Ω: part of the sphere $x^2+y^2+z^2=a^2$, laying in the first octant.

4.158. $\iint\limits_{\Omega} (x+y+z)\mathrm{d}S$, Ω: hemisphere $x^2+y^2+z^2=a^2$, $z\geqslant 0$.

4.159. $\iint\limits_{\Omega} (x^2+y^2+az)\mathrm{d}S$, Ω: hemisphere $x^2+y^2+z^2=a^2$, $z\leqslant 0$.

4.160. $\iint\limits_{\Omega} x^2y^2\mathrm{d}S$, Ω: hemisphere $z=\sqrt{a^2-x^2-y^2}$.

4.161. $\iint\limits_{\Omega} (y+z+\sqrt{a^2-x^2})\mathrm{d}S$, Ω: part of the lateral surface of the cylinder $x^2+y^2=a^2$, $0\leqslant z\leqslant h$.

4.162. $\oiint\limits_{\Omega}(x^2 + y^2 + z^2)\mathrm{d}S$, Ω: surface of the body $x^2 + y^2 \leqslant a^2$, $0 \leqslant z \leqslant h$.

4.163. $\iint\limits_{\Omega} xy^2z^3\mathrm{d}S$, Ω: part of the lateral surface of the cylinder $x^2 + y^2 = 2ax$, $z \geqslant 0$, inside the cone $y^2 + z^2 = x^2$.

4.164. $\iint\limits_{\Omega}(x - y^2 + z^3)\mathrm{d}S$, Ω: part of the lateral surface of the cylinder $x^2 = 2y$, $0 \leqslant x \leqslant 1$, between the planes $x + z = 0$ and $x - z = 0$.

4.165. $\iint\limits_{\Omega} xz\mathrm{d}S$, Ω: part of the lateral surface of the cylinder $x^2 + y^2 = 2ax$, between the cone $z = \sqrt{x^2 + y^2}$ and paraboloid $z = \dfrac{x^2 + y^2}{2a}$.

4.166. $\iint\limits_{\Omega} \sqrt{x}\mathrm{d}S$, Ω: part of the lateral surface of the cylinder $x^2 + y^2 = 2ax$, outside the hyperboloid $x^2 + y^2 - z^2 = a^2$.

4.167. $\iint\limits_{\Omega}(x - y)\mathrm{d}S$, Ω: part of the lateral surface of the cylinder $x^2 + y^2 = a^2$, inside the cylinder $z^2 = a(a - x)$.

4.168. $\iint\limits_{\Omega} y\mathrm{d}S$, Ω: part of the lateral surface of the cylinder $x = 2y^2 + 1$, $y > 0$, cut of by the surfaces $x = y^2 + z^2$, $x = 2$, $x = 3$.

4.169. $\oiint\limits_{\Omega} |xy|\mathrm{d}S$, Ω: surface of the body formed by the cylinders $x^2 + z^2 = a^2$ and $y^2 + z^2 = a^2$.

4.170. $\oiint\limits_{\Omega}(x^2 + y^2)\mathrm{d}S$, Ω: surface of the body $\sqrt{x^2 + y^2} \leqslant z \leqslant 1$.

4.171. $\iint\limits_{\Omega} \sqrt{x^2 + y^2}\mathrm{d}S$, Ω: part of the cone $z = \sqrt{x^2 + y^2}$, $0 \leqslant z \leqslant h$.

4.172. $\iint\limits_{\Omega}(3x^2 + 5y^2 + 3z^2 - 2)\mathrm{d}S$, Ω: part of the cone $y = \sqrt{x^2 + z^2}$, between the planes $y = 0$ and $y = b$.

4.173. $\iint\limits_{\Omega}(xy + yz + zx)\mathrm{d}S$, Ω: part of the cone $z = \sqrt{x^2 + y^2}$, inside the cylinder $x^2 + y^2 = 2ax$.

4.174. $\iint\limits_{\Omega}(x+y+z)\mathrm{d}S$, Ω: part of the cone $x^2=y^2+z^2$, inside the cylinder $x^2+y^2=2ax$.

4.175. $\iint\limits_{\Omega} xyz\mathrm{d}S$, Ω: part of the cone $z^2=2xy$, $z\geqslant 0$, inside the cylinder $x^2+y^2=a^2$.

4.176. $\oiint\limits_{\Omega}(x+y+z)\mathrm{d}S$, Ω: surface of the body formed by the plane $z=0$, hemisphere $z=\sqrt{a^2-x^2-y^2}$ and cone $z=\sqrt{x^2+y^2}$.

4.177. $\iint\limits_{\Omega} z^2\mathrm{d}S$, Ω: part of the cone $x=\rho\cos\varphi\sin\alpha$, $y=\rho\sin\varphi\sin\alpha$, $z=\rho\cos\alpha$, $0\leqslant\rho\leqslant R$, $0\leqslant\varphi\leqslant 2\pi$, $0<\alpha<\dfrac{\pi}{2}$, $\alpha=\mathrm{const}$.

4.178. $\iint\limits_{\Omega}|xyz|\mathrm{d}S$, Ω: part of the paraboloid $z=x^2+y^2$, below the plane $z=1$.

4.179. $\iint\limits_{\Omega}(x^2+y^2)\mathrm{d}S$, Ω: part of the paraboloid $2z=x^2+y^2$, below the plane $z=1$.

4.180. $\iint\limits_{\Omega}\sqrt{1+\dfrac{x^2}{p^2}+\dfrac{y^2}{q^2}}\mathrm{d}S$, Ω: part of the paraboloid $z=\dfrac{x^2}{2p}+\dfrac{y^2}{2q}$, $x\geqslant 0$, inside the cylinder $\left(\dfrac{x^2}{p^2}+\dfrac{y^2}{q^2}\right)^2=a^2\left(\dfrac{x^2}{p^2}-\dfrac{y^2}{q^2}\right)$.

4.181. $\iint\limits_{\Omega}\sqrt{a^2+y^2+z^2}\mathrm{d}S$, Ω: part of the paraboloid $ax=yz$ inside the cylinder $(y^2+z^2)^2=2b^2yz$.

4.182. $\iint\limits_{\Omega}\left(x^2+y^2+z-\dfrac{1}{2}\right)\mathrm{d}S$, Ω: part of the paraboloid $2z=2-$ $-x^2-y^2$ over the plane xOy.

4.183. $\iint\limits_{\Omega} y\mathrm{d}S$, Ω: $3x^2+3y^2+z^2=3a^2$, $z>0$.

4.184. $\iint\limits_{\Omega} z\mathrm{d}S$, Ω: part of the helicoid $z=u\cos v, y=u\sin v$, $z=v$, $0\leqslant u\leqslant a$, $0\leqslant v\leqslant 2\pi$.

4.185. $\iint\limits_{\Omega}(x+y+z)\mathrm{d}S$, Ω: part of the torus $x=(b+a\cos u)\cos v$, $y=(b+a\cos u)\sin v$, $z=a\sin u$, $x\geqslant 0$, $z\geqslant 0$, $b>a$.

4.186. $\iint\limits_{\Omega}\frac{\mathrm{d}S}{x+\sqrt{y^2+z^2}}$, Ω: surface formed by rotation of the parabola $x=a\cos^4 t$, $y=a\sin^4 t$ around the axis Ox.

4.187. $\iint\limits_{\Omega}(x^2+y^2+z^2)\mathrm{d}S$, Ω: surface formed by rotation of the cardioid $\rho=a(1+\cos\varphi)$ around the axis Ox.

4.188. $\iint\limits_{\Omega}(y+z)\mathrm{d}S$, Ω: surface formed by rotation of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$, $0\leqslant t\leqslant\frac{\pi}{2}$ around the axis Ox and laying in the first octant.

4.189. $\iint\limits_{\Omega}\frac{\mathrm{d}S}{\sqrt{2-y^2-z^2}}$, Ω: surface formed by rotation of the curve $y=\sin x$, $0\leqslant x\leqslant\pi$ around the axis Ox.

4.190. $\iint\limits_{\Omega}yz\mathrm{d}S$, Ω: part of the surface formed by rotation of the curve $y=\cos x$, $|x|\leqslant\frac{\pi}{2}$, around the axis Ox, satisfied the condition $0<y<z$.

4.191. Evaluate the difference between surface integrals

$$I_1=\iint\limits_{\Omega_1}(x^2+y^2+z^2)\mathrm{d}S \quad\text{and}\quad I_2=\iint\limits_{\Omega_2}(x^2+y^2+z^2)\mathrm{d}S,$$

where Ω_1 is the sphere $x^2+y^2+z^2=a^2$; Ω_2 is the surface of the octahedron $|x|+|y|+|z|=a$ inscribed onto this sphere.

4.192. Evaluate $\oiint\limits_{\Omega}\frac{\mathrm{d}S}{h}$, where Ω is ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and h is the distance from the origin to the tangent plane at a current point of the surface.

4.193. Prove Poison's formula

$$\iint\limits_{\Omega}f(ax+by+cz)\mathrm{d}S=2\pi\int\limits_{-1}^{1}f(u\sqrt{a^2+b^2+c^2})\mathrm{d}u,$$

where Ω is the sphere $x^2+y^2+z^2=1$.

4.194. Evaluate the integral

$$F(t) = \iint\limits_{x+y+z=t} f(x,y,z)\mathrm{d}S,$$

where

$$f(x,y,z) = \begin{cases} 1 - x^2 - y^2 - z^2, & \text{if} \quad x^2+y^2+z^2 \leqslant 1, \\ 0, & \text{if} \quad x^2+y^2+z^2 > 1. \end{cases}$$

4.195. Evaluate the integral

$$F(t) = \iint\limits_{x^2+y^2+z^2=t^2} f(x,y,z)\mathrm{d}S,$$

where

$$f(x,y,z) = \begin{cases} x^2+y^2, & \text{if} \quad z \geqslant \sqrt{x^2+y^2}, \\ 0, & \text{if} \quad z < \sqrt{x^2+y^2}. \end{cases}$$

4.196. Evaluate the integral

$$F(x,y,z,t) = \iint\limits_{\Omega} f(\xi,\eta,\zeta)\mathrm{d}S,$$

where Ω is the growing sphere

$$(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = t^2$$

and

$$f(\xi,\eta,\zeta) = \begin{cases} 1, & \text{if} \quad \xi^2+\eta^2+\zeta^2 < a^2, \\ 0, & \text{if} \quad \xi^2+\eta^2+\zeta^2 \geqslant a^2, \end{cases}$$

assuming that

$$r = \sqrt{x^2+y^2+z^2} > a > 0.$$

4.6. Applications of the surface integral of the first order

Suppose that a vector function $\vec{r}(u,v)$ specifies the position of a massive nonhomogeneous shell in E^3 and $\rho(\vec{r}(u,v))$ is its surface density function.

Then

a) the total mass of the shell:

$$\mu = \iint\limits_{\Omega} \rho(\vec{r})\mathrm{d}S;$$

b) the center of gravity:

$$\vec{R}_c = \frac{1}{\mu}\iint\limits_{\Omega} \vec{r}\rho(\vec{r})\mathrm{d}S;$$

c) the moments of inertia with respect to coordinate axes:

$$I_x = \iint\limits_{\Omega} (y^2+z^2)\rho(\vec{r})\mathrm{d}S, \quad I_y = \iint\limits_{\Omega} (x^2+z^2)\rho(\vec{r})\mathrm{d}S,$$

$$I_z = \iint\limits_{\Omega} (x^2+y^2)\rho(\vec{r})\mathrm{d}S;$$

d) the moments of inertia with respect to coordinate planes:

$$I_{yz} = \iint\limits_{\Omega} x^2\rho(\vec{r})\mathrm{d}S, \quad I_{xz} = \iint\limits_{\Omega} y^2\rho(\vec{r})\mathrm{d}S, \quad I_{xy} = \iint\limits_{\Omega} z^2\rho(\vec{r})\mathrm{d}S;$$

e) the moment of inertia with respect to the origin of coordinates:

$$I_0 = \iint\limits_{\Omega} (x^2+y^2+z^2)\rho(\vec{r})\mathrm{d}S.$$

The attraction force acting on a material point $M_0 = (x_{0,}y_0, z_0)$ of mass m by a massive shell:

$$\vec{F} = \gamma m \iint\limits_{\Omega} \frac{\vec{R}}{|\vec{R}|^3}\rho\,(x,y,z)\,\mathrm{d}S,$$

where $\vec{R} = (x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}$.

Electric field strength $\vec{E}$ and potential φ at the point $M_0 = (x_0, y_0, z_0)$, induced by a charge distributed over a surface Ω with surface density $\sigma(x,y,z)$:

$$\vec{E} = k\iint\limits_{\Omega} \frac{\vec{R}}{|\vec{R}|^3}\sigma(x,y,z)\mathrm{d}S, \quad \varphi = k\iint\limits_{\Omega} \frac{\sigma(x,y,z)}{|\vec{R}|}\mathrm{d}S,$$

where $\vec{R} = (x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)\vec{k}$.

Example 1. Find the inertia moment of the part of the homogeneous cylinder $x^2 + y^2 = ax$ inside the sphere $x^2 + y^2 + z^2 = a^2$ with respect to the plane xOz.

Solution. To parameterize the cylinder we primarily complete the squares in x variable:

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}.$$

Then the parametrization is

$$x = \frac{a}{2}(1 + \cos t) = a\cos^2\frac{t}{2}, \quad y = \frac{a}{2}\sin t, \quad z = z,$$

$$0 \leqslant t \leqslant 2\pi, \quad |z| < \sqrt{a^2 - x^2 - y^2} = a\sin\frac{t}{2}.$$

Element of the area is $\mathrm{d}S = \dfrac{a}{2}\mathrm{d}t\mathrm{d}z$. Therefore we can compute the total mass of the surface:

$$M = \iint\limits_{\Omega} \rho_0 \mathrm{d}S = \frac{a\rho_0}{2}\int\limits_0^{2\pi}\mathrm{d}t\int\limits_{-a\sin\frac{t}{2}}^{a\sin\frac{t}{2}}\mathrm{d}z = a^2\rho_0\int\limits_0^{2\pi}\sin\frac{t}{2}\mathrm{d}t = 4a^2\rho_0.$$

Finally the inertia moment is

$$I_{xz} = \iint\limits_{\Omega} y^2\rho_0\mathrm{d}S = \frac{a^3\rho_0}{8}\int\limits_0^{2\pi}\mathrm{d}t\int\limits_{-a\sin\frac{t}{2}}^{a\sin\frac{t}{2}}\sin^2 t\mathrm{d}z = \frac{a^4\rho_0}{4}\int\limits_0^{2\pi}\sin\frac{t}{2}\sin^2 t\mathrm{d}t =$$

$$= \frac{a^4\rho_0}{4}\left(\frac{1}{10}\cos\frac{5t}{2} - \frac{1}{6}\cos\frac{3t}{2} - \cos\frac{t}{2}\right)\Bigg|_0^{2\pi} = \frac{8}{15}a^4\rho_0 = \frac{2}{15}Ma^2.$$

Example 2. The charge density of the surface composed from the cone $\dfrac{x^2 + y^2}{a^2} - \dfrac{z^2}{c^2} = 0,\ z > 0$ and the plane $z = c$ is $\sigma = \alpha z,\ \alpha = \mathrm{const}$. Find the total charge.

Solution. The area of the plane part is πa^2, and charge density equals to αc, therefore the total charge is $q_0 = \alpha\pi c a^2$.

The total charge of the cone Ω_1 is the surface integral

$$q_1 = \iint\limits_{\Omega_1} \alpha z\mathrm{d}S.$$

We project the cone onto the plane xOy and express $z = \dfrac{c}{a}\sqrt{x^2 + y^2}$, $x^2 + y^2 \leqslant a^2$. The area element is

$$\mathrm{d}S = \sqrt{1 + {z'_x}^2 + {z'_y}^2}\mathrm{d}x\mathrm{d}y = \frac{\sqrt{a^2 + c^2}}{a}\mathrm{d}x\mathrm{d}y.$$

Then

$$q_1 = \alpha \iint\limits_{\Omega_1} z\mathrm{d}S = \frac{\alpha c}{a} \iint\limits_{x^2+y^2 \leqslant a^2} \sqrt{x^2 + y^2}\frac{\sqrt{a^2 + c^2}}{a}\mathrm{d}x\mathrm{d}y.$$

Integrating in the polar coordinates $x = \rho\cos\varphi,\ y = \rho\sin\varphi$, $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant r \leqslant a$ we obtain

$$q_1 = \frac{\alpha c}{a}\frac{\sqrt{a^2 + c^2}}{a}\int\limits_0^{2\pi}\mathrm{d}\varphi\int\limits_0^a \rho^2\mathrm{d}\rho = \frac{2}{3}\alpha\pi ac\sqrt{a^2 + c^2}.$$

Finally the total charge is

$$q = \alpha\pi ac\left(\frac{2}{3}\sqrt{a^2 + c^2} + a\right).$$

Exercises

4.197. Evaluate the mass of the part of the homogeneous paraboloid $2z = x^2 + y^2$, $0 \leqslant z \leqslant 1$.

4.198. Evaluate the mass of the part of the paraboloid $2z = x^2 + y^2$, $0 \leqslant z \leqslant 1$, if the surface density is $\rho(x, y, z) = z$.

4.199. Evaluate the mass of the part of the cylinder $x^2 + z^2 = 2az$ inside the cone $x^2 + y^2 = z^2$, if the surface density is $\rho(x, y, z) = |y|$.

4.200. Evaluate the mass of the part of the cone $x^2 = y^2 + z^2$ inside the cylinder $x^2 + y^2 = 2ax$, if the density of the cone is $\rho(x, y, z) = x$.

4.201. Evaluate the mass of the part of the cone $x^2 + y^2 = z^2$, $0 \leqslant z \leqslant 4$, if the density at each point is equal to square of the distance from the point to the cone vertex.

4.202. Evaluate the mass of a sphere of radius a, if its density is equal to square of the distance from a point on the sphere to a fixed diameter.

4.203. Evaluate the mass of the upper hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$, with density $\rho(x, y, z) = z/a$.

4.204. Evaluate the moment of inertia of the homogeneous surface $x^2 + y^2 = 2ax,\ x^2 \geqslant y^2 + z^2$, with respect to the axis Oz.

4.205. Evaluate the moment of inertia of the homogeneous surface formed by rotation of the first arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ around the axis Ox with respect to this axis.

4.206. Evaluate the moment of inertia of the homogeneous sphere $x^2 + y^2 + z^2 = a^2$ with respect to the axis Oz.

4.207. Evaluate the moment of inertia of the homogeneous cone $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 0$, $0 \leqslant z \leqslant b$, with respect to the straight line $\frac{x}{1} = \frac{y}{0} = \frac{z-b}{0}$.

4.208. Evaluate the moment of inertia of the part of the homogeneous upper hemisphere $x^2+y^2+z^2 = a^2$, $z \geqslant 0$, inside the cylinder $x^2+y^2 = ax$, with respect to the plane yOz.

4.209. Evaluate the moment of inertia of the homogeneous cone $x^2 + y^2 = z^2 \tan^2 \alpha$, $x^2+y^2 \leqslant R^2$, $0 < \alpha < \frac{\pi}{2}$ with respect to the plane xOy.

4.210. Evaluate the moment of inertia of the homogeneous surface $\max(|x|, |y|, |z|) = a$ with respect to the origin.

4.211. Evaluate the moments of inertia of the homogeneous triangle plate $x + y + z = 1$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, with respect to coordinate planes.

4.212. Evaluate the moment of inertia of the homogeneous torus $x = (b + a\cos u)\cos v$, $y = (b + a\cos u)\sin v$, $z = a\sin u$, $b > a$, with respect to the coordinate axes.

4.213. Evaluate the moment of inertia of the homogeneous paraboloid $x^2 + y^2 = 2az$, $0 \leqslant z \leqslant a$, with respect to the axis Oz.

4.214. Evaluate the moment of inertia of the homogeneous segment of the sphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant h$, $h < a$, with respect to the axis Oz.

4.215. Evaluate the center of gravity of the homogeneous hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$.

4.216. Evaluate the center of gravity of the part of the homogeneous sphere $x^2 + y^2 + z^2 = a^2$ laying in the first octant.

4.217. Evaluate the center of gravity of the upper hemisphere $x^2+y^2+z^2 = a^2$, $z \geqslant 0$, if the density is equal to the distance between a point on the sphere and the Oz axis.

4.218. Evaluate the center of gravity of the homogeneous surface formed by rotation of the parabola $y^2 = 2px$, $0 \leqslant x \leqslant p$, around the axis Ox.

4.219. Evaluate the center of gravity of the homogeneous paraboloid $x^2 + y^2 = 2az$, $0 \leqslant z \leqslant a$.

4.220. Evaluate the center of gravity of the homogeneous cone $x^2+y^2 = \frac{a^2}{h^2}z^2,\ 0 \leqslant z \leqslant h$.

4.221. Evaluate the center of gravity of the homogeneous helicoid $x = u\cos v,\ y = u\sin v,\ z = av,\ 0 \leqslant u \leqslant a,\ 0 \leqslant v \leqslant \pi$.

4.222. Find the attractive force exerted by a homogeneous cylinder of radius a, height h and density ρ_0 on the point mass m placed in the center of the cylinder base.

4.223. Find the force exerted by a uniformly charged truncated cone $x = \rho\cos t,\ y = \rho\sin t,\ z = \rho,\ 0 \leqslant t \leqslant 2\pi,\ 0 < b \leqslant \rho \leqslant a$, with total charge Q on a point charge q placed in the cone vertex.

4.224. The surface charge density of the sphere $x^2 + y^2 + z^2 = R^2$ is $\sigma(\vec{r}) = \vec{a} \cdot \vec{r}$, where $\vec{a}$ is a constant vector a $\vec{r}$ is a radius-vector. Find the electric field strength in the center of the sphere.

4.7. Surface integrals of the second kind

A smooth surface is called *oriented* or *two-sided* if a unit normal vector to the surface can be chosen in such a manner that it varies continuously as it moves about the surface, otherwise it is called *non-oriented* (*one-sided*). Möbius strip and Klein bottle are examples of one-sided surfaces. Further only the oriented surfaces will be considered. The side the surface is determined by the direction on the normal vector.

Consider a simple smooth surface Ω defined by the equation $\vec{r} = \vec{r}(u, v)$, $(u, v) \in D$, and $\partial D = \{(u, v) :\ u = u(t),\ v = v(t),\ t \in [\alpha, \beta]\}$ is boundary of the domain D. Boundary $\partial\Omega = \{\vec{r}(t) = \vec{r}(u(t), v(t)),\ t \in [\alpha, \beta]\}$ of the surface Ω has the same orientation as the ∂D.

Orientation of the surface is associated with orientation of its boundary by right-hand rule: while a point moves along the boundary in the positive direction the positive side of the surface must lie to its left. The piecewise smooth surface is oriented by choosing the normal vector on each smooth portion of the surface in such a way that along a common boundary of two portions, the positive direction of the boundary relative to one portion is opposite to the direction of the boundary relative to another portion.

The scalar
$$\iint\limits_{\Omega} \vec{a} \cdot \mathrm{d}\vec{S} = \iint\limits_{\Omega} (\vec{a} \cdot \vec{n})\mathrm{d}S,$$

where $\mathrm{d}\vec{S} = \vec{n}\mathrm{d}S$, is called *the surface integral of the second kind* of a continuous vector field $\vec{a}(M)$ on a surface Ω, oriented by unit normals $\vec{n}$.

Suppose that in cartesian coordinates there given a continuous in the neighborhood of a surface Ω vector field

$$\vec{a}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}.$$

Then the surface integral of the second kind takes the form of

$$\iint\limits_{\Omega} \vec{a}\cdot\mathrm{d}\vec{S} = \iint\limits_{\Omega} P(x,y,z)\mathrm{d}y\mathrm{d}z + Q(x,y,z)\mathrm{d}x\mathrm{d}z + R(x,y,z)\mathrm{d}x\mathrm{d}y.$$

In physics the surface integral of the second kind is called the flux of a vector field through an oriented surface. In case of evaluating the flux through a closed surface it is common to orientate it in outward direction.

Properties

1. *Linearity.* If there exist surface integrals of vector fields $\vec{a}(M)$ and $\vec{b}(M)$ on a surface Ω, then for all real numbers α and β

$$\iint\limits_{\Omega} (\alpha\vec{a}(M) + \beta\vec{b}(M))\cdot\mathrm{d}\vec{S} = \alpha\iint\limits_{\Omega} \vec{a}(M)\cdot\mathrm{d}\vec{S} + \beta\iint\limits_{\Omega} \vec{b}(M)\cdot\mathrm{d}\vec{S}.$$

2. *Additivity.* If a surface Ω is the union of two nonintersecting surfaces Ω_1 and Ω_2, i. e. $\Omega = \Omega_1 + \Omega_2$, and for a vector field $\vec{a}(M)$ there exists the surface integral on the surface Ω then

$$\iint\limits_{\Omega} \vec{a}(M)\cdot\mathrm{d}\vec{S} = \iint\limits_{\Omega_1} \vec{a}(M)\cdot\mathrm{d}\vec{S} + \iint\limits_{\Omega_2} \vec{a}(M)\cdot\mathrm{d}\vec{S}.$$

3. Surface integrals of the second kind on surfaces with opposite orientation differ only in sign:

$$\iint\limits_{\Omega_+} \vec{a}(M)\cdot\mathrm{d}\vec{S} = -\int\limits_{\Omega_-} \vec{a}(M)\cdot\mathrm{d}\vec{S}.$$

Evaluation methods

a) A surface is defined by the parametric equation:

$$\vec{r} = \vec{r}(u,v), \quad (u,v)\in D_{uv},$$

$$\iint\limits_{\Omega} \vec{a}\cdot\mathrm{d}\vec{S} = \pm\iint\limits_{D_{uv}} (\vec{a}\cdot(\vec{r}_u\times\vec{r}_v))\Bigg|_{\substack{x=x(u,v)\\ y=y(u,v)\\ z=z(u,v)}} \mathrm{d}u\mathrm{d}v. \tag{4.1}$$

b) A surface is defined by the explicit equation:

- $z = f(x,y), \quad (x,y) \in D_{xy}, \quad \vec{N}_{xy} = \pm(-f'_x\vec{i} - f'_y\vec{j} + \vec{k}),$

$$\iint\limits_{\Omega} \vec{a} \cdot \mathrm{d}\vec{S} = \iint\limits_{D_{xy}} (\vec{a} \cdot \vec{N}_{xy})\Big|_{z=f(x,y)} \mathrm{d}x\mathrm{d}y; \tag{4.2}$$

- $y = g(x,z), \quad (x,z) \in D_{xz}, \quad \vec{N}_{xz} = \pm(-g'_x\vec{i} + \vec{j} - g'_z\vec{k}),$

$$\iint\limits_{\Omega} \vec{a} \cdot \mathrm{d}\vec{S} = \iint\limits_{D_{xz}} (\vec{a} \cdot \vec{N}_{xz})\Big|_{y=g(x,z)} \mathrm{d}x\mathrm{d}z; \tag{4.3}$$

- $x = h(y,z), \quad (y,z) \in D_{yz}, \quad \vec{N}_{yz} = \pm(\vec{i} - h'_y\vec{j} - h'_z\vec{k}),$

$$\iint\limits_{\Omega} \vec{a} \cdot d\vec{S} = \iint\limits_{D_{yz}} (\vec{a} \cdot \vec{N}_{yz})\Big|_{x=h(y,z)} \mathrm{d}y\mathrm{d}z. \tag{4.4}$$

The surface integrals

$$\iint\limits_{\Omega_+} R(x,y,z)\mathrm{d}x\mathrm{d}y \quad \text{and} \quad \iint\limits_{\Omega_-} R(x,y,z)\mathrm{d}x\mathrm{d}y$$

are evaluated by the formulas

$$\iint\limits_{\Omega_+} R(x,y,z)\mathrm{d}x\mathrm{d}y = \iint\limits_{\Omega} R(x,y,z)\cos(\widehat{\vec{n},\vec{k}})\mathrm{d}S,$$

$$\iint\limits_{\Omega_-} R(x,y,z)\mathrm{d}x\mathrm{d}y = \iint\limits_{\Omega} R(x,y,z)\cos(\widehat{-\vec{n},\vec{k}})\mathrm{d}S,$$

where $(\widehat{\vec{n},\vec{k}})$ and $(\widehat{-\vec{n},\vec{k}})$ are angles between corresponding vectors. If $R(x,y,z) \equiv 1$ and the surface equation is $z = z(x,y)$, then the surface integral $\iint\limits_{\Omega} \mathrm{d}x\mathrm{d}y$ is equal to area of the projection D_{xy} of the surface onto the plane xOy taken with positive sign if the angle $(\widehat{\vec{n},\vec{k}})$ is acute for all surface points and with negative sign otherwise

$$\iint\limits_{\Omega} \mathrm{d}x\mathrm{d}y = \pm \iint\limits_{Dxy} \mathrm{d}x\mathrm{d}y = \pm S(D_{xy}).$$

If a surface Ω is orthogonal to the plane xOy then

$$\iint\limits_{\Omega} R(x, y, z)\mathrm{d}x\mathrm{d}y = 0.$$

If the surface Ω is symmetric with respect to xOy and a function $R(x, y, z)$ is odd with respect to z, then

$$\iint\limits_{\Omega} R(x, y, z)\mathrm{d}x\mathrm{d}y = 2\iint\limits_{\widetilde{\Omega}} R(x, y, z)\mathrm{d}x\mathrm{d}y,$$

where $\widetilde{\Omega}$ is upper (lower) half of Ω; if the function $R(x, y, z)$ is even with respect to z, then

$$\iint\limits_{\Omega} R(x, y, z)\mathrm{d}x\mathrm{d}y = 0.$$

c) A surface Ω is the part of the *cylinder* $x^2 + y^2 = R^2$ bounded by surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$ $(f_1(x, y) \leqslant f_2(x, y))$ and half-planes $\varphi = \varphi_1$ and $\varphi = \varphi_2$ then

$$x = R\cos\varphi, \quad y = R\sin\varphi, \quad z = z;$$

$$\vec{n} = \frac{x\vec{i} + y\vec{j}}{R} = \vec{i}\cos\varphi + \vec{j}\sin\varphi;$$

$$\iint\limits_{\Omega} \vec{a}\cdot\mathrm{d}\vec{S} = R\int\limits_{\varphi_1}^{\varphi_2}\mathrm{d}\varphi \int\limits_{f_1(R\cos\varphi, R\sin\varphi)}^{f_2(R\cos\varphi, R\sin\varphi)} (\vec{a}\cdot\vec{n})\Big|_{\substack{x=R\cos\varphi\\ y=R\sin\varphi}} \mathrm{d}z. \tag{4.5}$$

d) A surface Ω is the part of the *sphere* $x^2 + y^2 + z^2 = R^2$ bounded by conical surfaces $\theta = f_1(\varphi)$ and $\theta = f_2(\varphi)$ $(f_1(\varphi) \leqslant f_2(\varphi))$ and half-planes $\varphi = \varphi_1$ and $\varphi = \varphi_2$:

$$x = R\cos\varphi\sin\theta, \quad y = R\sin\varphi\sin\theta, \quad z = R\cos\theta;$$

$$\vec{n} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{R} = \vec{i}\cos\varphi\sin\theta + \vec{j}\sin\varphi\sin\theta + \vec{k}\cos\theta;$$

$$\iint\limits_{\Omega} \vec{a}\cdot\mathrm{d}\vec{S} = R^2\int\limits_{\varphi_1}^{\varphi_2}\mathrm{d}\varphi \int\limits_{f_1(\varphi)}^{f_2(\varphi)} (\vec{a}\cdot\vec{n})\Big|_{\substack{x=R\cos\varphi\sin\theta\\ y=R\sin\varphi\sin\theta\\ z=R\cos\theta}} \sin\theta\mathrm{d}\theta. \tag{4.6}$$

Example 1. Evaluate the surface integral of the second kind

$$\iint\limits_{\Omega} \vec{a}\cdot \mathrm{d}\vec{S} = \iint\limits_{\Omega} \frac{x}{\sqrt{x^2+y^y}}\mathrm{d}y\mathrm{d}z + \frac{y}{\sqrt{x^2+y^y}}\mathrm{d}x\mathrm{d}z,$$

on the upper side of the conical surface

$$\vec{r}\,(u,v) = u\cos v\sin\alpha\vec{i} + u\sin v\sin\alpha\vec{j} + u\cos\alpha\vec{k},$$

$$0 \leqslant v \leqslant 2\pi,\ 0 \leqslant u \leqslant 1; \qquad 0 < \alpha < \frac{\pi}{2}.$$

Solution. The domain D_{uv} is the rectangle

$$D_{uv} = \{(u,v):\ 0 \leqslant v \leqslant 2\pi,\ 0 \leqslant u \leqslant 1\}.$$

According to the formula (4.1) we are to find tangent vectors $\vec{r}_u,\ \vec{r}_v$, then find the scalar triple product $\vec{a}\vec{r}_u\vec{r}_v$, choose the normal vector direction, and finally substitute all in the surface integral. Let's do it consequently.

Tangent vectors:

$$\vec{r}_u = \cos v\sin\alpha\,\vec{i} + \sin v\sin\alpha\,\vec{j} + \cos\alpha\vec{k},$$

$$\vec{r}_v = -u\sin v\sin\alpha\,\vec{i} + u\cos v\sin\alpha\,\vec{j}.$$

Scalar triple product:

$$\vec{a}\cdot(\vec{r}_u\times\vec{r}_v)\Big|_{(x,y,z)\in\Omega} =$$

$$= \begin{vmatrix} \cos v & \sin v & 0 \\ \cos v\sin\alpha & \sin v\sin\alpha & \cos\alpha \\ -u\sin v\sin\alpha & u\cos v\sin\alpha & 0 \end{vmatrix} = -\frac{u}{2}\sin 2\alpha.$$

The term "upper side of the surface" assumes that z-component of the normal vector is positive. In our case it is

$$n_z = \pm\begin{vmatrix} \cos v\sin\alpha & \sin v\sin\alpha \\ -u\sin v\sin\alpha & u\cos v\sin\alpha \end{vmatrix} = \pm u\sin^2\alpha.$$

Since the value $u\sin^2\alpha$ in the domain D_{uv} is positive then we choose the sign "+".

Finally the integral is

$$\iint\limits_{\Omega} \vec{a} \cdot \mathrm{d}\vec{S} = +\iint\limits_{D_{uv}} \vec{a} \cdot (\vec{r}_u \times \vec{r}_v)\Big|_{(x,y,z)\in\Omega} \mathrm{d}u\mathrm{d}v =$$

$$= -\int\limits_0^1 \frac{u}{2} \sin 2\alpha \mathrm{d}u \int\limits_0^{2\pi} \mathrm{d}v = -\frac{\pi}{2} \sin 2\alpha.$$

Example 2. Evaluate the surface integral of the second kind

$$\oiint\limits_{\Omega} xy\mathrm{d}y\mathrm{d}z + 2z^2\mathrm{d}x\mathrm{d}z + x^2\mathrm{d}x\mathrm{d}y$$

on the outward side of the tetrahedron $4x + 3y + 4z \leqslant 12$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$.

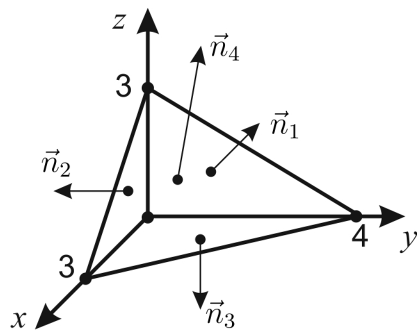

Fig. 4.11

Solution. Since Ω is a piecewise smooth surface we divide it into the four smooth portions Ω_i, $i = \overline{1, 4}$ with the corresponding orientation then (fig. 4.11). Then we consequently evaluate the integrals on each portion.

The surface Ω_1 is the domain of the plane $x = 0$:

$$\Omega_1 = D_{yz} = \{(y, z) : \ 3y + 4z \leqslant 12, \ y \geqslant 0, \ z \geqslant 0\}.$$

The outward unit normal vector is $\vec{n}_1 = -\vec{i}$. Then

$$\iint\limits_{\Omega_1} \vec{a} \cdot \mathrm{d}\vec{S} = \iint\limits_{\Omega_1} (\vec{a} \cdot \vec{n}_1)\mathrm{d}S = -\iint\limits_{D_{yz}} xy\big|_{x=0} \mathrm{d}y\mathrm{d}z = 0.$$

The surface Ω_2 is the subset of the plane $y = 0$:

$$D_{xz} = \{(x, z) : \ x + z \leqslant 3, \ x \geqslant 0, \ z \geqslant 0\}.$$

The outward unit normal vector is $\vec{n}_2 = -\vec{j}$ and the integral is

$$\iint\limits_{\Omega_2} \vec{a} \cdot \mathrm{d}\vec{S} = \iint\limits_{\Omega_2} (\vec{a} \cdot \vec{n}_2)\, \mathrm{d}S = -\iint\limits_{D_{xz}} 2z^2\big|_{y=0} \mathrm{d}x\mathrm{d}z =$$

$$= -2\int\limits_0^3 z^2 \mathrm{d}z \int\limits_0^{3-z} \mathrm{d}x = -\frac{27}{2}.$$

Then the surface Ω_3 is the domain of the plane $z = 0$:

$$D_{xy} = \{(x, y) :\ 4x + 3y \leqslant 12,\ x \geqslant 0,\ y \geqslant 0\}.$$

The outward unit normal vector is $\vec{n}_3 = -\vec{k}$ and

$$\iint\limits_{\Omega_3} \vec{a} \cdot \mathrm{d}\vec{S} = \iint\limits_{\Omega_3} (\vec{a} \cdot \vec{n}_3)\mathrm{d}S = -\iint\limits_{D_{xy}} x^2\big|_{z=0} \mathrm{d}x\mathrm{d}y =$$

$$= -\int\limits_0^3 x^2 \mathrm{d}x \int\limits_0^{4-\frac{4}{3}x} \mathrm{d}y = -9.$$

And finally to evaluate the integral on the surface Ω_4 we use the formula (4.3). Expressing y from the implicit surface equation we obtain that

$$y = 4 - \frac{4}{3}x - \frac{4}{3}z, \quad (x, z) \in D_{xz},$$

and the outward unit normal vector is

$$\vec{N}_{xz} = \frac{4}{3}\vec{i} + \vec{j} + \frac{4}{3}\vec{k}.$$

Substitution into integral yields

$$\iint\limits_{\Omega_3} \vec{a} \cdot d\vec{S} = \iint\limits_{D_{xz}} (\vec{a} \cdot \vec{N}_{xz})\Big|_{y=4-\frac{4}{3}x-\frac{4}{3}z} \mathrm{d}x\mathrm{d}z =$$

$$= \int\limits_0^3 \mathrm{d}z \int\limits_0^{3-z} \left(-\frac{4}{9}x^2 + \frac{16}{3}x + 2z^2 - \frac{16}{9}zx\right) \mathrm{d}x = \frac{57}{2}.$$

Therefore the value of the integral on the whole surface is

$$\oiint\limits_{\Omega} \vec{a} \cdot \mathrm{d}\vec{S} = \sum_{i=1}^{4} \iint\limits_{\Omega_i} \vec{a} \cdot \mathrm{d}\vec{S} = 0 - \frac{27}{2} - 9 + \frac{57}{2} = 6.$$

Example 3. Evaluate the surface integral of the second kind

$$\iint\limits_{\Omega} y^2 \mathrm{d}y\mathrm{d}z + x\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y$$

on the outward side of the paraboloid $x = y^2 + z^2 - 1,\ x \leqslant 3$.

Solution. We can uniquely project this paraboloid onto the plane yOz and use the formula (4.4). In our case $D_{yz}:\ y^2 + z^2 \leqslant 4$, and

$$\vec{N}_{yz} = \pm(\vec{i} - x'_y\vec{j} - x'_z\vec{k}) = \pm(\vec{i} - 2y\vec{j} - 2z\vec{k}).$$

Since the outward side of the given paraboloid corresponds to negative value of N_x, then we are to chose the minus sign, i. e. $\vec{N}_{yz} = -\vec{i} + 2y\vec{j} + 2z\vec{k}$. Then

$$\iint\limits_{\Omega} y^2\mathrm{d}y\mathrm{d}z + x\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y =$$

$$= \iint\limits_{D_{yz}} (y^2\vec{i} + x\vec{j} + z^2\vec{k}) \cdot (-\vec{i} + 2y\vec{j} + 2z\vec{k})\Big|_{x=y^2+z^2-1} \mathrm{d}y\mathrm{d}z =$$

$$= \iint\limits_{y^2+z^2\leqslant 4} (2y^3 + 2z^3 + 2yz^2 - y^2 - 2y)\mathrm{d}y\mathrm{d}z = \begin{bmatrix} y = \rho\cos\varphi \\ z = \rho\sin\varphi \\ \mathrm{d}y\mathrm{d}z = \rho\mathrm{d}\rho\mathrm{d}\varphi \end{bmatrix} =$$

$$= \int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^2 \rho\mathrm{d}\rho\big(2\rho^3(\cos^3\varphi + \sin^3\varphi + \sin^2\varphi\cos\varphi) -$$

$$-\rho^2\cos^2\varphi - 2\rho\cos\varphi\big) = -4\pi.$$

Example 4. Evaluate the surface integral of the second kind

$$\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$$

on the outward side of the lateral surface of the cylinder $x^2 + y^2 = 1$ inside the cylinder $y^2 + z^2 = 1$.

Solution. In this example it is reasonable to pass to the cylindrical coordinates $x = \cos\varphi,\ y = \sin\varphi,\ z = z$ and use the formula (4.5). The integration domain becomes $0 \leqslant \varphi \leqslant 2\pi, -\sqrt{1-y^2} \leqslant z \leqslant \sqrt{1-y^2}$. Then we are to find the dot product

$$\vec{a} \cdot \vec{n} = (x\vec{i} + y\vec{j} + z\vec{k}) \cdot (x\vec{i} + y\vec{j}) = x^2 + y^2.$$

Taking into account that $x^2 + y^2 = 1$ on the cylinder, and $\sqrt{1 - y^2} =$ $= \sqrt{1 - \sin^2 \varphi} = |\cos \varphi|$, we obtain

$$\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y = \int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_{-|\cos\varphi|}^{|\cos\varphi|} \mathrm{d}z = 2\int\limits_0^{2\pi} |\cos\varphi|\mathrm{d}\varphi = 8.$$

Example 5. Evaluate the surface integral of the second kind

$$\iint\limits_{\Omega} (x - 2y + 1)\,\mathrm{d}y\mathrm{d}z + (2x + y - 3z)\,\mathrm{d}x\mathrm{d}z + (2y + z)\,\mathrm{d}x\mathrm{d}y$$

on the outward side of the part of the sphere $x^2 + y^2 + z^2 = 1$ laying in the first octant.

Solution. To use the formula (4.6) we pass to the spherical coordinates:

$$x = \cos\varphi \sin\theta, \quad y = \sin\varphi \sin\theta, \quad z = \cos\theta.$$

The first octant corresponds to the domain $0 \leqslant \varphi \leqslant \dfrac{\pi}{2}, 0 \leqslant \theta \leqslant \dfrac{\pi}{2}$. The outward unit normal vector is $\vec{n} = x\vec{i} + y\vec{j} + z\vec{k}$. Then

$$\vec{a} \cdot \vec{n} = ((x - 2y + 1)\vec{i} + (2x + y - 3z)\vec{j} + (2y + z)\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) =$$

$$= x^2 + y^2 + z^2 + x - yz$$

and

$$\vec{a} \cdot \vec{n}\big|_{\Omega} = 1 + \cos\varphi \sin\theta - \sin\varphi \sin\theta \cos\theta.$$

Finally

$$\iint\limits_{\Omega} \vec{a} \cdot \mathrm{d}\vec{S} = . \int\limits_0^{\frac{\pi}{2}} \mathrm{d}\varphi \int\limits_0^{\frac{\pi}{2}} (1 + \cos\varphi \sin\theta - \sin\varphi \sin\theta \cos\theta) \sin\theta \mathrm{d}\theta =$$

$$= -\ \varphi|_0^{\frac{\pi}{2}} \cos\theta|_0^{\frac{\pi}{2}} + \sin\varphi|_0^{\frac{\pi}{2}} \cdot \left.\frac{2\theta - \sin 2\theta}{4}\right|_0^{\frac{\pi}{2}} +$$

$$+ \cos\varphi|_0^{\frac{\pi}{2}} \cdot \left.\frac{\sin^3\theta}{3}\right|_0^{\frac{\pi}{2}} = \frac{3\pi}{4} - \frac{1}{3}.$$

Exercises

Evaluate the surface integral of the second kind on the given side of the surface Ω. Consider all parameters like a, b, p etc. positive.

4.225. $\oiint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cube surface $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$.

4.226. $\oiint\limits_{\Omega} f(x)\mathrm{d}y\mathrm{d}z + g(y)\mathrm{d}x\mathrm{d}z + h(z)\mathrm{d}x\mathrm{d}y$, Ω: outward sine of the parallelepiped surface $0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b,\ 0 \leqslant z \leqslant c,\ f(x), g(y), h(z)$ are continuous functions.

4.227. $\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: upper side of the plane $x + y + {}$ $+2z = 1$, laying in the first octant.

4.228. $\iint\limits_{\Omega} 2xy\mathrm{d}y\mathrm{d}z + (x - 3y)\mathrm{d}x\mathrm{d}z + y^2\mathrm{d}x\mathrm{d}y$, Ω: upper side of the plane $-3x + y + 6z = 3$, laying in the second octant.

4.229. $\oiint\limits_{\Omega} (3x - 1)\mathrm{d}y\mathrm{d}z + (y - x + z)\mathrm{d}x\mathrm{d}z + 4z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the tetrahedron formed by the plane $2x - y - 2z = 2$ and the coordinate planes.

4.230. $\iint\limits_{\Omega} x^2\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}z\mathrm{d}x + z^2\mathrm{d}x\mathrm{d}y$, Ω: upper side of the paraboloid $z = x^2 + y^2,\ z \leqslant h$.

4.231. $\iint\limits_{\Omega} 2x\mathrm{d}y\mathrm{d}z - 3y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: upper side of the paraboloid $z = 9 - x^2 - y^2,\ z \geqslant 0$.

4.232. $\iint\limits_{\Omega} y\mathrm{d}y\mathrm{d}z - x\mathrm{d}x\mathrm{d}z + 3z\mathrm{d}x\mathrm{d}y$, Ω: bottom side of the paraboloid $z = 8 - x^2 - y^2,\ z \geqslant 2$.

4.233. $\iint\limits_{\Omega} x^2\mathrm{d}y\mathrm{d}z - y^2\mathrm{d}x\mathrm{d}z + (z - 1)\,\mathrm{d}x\mathrm{d}y$, Ω: bottom side of the paraboloid $z = 6 - x^2 - y^2,\ z \geqslant 2$.

4.234. $\iint\limits_{\Omega} xz\mathrm{d}y\mathrm{d}z - x\mathrm{d}x\mathrm{d}z + 4z\mathrm{d}x\mathrm{d}y$, Ω: left side of the paraboloid $y = 1 + x^2 + z^2,\ y \leqslant 2$.

4.235. $\iint\limits_{\Omega} y\mathrm{d}y\mathrm{d}z + 2xy\mathrm{d}x\mathrm{d}z - x\mathrm{d}x\mathrm{d}y$, Ω: outward side of the paraboloid $y = 4 - x^2 - z^2,\ y \geqslant 0$.

4.236. $\iint\limits_{\Omega} y^2\mathrm{d}y\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}z + x^2\mathrm{d}x\mathrm{d}y$, Ω: inward side of the paraboloid $x = y^2 + z^2 - 1,\ x \leqslant 3$.

4.237. $\iint\limits_{\Omega} (y^2 + z^2)\mathrm{d}x\mathrm{d}z$, Ω: inward side of the paraboloid $x = a^2 - -y^2 - z^2,\ x \geqslant 0$.

4.238. $\iint\limits_{\Omega} x^3\mathrm{d}y\mathrm{d}z + y^3\mathrm{d}x\mathrm{d}z + z^3\mathrm{d}x\mathrm{d}y$, Ω: upper side of the paraboloid $x^2 + y^2 = 2 - z,\ z \geqslant 0$.

4.239. $\iint\limits_{\Omega} (x^4 + y^4 + 2a^2z^2)\mathrm{d}x\mathrm{d}y$, Ω: bottom side of the paraboloid $az = xy$, laying in the first octant and inside the cylinder $(x^2+y^2)^2 = b^2xy$.

4.240. $\iint\limits_{\Omega} xy\mathrm{d}y\mathrm{d}z + yz\mathrm{d}x\mathrm{d}z + zx\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cone $z = \sqrt{x^2 + y^2},\ 0 \leqslant z \leqslant h$.

4.241. $\iint\limits_{\Omega} (y - z)\mathrm{d}y\mathrm{d}z + (z - x)\mathrm{d}x\mathrm{d}z + (x - y)\mathrm{d}x\mathrm{d}y$, Ω: inward side of the cone $z = \sqrt{x^2 + y^2},\ 0 \leqslant z \leqslant h$.

4.242. $\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: bottom side of the cone $z = 1 - \sqrt{x^2 + y^2},\ 0 \leqslant z \leqslant 1$.

4.243. $\iint\limits_{\Omega} y\mathrm{d}y\mathrm{d}z - x\mathrm{d}x\mathrm{d}z + 2z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cone $z = \sqrt{x^2 + y^2},\ 0 \leqslant z \leqslant 4$.

4.244. $\iint\limits_{\Omega} (x^2 + y^2 + z^2)\mathrm{d}x\mathrm{d}z$, Ω: upper side of the cone $y = \sqrt{x^2 + z^2}$, $0 \leqslant y \leqslant b$.

4.245. $\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z - 3y^2\mathrm{d}x\mathrm{d}z - z\mathrm{d}x\mathrm{d}y$, Ω: right side of the cone $y = -\sqrt{x^2 + z^2},\ -3 \leqslant y \leqslant 0$.

4.246. $\iint\limits_{\Omega} yz\mathrm{d}y\mathrm{d}z - x\mathrm{d}x\mathrm{d}z - y\mathrm{d}x\mathrm{d}y$, Ω: inward side of the cone $z = \sqrt{x^2+y^2}$, $0 \leqslant z \leqslant 1$.

4.247. $\iint\limits_{\Omega} x^2\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cone $2z = \sqrt{x^2+y^2}$, $0 \leqslant z \leqslant 1$.

4.248. $\iint\limits_{\Omega} y^2\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y$, Ω: inward side of the cone $x = \sqrt{y^2+z^2}$, $0 \leqslant x \leqslant 3$.

4.249. $\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z - y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: inward side of the cone $z^2 = x^2 + y^2$, laying upper than the plane $z = 0$ and inside the cylinder $x^2+y^2 = a^2$.

4.250. $\iint\limits_{\Omega} (xz^2+y^2)\mathrm{d}y\mathrm{d}z + (yx^2+z^2)\mathrm{d}x\mathrm{d}z + (zy^2+x^2)\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cone $1 - z = \sqrt{x^2+y^2}$, $z \geqslant 0$.

4.251. $\iint\limits_{\Omega} (y^2+z^2)\mathrm{d}x\mathrm{d}y$, Ω: upper side of the cylinder $z = \sqrt{a^2-x^2}$, $0 \leqslant y \leqslant b$.

4.252. $\iint\limits_{\Omega} (x^2+z^2)\mathrm{d}y\mathrm{d}z$, Ω: outward side of the cylinder $x = \sqrt{9-y^2}$, $0 \leqslant z \leqslant 2$.

4.253. $\iint\limits_{\Omega} yz\mathrm{d}y\mathrm{d}z + zx\mathrm{d}x\mathrm{d}z + xy\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cylinder $x^2+y^2 = a^2$, $0 \leqslant z \leqslant h$.

4.254. $\iint\limits_{\Omega} (x+y^2)\mathrm{d}y\mathrm{d}z + (y+z^2)\mathrm{d}x\mathrm{d}z + (z+x^2)\mathrm{d}x\mathrm{d}y$, Ω: inward side of the cylinder $x^2+y^2 = a^2$, $0 \leqslant z \leqslant h$.

4.255. $\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cylinder $x^2+y^2 = 1$, bounded by the planes $x+y+z = 1$ and $x+y+z = 2$.

4.256. $\iint\limits_{\Omega} y\mathrm{d}y\mathrm{d}z + x\mathrm{d}x\mathrm{d}z - e^{xyz}\mathrm{d}x\mathrm{d}y$, Ω: inward side of the cylinder $x^2+y^2 = 4$, bounded by the planes $z = 0$ and $x+y+z = 4$.

4.257. $\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z - xy\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cylinder $x^2+y^2=a^2$, bounded by the plane $y=1$ and $x+y=4$.

4.258. $\oiint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $x^2+y^2 \leqslant a^2,\ -h \leqslant z \leqslant h$.

4.259. $\oiint\limits_{\Omega} (x+y^2)\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y$, Ω: inward side of the surface of the body $x^2+y^2 \leqslant a^2,\ -h \leqslant z \leqslant h$.

4.260. $\iint\limits_{\Omega} x^3\mathrm{d}y\mathrm{d}z - y^3\mathrm{d}x\mathrm{d}z + xz^3\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cylinder $x^2+y^2=9$ inside the sphere $x^2+y^2+z^2=25$.

4.261. $\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z - y\mathrm{d}x\mathrm{d}z + xyz^3\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cylinder $x^2+y^2=1$, bounded by the plane $z=0$ and paraboloid $z=x^2-y^2$.

4.262. $\iint\limits_{\Omega} (xy-y^2)\mathrm{d}y\mathrm{d}z + (2x-x^2+xy)\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: inward side of the cylinder $x^2+y^2=1$, bounded by the elliptic cone $z^2=\dfrac{x^2}{2}+y^2$.

4.263. $\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cylinder $x^2+y^2=1$, bounded by the cylinder $y^2+z^2=1$.

4.264. $\iint\limits_{\Omega} (x^2+6z-2y^2)\mathrm{d}x\mathrm{d}y$, Ω: bottom side of the cylinder $y^2=6z,\ 0 \leqslant x \leqslant 3,\ 0 \leqslant z \leqslant 6$.

4.265. $\iint\limits_{\Omega} (ax^2+by^2+cz^2)\mathrm{d}y\mathrm{d}z$, Ω: outward side of the cylinder $y^2=2px,\ 0 \leqslant z \leqslant q,\ 0 \leqslant x \leqslant 2p$.

4.266. $\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cylinder $y^2+x=1,\ 0 \leqslant z \leqslant 2,\ x \geqslant 0$.

4.267. $\iint\limits_{\Omega} (4x^2+z^2)\mathrm{d}y\mathrm{d}z + 4xy\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y$, Ω: front side of the cylinder $4x^2-y^2=a^2$, bounded by the cone $x=\sqrt{y^2+z^2},\ a>0$.

4.268. $\iint\limits_{\Omega} (x-2y+z)\,\mathrm{d}y\mathrm{d}z + (2x+y-3z)\,\mathrm{d}x\mathrm{d}z + (2y+z)\,\mathrm{d}x\mathrm{d}y$, Ω: outward side of the sphere $x^2+y^2+z^2=1$, laying in the first octant.

4.269. $\iint\limits_{\Omega} x^3\mathrm{d}y\mathrm{d}z - y^3\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the sphere $x^2+y^2+z^2=1$, bounded by the cone $z=\sqrt{x^2+y^2}$.

4.270. $\iint\limits_{\Omega} yz\mathrm{d}y\mathrm{d}z + xz\mathrm{d}x\mathrm{d}z + xy\mathrm{d}x\mathrm{d}y$, Ω: outward side of the sphere $x^2+y^2+z^2=a^2$, laying in the first octant.

4.271. $\iint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: inward side of the sphere x^2+ $+y^2+z^2=2$, bounded by the planes $z=0$ and $z=y,\ y>0$.

4.272. $\iint\limits_{\Omega} xz\mathrm{d}y\mathrm{d}z + yz\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y$, Ω: outward side of the sphere $x^2+y^2+z^2=2,\ z\geqslant 1$.

4.273. $\oiint\limits_{\Omega} x^2\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y$, Ω: outward side of the sphere $(x-a)^2+(y-b)^2+(z-c)^2=R^2$.

4.274. $\iint\limits_{\Omega} x^2\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y$, Ω: outward side of the sphere $x^2+y^2+z^2=a^2$, laying in the first octant.

4.275. $\oiint\limits_{\Omega} x^3\mathrm{d}y\mathrm{d}z + y^3\mathrm{d}x\mathrm{d}z + z^3\mathrm{d}x\mathrm{d}y$, Ω: outward side of the sphere $x^2+y^2+z^2=x$.

4.276. $\iint\limits_{\Omega} (y-z)\,\mathrm{d}y\mathrm{d}z + (z-x)\,\mathrm{d}x\mathrm{d}z + (x-y)\,\mathrm{d}x\mathrm{d}y$, Ω: outward side of the upper hemisphere $x^2+y^2+z^2=2Rx$, inside the cylinder $x^2+y^2=2ax,\ a<R$.

4.277. $\oiint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: inward side of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

4.278. $\oiint\limits_{\Omega} \dfrac{\mathrm{d}y\mathrm{d}z}{x} + \dfrac{\mathrm{d}x\mathrm{d}z}{y} + \dfrac{\mathrm{d}x\mathrm{d}y}{z}$, Ω: outward side of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

4.279. $\displaystyle\oint\!\!\!\oint_{\Omega} \frac{\cos\alpha}{x^2}\mathrm{d}y\mathrm{d}z + \frac{\cos\beta}{y^2}\mathrm{d}x\mathrm{d}z + \frac{\cos\gamma}{z^2}\mathrm{d}x\mathrm{d}y$, Ω: outward side of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} + \dfrac{z^2}{c^2} = 1$; $(\cos\alpha,\ \cos\beta,\ \cos\gamma)$ are direction cosines of the radius vector.

4.280. $\displaystyle\oint\!\!\!\oint_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the ellipsoid $\dfrac{x^2}{9} + \dfrac{y^2}{4} + z^2 = 1$.

4.281. $\displaystyle\iint_{\Omega} x^3\mathrm{d}y\mathrm{d}z + y^3\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: inward side of the hyperboloid $x^2 + y^2 - z^2 = 1,\ 0 \leqslant z \leqslant 3$.

4.282. $\displaystyle\iint_{\Omega} (x^2 + y^2)\mathrm{d}y\mathrm{d}z + (y^2 + z^2)\mathrm{d}x\mathrm{d}z + (z^2 + x^2)\mathrm{d}x\mathrm{d}y$, Ω: the disk $z = 0,\ x^2 + y^2 \leqslant 1$, which side is defined by the normal $\vec{k}$.

4.283. $\displaystyle\iint_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: upper side of the disk $z = a$, $\sqrt{x^2 + y^2} \leqslant a$.

4.284. $\displaystyle\iint_{\Omega} \sqrt{x^2 + y^2}\mathrm{d}y\mathrm{d}z + \sqrt{x^2 + y^2}\mathrm{d}x\mathrm{d}z + \sqrt{z}\mathrm{d}x\mathrm{d}y$, Ω: right side of the surface of the body $x^2 + y^2 \leqslant z^2,\ x^2 + y^2 \leqslant 2 - z,\ z \geqslant 0,\ x \geqslant 0$.

4.285. $\displaystyle\iint_{\Omega} y^2\mathrm{d}y\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}z - x^2\mathrm{d}x\mathrm{d}y$, Ω: surface of the body $2x^2 + 2y^2 \leqslant az \leqslant x^2 + y^2 + a^2,\ y \geqslant 0,\ a > 0$, the normal vector at the point $M\left(0, \dfrac{a}{2}, \dfrac{5a}{4}\right)$ forming acute angle with the axis Oz.

4.286. $\displaystyle\iint_{\Omega} y\mathrm{d}y\mathrm{d}z - x\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: upper side of the helicoid $x = u\cos v,\ y = u\sin v,\ z = av,\ 0 \leqslant v \leqslant 2\pi,\ 0 \leqslant u \leqslant 1$.

4.287. $\displaystyle\iint_{\Omega} xy\mathrm{d}y\mathrm{d}z + yz\mathrm{d}x\mathrm{d}z + zx\mathrm{d}x\mathrm{d}y$, Ω: left side of the surface $x = 2u + v^2,\ y = u^2 - 2v,\ z = 2uv,\ 0 \leqslant v \leqslant 1,\ 0 \leqslant u \leqslant 1$.

4.8. Green's theorem in the plane

A domain $D \subset E^2$ is said to be *simply connected* (also called *1-connected*) if any simple closed curve can be shrunk to a point continuously in the set, otherwise it is said to be *multiply connected*.

Let $P(x,y)$, $Q(x,y)$, $\dfrac{\partial P}{\partial y}$, $\dfrac{\partial Q}{\partial x}$ be continuous functions in a closed simply connected domain D, bounded by a piecewise smooth simple closed curve ∂D, then

$$\oint\limits_{\partial D} P\mathrm{d}x + Q\mathrm{d}y = \iint\limits_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y, \tag{4.7}$$

where line integral is taken in positive (counterclockwise) direction of ∂D. Formula (4.7) is called Green's formula. In case of multiply connected domain D the curve ∂D is a union of all simple piecewise smooth closed curves bounding the domain taken in the positive direction.

Green's formula can be used to evaluate the area of a domain:

$$S_D = \frac{1}{2} \int\limits_{\partial D} x\mathrm{d}y - y\mathrm{d}x.$$

Example 1. Evaluate the given line integral using Green's formula:

$$\oint\limits_{\mathcal{L}} (x^2y + x + y)\mathrm{d}x + (xy^2 + x - y)\mathrm{d}y,$$

where $\mathcal{L}$ is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ taken in the positive direction.

Solution. According to the formula (4.7) we are to find derivatives $\dfrac{\partial P}{\partial y}$ and $\dfrac{\partial Q}{\partial x}$ first. Since $P = x^2y + x + y$, $Q = xy^2 + x - y$, then we obtain

$$\frac{\partial P}{\partial y} = x^2 + 1, \quad \frac{\partial Q}{\partial x} = y^2 + 1.$$

Then

$$I = \oint\limits_{\mathcal{L}} (x^2y + x + y)\mathrm{d}x + (xy^2 + x - y)\mathrm{d}y = \iint\limits_{D} ((y^2 + 1) - (x^2 + 1))\mathrm{d}x\mathrm{d}y,$$

where $D = \left\{(x,y): \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leqslant 1\right\}$. To evaluate the double integral we change cartesian coordinates to generalized polar coordinates:

$$x = a\rho\cos\varphi, \quad y = b\rho\sin\varphi, \quad 0 \leqslant \varphi \leqslant \pi,\ 0 \leqslant \rho \leqslant 1,$$

and

$$\mathrm{d}x\mathrm{d}y = ab\rho\mathrm{d}\rho\mathrm{d}\varphi.$$

Then

$$I = \iint\limits_D (y^2 - x^2)\mathrm{d}x\mathrm{d}y = ab\int\limits_0^{2\pi}\mathrm{d}\varphi\int\limits_0^1 (b^2\rho^2\sin^2\varphi - a^2\rho^2\cos^2\varphi)\rho\mathrm{d}\rho =$$

$$= ab\pi(b^2 - a^2)\int\limits_0^1 \rho^3\mathrm{d}\rho = \frac{\pi ab}{4}(b^2 - a^2).$$

Example 2. Evaluate the given line integral using Green's formula:

$$\int\limits_{\mathcal{L}} (y + x\ln y)\mathrm{d}x + \left(\frac{x^2}{2y} + x + 1\right)\mathrm{d}y,$$

where $\mathcal{L}$ is the semicircle $x^2 + y^2 = 2y,\ y \geqslant 1$ from the point $(1,1)$ to $(-1,1)$.

Solution. To apply Green's formula we close the curve $\mathcal{L}$ with the segment $\mathcal{L}_1: \ y = 1,\ -1 \leqslant x \leqslant 1$ (fig. 4.12). The required integral is equal to the integral along the close curve $\mathcal{L} + \mathcal{L}_1$ to be evaluated by Green's formula minus the integral along the segment $\mathcal{L}_1$.

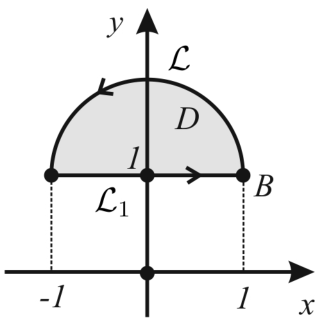

Fig. 4.12

Since

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y + x\ln y) = 1 + \frac{x}{y},$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2}{2y} + x + 1\right) = 1 + \frac{x}{y}.$$

then

$$\oint\limits_{\mathcal{L}+\mathcal{L}_1} (y + x\ln y)\mathrm{d}x + \left(\frac{x^2}{2y} + x + 1\right)\mathrm{d}y = 0.$$

The curve $\mathcal{L}_1$ can be parameterised as $x = t,\ y = 1,\ t \in [-1, 1]$. Thus,

$$\int\limits_{\mathcal{L}_1} (y + x \ln y)\mathrm{d}x + \left(\frac{x^2}{2y} + x + 1\right) \mathrm{d}y = \int\limits_{-1}^{1} (1 + t \ln 1)\mathrm{d}t = 2.$$

Finally we obtain

$$\int\limits_{\mathcal{L}} (y + x \ln y)\mathrm{d}x + \left(\frac{x^2}{2y} + x + 1\right) \mathrm{d}y = -2.$$

Example 3. Evaluate the are of the domain bounded by the curve

$$(x^2 + y^2)^2 = a^2(x^2 - y^2), \quad x \geqslant 0.$$

Solution. The given curve is the Bernoulli's lemniscate which equation in polar coordinate $x = \rho \cos \varphi,\ y = \rho \sin \varphi$ is

$$\rho^2 = a^2 \cos 2\varphi, \quad -\frac{\pi}{4} \leqslant \varphi \leqslant \frac{\pi}{4}.$$

Then we find

$$x' = \rho' \cos \varphi - \rho \sin \varphi = -a \frac{\sin 3\varphi}{\sqrt{\cos 2\varphi}},$$

$$y' = \rho' \sin \varphi + \rho \cos \varphi = a \frac{\cos 3\varphi}{\sqrt{\cos 2\varphi}}.$$

Thus the area is

$$S_D = \frac{1}{2} \int\limits_{\partial D} x\mathrm{d}y - y\mathrm{d}x =$$

$$= \frac{1}{2} a^2 \int\limits_{-\pi/4}^{\pi/4} \left(\sqrt{\cos 2\varphi} \cos \varphi \frac{\cos 3\varphi}{\sqrt{\cos 2\varphi}} + \sqrt{\cos 2\varphi} \sin \varphi \frac{\sin 3\varphi}{\sqrt{\cos 2\varphi}}\right) \mathrm{d}\varphi =$$

$$= \frac{1}{2} a^2 \int\limits_{-\pi/4}^{\pi/4} \cos 2\varphi \mathrm{d}\varphi = \frac{1}{2} a^2.$$

Exercises

Evaluate the given line integrals of the second kind using Green's formula. Consider all parameters like a, b etc. positive.

4.288. $\oint\limits_{\mathcal{L}}(x+y)^2\mathrm{d}x-(x^2+y^2)\mathrm{d}y$, $\mathcal{L}$: triangle with vertexes at $A(1,1)$, $B(3,2)$, $C(2,5)$ traced in the positive direction.

4.289. $\oint\limits_{\mathcal{L}}(x^2-y^2)\mathrm{d}x+2xy\mathrm{d}y$, $\mathcal{L}$: triangle with vertexes at $A(1,1)$, $B(3,1)$, $C(3,3)$ traced in the positive direction.

4.290. $\oint\limits_{\mathcal{L}}xy\mathrm{d}x+2xy^2\mathrm{d}y$, $\mathcal{L}$: triangle with vertexes at $A(1,0)$, $B(0,1)$, $C(1,1)$ traced in the negative direction.

4.291. $\oint\limits_{\mathcal{L}}\dfrac{x}{x+y}\mathrm{d}x-\dfrac{y}{x+y}\mathrm{d}y$, $\mathcal{L}$: square with vertexes $A(1,1)$, $B(2,2)$, $C(1,3)$, $D(0,2)$ traced in the positive direction.

4.292. $\oint\limits_{\mathcal{L}}(y-x^2)\mathrm{d}x+(x+y^2)\mathrm{d}y$, $\mathcal{L}$: union of the circle arc $x=a\cos\varphi$, $y=a\sin\varphi$, $0\leqslant\varphi\leqslant\dfrac{\pi}{2}$, and segments of the coordinate axes traced in the positive direction.

4.293. $\oint\limits_{\mathcal{L}}\sqrt{x^2+y^2}\mathrm{d}x+y(xy+\ln(x+\sqrt{x^2+y^2}))\mathrm{d}y$, $\mathcal{L}$: circle $x^2+y^2=$ $=a^2$ traced in the negative direction.

4.294. $\oint\limits_{\mathcal{L}}(e^x\sin y-y)\mathrm{d}x+(e^x\cos y-1)\mathrm{d}y$, $\mathcal{L}$: union of the circle arc $x^2+y^2=ax$, $y\geqslant 0$, and the segment of the Ox axis traced in the positive direction.

4.295. $\oint\limits_{\mathcal{L}}(xy+x+y)\mathrm{d}x+(xy+x-y)\mathrm{d}y$, $\mathcal{L}$: positively oriented circle $x^2+y^2=ax$.

4.296. $\oint\limits_{\mathcal{L}}(2xy-y)\mathrm{d}x+x^2\mathrm{d}y$, $\mathcal{L}$: positively oriented ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

4.297. $\oint\limits_{\mathcal{L}}e^{y^2-x^2}\cos 2xy\mathrm{d}x+e^{y^2-x^2}\sin 2xy\mathrm{d}y$, $\mathcal{L}$: negatively oriented circle $x^2+y^2=a^2$.

4.298. $\oint\limits_{\mathcal{L}} y^{5/3}\mathrm{d}x - x^{5/3}\mathrm{d}y$, $\mathcal{L}$: positively oriented astroid $x^{2/3} + y^{2/3} = $ $= a^{2/3}$.

4.299. $\oint\limits_{\mathcal{L}} xy^2\mathrm{d}x + (y^2 - x^2)\mathrm{d}y$, $\mathcal{L}$: curve in polar coordinates $\rho = a(1 + \cos\varphi)$ traced in the positive direction.

Evaluate the line integral of the second kind along the given non-closed curve using Green's formula.

4.300. $\int\limits_{\mathcal{L}} (x - y)^2\mathrm{d}x + (x + y)^2\mathrm{d}y$, $\mathcal{L}$: polyline ABC with vertexes at $A(0,0)$, $B(2,2)$, $C(0,1)$.

4.301. $\int\limits_{\mathcal{L}} x^3y^3\mathrm{d}x + (x - y)^2\mathrm{d}y$, $\mathcal{L}$: polyline ABC with vertexes at $A(2,1)$, $B(0,3)$, $C(-2,1)$.

4.302. $\int\limits_{\mathcal{L}} (4xy - 15x^2y)\mathrm{d}x + (2x^2 - 5x^3 + 7)\mathrm{d}y$, $\mathcal{L}$: curve $y = x^3 - 3x^2 + 2$ from the point $A(1 - \sqrt{3}, 0)$ to $B(1,0)$.

4.303. $\int\limits_{\mathcal{L}} y\mathrm{d}x + x\mathrm{d}y$, $\mathcal{L}$: curve

$$y = \begin{cases} x^2 \sin\dfrac{1}{x} + \dfrac{4}{\pi^2}, & x \neq 0, \\ \dfrac{4}{\pi^2}, & x = 0, \end{cases}$$

from the $A\left(0, \dfrac{4}{\pi^2}\right)$ to $B\left(\dfrac{2}{\pi}, \dfrac{8}{\pi^2}\right)$.

4.304. $\int\limits_{\mathcal{L}} (xy + x + y)\mathrm{d}x + (xy + x - y)\mathrm{d}y$, $\mathcal{L}$: arc of the circle $x^2 + y^2 = ax$, $x \leqslant \dfrac{a}{2}$, from the point $A\left(\dfrac{a}{2}, -a\right)$ to $B\left(\dfrac{a}{2}, a\right)$.

4.305. $\int\limits_{\mathcal{L}} \left(1 - \dfrac{y}{2}\right)\mathrm{d}x + \dfrac{x}{2}\mathrm{d}y$, $\mathcal{L}$: upper semicircle $x^2 + y^2 = a^2$, $y \geqslant 0$, from the point $A(a,0)$ to $B(-a,0)$.

4.306. $\int\limits_{\mathcal{L}} (e^{-x}\cos y - y^2)\mathrm{d}x + (e^{-x}\sin y - x^2)\mathrm{d}y$, $\mathcal{L}$: semicircle $x^2+y^2 = 2ax$, $x \geqslant a$, from the point $A(a,a)$ to $B(a,-a)$.

4.307. $\int\limits_{\mathcal{L}} x^2y\mathrm{d}x - y^2x\mathrm{d}y$, $\mathcal{L}$: arc of the lemniscate $(x^2+y^2)^2 = a^2(x^2-y^2)$, $x \geqslant 0$, $y \geqslant 0$, from the $A(0,0)$ to $B(a,0)$.

Evaluate the area of the domain, bounded by the given curve. Consider all parameters like a, b, p etc. positive.

4.308. $x = a\cos^3 t,\ y = b\sin^3 t$.

4.309. $x = a\cos t,\ y = a\sin 2t,\ x \geqslant 0$.

4.310. $x = a(2\cos t - \cos 2t),\ y = a(2\sin t - \sin 2t)$.

4.311. $y^2 = x^2 - x^4$.

4.312. $9y^2 = 4x^3 - x^4$.

4.313. $(x^2+y^2)^2 = a^2(x^2-y^2)$.

4.314. $(x^2+y^2)^2 = 2ax^3$.

4.315. $(x+y)^2 = 2ax$ and $x = 0$.

4.316. $x = \dfrac{3t}{1+t^3},\ y = \dfrac{3t^2}{1+t^3},\ 0 \leqslant t < \infty$.

4.317. $x^3+y^3 = x^2+y^2,\ x = 0,\ y = 0$.

4.318. $(\sqrt{x}+\sqrt{y})^{12} = xy$.

4.319. $(x+y)^{n+m+1} = ax^ny^m,\ n > 0,\ m > 0$.

4.320. $\left(\dfrac{x}{a}\right)^{2n+1} + \left(\dfrac{y}{b}\right)^{2n+1} = c\left(\dfrac{x}{a}\right)^n\left(\dfrac{y}{b}\right)^n,\ n > 0$.

4.9. Stokes' theorem

Let Ω be a piecewise smooth orientable surface bounded by a piecewise smooth closed curve $\partial\Omega$. If $\vec{a}(\vec{r})$ is continuously differentiable in the neighborhood of the surface vector field then

$$\oint\limits_{\partial\Omega} \vec{a}\cdot\mathrm{d}\vec{r} = \iint\limits_{\Omega} \operatorname{curl}\vec{a}\cdot\mathrm{d}\vec{S}. \tag{4.8}$$

Orientation of the surface Ω is associated with orientation of its boundary $\partial\Omega$. The formula (4.8) is called *Stokes' formula.*

In cartesian coordinates Stokes' formula takes the form

$$\oint\limits_{\partial\Omega} P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint\limits_{\Omega} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y\mathrm{d}z +$$

$$+\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z\mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y,$$

or

$$\oint\limits_{\partial\Omega} P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint\limits_{\Omega} \begin{vmatrix} \cos(\widehat{\vec{n},\vec{i}}) & \cos(\widehat{\vec{n},\vec{j}}) & \cos(\widehat{\vec{n},\vec{k}}) \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \mathrm{d}S.$$

Example 1. Evaluate the circulation of the vector field $\vec{a} = (y-z)\vec{i} + (z-x)\vec{j} + (x-y)\vec{k}$ around the ellipse $x^2+y^2 = a^2$, $\dfrac{x}{a} + \dfrac{z}{b} = 1$, $a > 0,\ b > 0$, traced in the positive direction on the upper side of the plane.

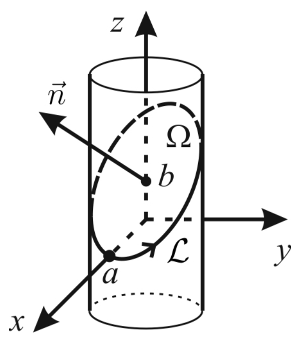

Fig. 4.13

Solution. To apply Stokes' formula to the line integral we are to choose the surface which border is the integration curve. It is better to chose the simplest surface and in our case it is the part of the plane $\dfrac{x}{a} + \dfrac{z}{b} = 1$ inside the cylinder $x^2 + y^2 = a^2$ (fig. 4.13).

Then we find $\operatorname{curl}\vec{a}$:

$$\operatorname{curl}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix} = -2(\vec{i} + \vec{j} + \vec{k}).$$

Thus

$$\oint\limits_{\mathcal{L}} \vec{a}\cdot \mathrm{d}\vec{r} = \iint\limits_{\Omega} \operatorname{curl}\vec{a}\cdot\vec{n}\,\mathrm{d}S = -2\iint\limits_{\Omega}\mathrm{d}y\mathrm{d}z - 2\iint\limits_{\Omega}\mathrm{d}x\mathrm{d}z - 2\iint\limits_{\Omega}\mathrm{d}x\mathrm{d}y.$$

The second integral in above formula is equal to zero since the plane Ω is perpendicular to the plane xOz. Other integrals are equal to the area of the plane Ω projection onto corresponding coordinate planes: the circle of the radius a onto the plane xOy and the ellipse with axes a and b onto the plane yOz. Therefore finally we have

$$\oint\limits_{\mathcal{L}} \vec{a}\cdot \mathrm{d}\vec{r} = -2\iint\limits_{D_{yz}}\mathrm{d}y\mathrm{d}z - 2\iint\limits_{D_{xy}}\mathrm{d}x\mathrm{d}z = -2\pi ab - 2\pi a^2 = -2\pi a(b+a).$$

Example 2. Evaluate the circulation of the vector field $\vec{a} = y^2\vec{i} + z^2\vec{j}$ around the curve $\mathcal{L}:\ x^2+y^2=9,\ 3y+4z=5$, traced in the positive direction on the upper side of the plane.

Solution. We chose as the simples surface Ω with the boundary $\mathcal{L}$ the part of the plane $3y+4z=5$ inside the cylinder $x^2+y^2=9$. First we find $\operatorname{curl}\vec{a}$:

$$\operatorname{curl}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & z^2 & 0 \end{vmatrix} = -2z\vec{i} - 2y\vec{k}.$$

To evaluate the surface integral $\iint\limits_{\Omega}\operatorname{curl}\vec{a}\cdot \mathrm{d}\vec{S}$ we project the surface onto the plane xOy. Then the explicit equation of the surface is

$$z = \frac{5-3y}{4}, \quad x^2+y^2 \leqslant 9.$$

Finding the normal to the plane we take into account the upper orientation of the plane that corresponds the curve orientation:

$$\vec{N}_{xy} = -z'_x\vec{i} - z'_y\vec{j} + \vec{k} = \frac{3}{4}\vec{j} + \vec{k}.$$

Finally combining all results together we obtain

$$\oint\limits_{\mathcal{L}} \vec{a}\cdot \mathrm{d}\vec{r} = \iint\limits_{x^2+y^2\leqslant 9} (-2y)\mathrm{d}x\mathrm{d}y = 0,$$

since the integrand is odd function, and the domain of integration is even with respect to the Ox axis.

Example 3. Evaluate the circulation of the vector field $\vec{a} = yz\vec{i} - xz\vec{j} + xy\vec{k}$ around the curve $\mathcal{L}$ of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ with coordinate planes, the curve laying in the first octant (fig. 4.14). The direction of the curve is counterclockwise as viewed from the point $(2a, 2a, 2a)$, $a > 0$.

Solution. The most appropriate surface Ω bounded by the oriented closed curve $\mathcal{L}$ is the outward side of the sphere $x^2 + y^2 + z^2 = a^2$, laying in the first octant. Then we evaluate

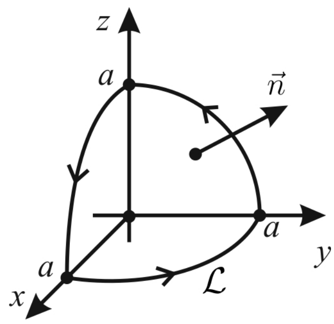

Fig. 4.14

$$\operatorname{curl} \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & -zx & xy \end{vmatrix} = 2x\vec{i} - 2z\vec{k}.$$

The unit outward normal vector to the sphere takes the form

$$\vec{n} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}.$$

Then we parameterize the surface with spherical coordinates θ and φ

$$x = a\cos\varphi\sin\theta, \quad y = a\sin\varphi\sin\theta, \quad z = a\cos\theta,$$

$$0 \leqslant \theta \leqslant \frac{\pi}{2}, \quad 0 \leqslant \varphi \leqslant \frac{\pi}{2}.$$

Thus

$$\mathrm{d}S = a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi,$$

$$\vec{n} \cdot \operatorname{curl} \vec{a} = \frac{2(x^2 - z^2)}{a} = 2a(\cos^2\varphi \sin^2\theta - \cos^2\theta),$$

and finally

$$\oint_{\mathcal{L}} \vec{a} \cdot \mathrm{d}\vec{r} = \iint_{\Omega} \vec{n} \cdot \operatorname{curl} \vec{a} \mathrm{d}S = 2a^3 \int_0^{\pi/2} \mathrm{d}\varphi \int_0^{\pi/2} (\cos^2\varphi \sin^2\theta - \cos^2\theta) \sin\theta \mathrm{d}\theta =$$

$$= 2a^3 \int_0^{\pi/2} \left(\frac{\pi}{4}\sin^2\theta - \frac{\pi}{2}\cos^2\theta\right) \sin\theta \mathrm{d}\theta = 0.$$

Exercises

Evaluate the given line integrals of the second kind using Stokes' formula. Consider all parameters like a, b etc. positive.

4.321. $\oint\limits_{\mathcal{L}} z^2\mathrm{d}x - y^2\mathrm{d}y + x^2\mathrm{d}z$, $\mathcal{L}$: circle $z = 3(x^2+y^2)+1$, $z = 4$, positively oriented on the upper side of the plane.

4.322. $\oint\limits_{\mathcal{L}} zy^2\mathrm{d}x + xz^2\mathrm{d}y + yx^2\mathrm{d}z$, $\mathcal{L}$: circle $x = y^2+z^2$, $x = 9$, taken in the counterclockwise direction as viewed from the point $M(0,0,10)$.

4.323. $\oint\limits_{\mathcal{L}} y^2\mathrm{d}x + x^2\mathrm{d}y + z^2\mathrm{d}z$, $\mathcal{L}$: circle $z = \sqrt{25-x^2-y^2}$, $x^2+y^2 = 16$, taken in the clockwise direction as viewed from the point $M(0,0,5)$.

4.324. $\oint\limits_{\mathcal{L}} z^2\mathrm{d}x + y\mathrm{d}y - 2xy\mathrm{d}z$, $\mathcal{L}$: circle $y = \sqrt{x^2+z^2}$, $y = 3$, taken in the counterclockwise direction as viewed from the point $M(0,0,0)$.

4.325. $\oint\limits_{\mathcal{L}} z\mathrm{d}x - y\mathrm{d}z$, $\mathcal{L}$: ellipse $x^2+y^2 = 4$, $x+2z = 5$, positively oriented on the upper side of the plane.

4.326. $\oint\limits_{\mathcal{L}} y\mathrm{d}x - z\mathrm{d}y + x\mathrm{d}z$, $\mathcal{L}$: ellipse $x^2+y^2+2z^2 = 2a^2$, $y - x = 0$, taken in the counterclockwise direction as viewed from the point $M(a,0,0)$.

4.327. $\oint\limits_{\mathcal{L}} (x^2+y)\mathrm{d}x + (y^2+z)\mathrm{d}y + (z^2+x)\mathrm{d}z$, $\mathcal{L}$: ellipse $x^2+y^2 = 4$, $x+z = 2$, positively oriented on the upper side of the plane.

4.328. $\oint\limits_{\mathcal{L}} 2xy\mathrm{d}x + z^2\mathrm{d}y + x^2\mathrm{d}z$, $\mathcal{L}$: ellipse $2x^2+2y^2 = z^2$, $x+z = a$, positively oriented on the upper side of the plane.

4.329. $\oint\limits_{\mathcal{L}} z^3\mathrm{d}x + x^3\mathrm{d}y + y^3\mathrm{d}z$, $\mathcal{L}$: curve $2x^2-y^2+z^2 = a^2$, $x + y = 0$. taken in the counterclockwise direction as viewed from the point $M(a,0,0)$.

4.330. $\oint\limits_{\mathcal{L}} (y^2+z^2)\mathrm{d}x + (z^2+x^2)\mathrm{d}y + (x^2+y^2)\mathrm{d}z$, $\mathcal{L}$: curve $x^2+y^2+z^2 = 2ax$, $x^2+y^2 = 2bx$, $z > 0$, $0 < b < a$, taken in the counterclockwise direction as viewed from the point $M(0,0,2a)$.

4.331. $\oint\limits_{\mathcal{L}}(y-z)\mathrm{d}x+(z-x)\mathrm{d}y+(x-y)\mathrm{d}z$, $\mathcal{L}$: circle $x^2+y^2+z^2=a^2$, $y = x\tan\alpha$, $0<\alpha<\dfrac{\pi}{2}$, taken in the counterclockwise direction as viewed from the point $M(2a,0,0)$.

4.332. $\oint\limits_{\mathcal{L}}(y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2+y^2)\mathrm{d}z$, $\mathcal{L}$: curve of intersection of the boundary of the cube $\{0\leqslant x\leqslant a,\ 0\leqslant y\leqslant a,\ 0\leqslant z\leqslant a\}$ and the plane $x+y+z=\dfrac{3a}{2}$, positively oriented on the upper side of the plane.

4.333. Let K be the cube $\{0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 1,\ 0\leqslant z\leqslant 1\}$. Evaluate $\oint\limits_{\mathcal{L}} y^2\mathrm{d}x+z^2\mathrm{d}y+x^2\mathrm{d}z$, where $\mathcal{L}$ is a) curve of intersection of the boundary of the cube K and the plane passing through the points $O(0,0,0)$, $A(1,1,0)$, $B(0,0,1)$, positively oriented on the right side of the plane; b) curve of intersection of the boundary of the cube K and the plane passing through the points $P(1,0,0)$, $Q(0,1,0)$, $R(1,0,1)$, positively oriented on the right side of the plane.

4.334. $\oint\limits_{\mathcal{L}}(xy+z)\mathrm{d}x+(yz+x)\mathrm{d}y+(xz+y)\mathrm{d}z$, $\mathcal{L}$: circle $x^2+y^2+z^2=a^2$, $x+y+z=0$, positively oriented on the upper side of the plane.

4.335. $\oint\limits_{\mathcal{L}}(z^2-y^2)\mathrm{d}x+(x^2-z^2)\mathrm{d}y+(y^2-x^2+x)\mathrm{d}z$, $\mathcal{L}$: ellipse $x^2+y^2= = 8x$, $x+y+z=0$, positively oriented on the upper side of the plane.

4.336. $\oint\limits_{\mathcal{L}}\dfrac{x\mathrm{d}y-y\mathrm{d}x}{x^2+y^2}+z\mathrm{d}z$, $\mathcal{L}$: circle $x^2+y^2+z^2=a^2$, $x+y+z=t$, $0\leqslant t\leqslant\sqrt{3}a$, positively oriented on the upper side of the plane.

4.337. $\oint\limits_{\mathcal{L}}(x+y)\mathrm{d}x+(y+z)\mathrm{d}y+(x+z)\mathrm{d}z$, $\mathcal{L}$: circle $x^2+y^2+z^2=a^2$, $x+y+z=t$, $0\leqslant t\leqslant\sqrt{3}a$, $a>0$, positively oriented on the upper side of the plane.

4.338. $\oint\limits_{\mathcal{L}}(y^2+z^2)\mathrm{d}x+(x^2+z^2)\mathrm{d}y+(x^2+y^2)\mathrm{d}z$, $\mathcal{L}$: curve $x^2+y^2=2x$, $x^2+y^2+z^2=4z$, $z\geqslant 2$, positively oriented on the outward side of the upper hemisphere.

4.339. $\oint\limits_{\mathcal{L}}(z-x^2-y)\mathrm{d}x+(x+y+z)\mathrm{d}y+(y+2x+z^3)\mathrm{d}z$, $\mathcal{L}$: curve

$x = \sqrt{y^2 + z^2}$, $x^2 + y^2 + z^2 = 2az$, positively oriented on the outward side of the right hemisphere.

4.340. $\oint\limits_{\mathcal{L}} z^2\mathrm{d}x + x^2\mathrm{d}y + y^2\mathrm{d}z$, $\mathcal{L}$: curve $x^2 + y^2 = 2ax$, $z = \sqrt{\dfrac{x^2 + y^2}{3}}$, positively oriented on the outward side of the cone.

4.341. $\oint\limits_{\mathcal{L}} z^2x\mathrm{d}x + (x + y + z)\mathrm{d}y + y^2z\mathrm{d}z$, $\mathcal{L}$: curve $x^2 + y^2 = ax$, $x^2 = y^2 + z^2$, positively oriented on the outward side of the cylinder.

4.342. $\oint\limits_{\mathcal{L}} xyz\mathrm{d}x + y^2z\mathrm{d}y + zx^2\mathrm{d}z$, $\mathcal{L}$: curve $x^2 + y^2 = a^2$, $y^2 + z^2 = a^2$, $x \geqslant 0$, positively oriented on the outward side of the former cylinder.

4.343. $\oint\limits_{\mathcal{L}} (xy + z)\mathrm{d}x + (yz + x)\mathrm{d}y + y\sqrt{a^2 - x^2}\mathrm{d}z$, $\mathcal{L}$: curve $x^2 + y^2 + z^2 = 2ax$, $x^2 + y^2 = a^2$, $x \geqslant 0$, positively oriented on the inward side of the cylinder.

4.10. Gauss – Ostrogradsky theorem

A domain $G \subset E^e$ is said to be *simply connected* (also called *1-connected*) if any simple smooth closed surface in G can be shrunk to a point continuously in the set, otherwise it is said to be *multiply connected*.

Let $G \subset E^3$ be a simply connected domain bounded by a piecewise smooth close surface ∂G oriented by outward unit normal vector $\vec{n}$ and $\vec{a} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a continuously differentiable in $\bar{G} = G \bigcup \partial G$ vector field. Then the flux of the vector field $\vec{a}$ through the boundary ∂G is equal to the volume integral of $\operatorname{div}\vec{a}$ over the domain G:

$$\oint\limits_{\partial G} \vec{a} \cdot \mathrm{d}\vec{S} = \iiint\limits_{G} \operatorname{div}\vec{a}\,\mathrm{d}V, \tag{4.9}$$

or in cartesian coordinates:

$$\oint\limits_{\partial G} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}x\mathrm{d}z + R\mathrm{d}x\mathrm{d}y = \iiint\limits_{G} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

The formula (4.9) is called Gauss – Ostrogradsky formula.

In the case of multiply connected domain the boundary ∂G consists of finite number of partial boundaries with outward orientation.

Example 1. Evaluate the flux of the vector field $\vec{a} = (1+2x)\vec{i}+y\vec{j}+z\vec{k}$ through the closed surface $\Omega = \{(x, y, z): \ x^2 + y^2 = z^2, \ 0 \leqslant z \leqslant 4\}$.

Solution. Since the vector field is continuously differentiable on the surface and in the domain bounded by this surface we can apply Gauss – Ostrogradsky formula. Computing the divergence $\operatorname{div}\vec{a} = 4$, we obtain

$$\oiint\limits_{S} \vec{a} \cdot \mathrm{d}\vec{S} = \iiint\limits_{V} \operatorname{div}\vec{a}\mathrm{d}V = 4\iiint\limits_{V} \mathrm{d}V = 4V_K,$$

i.e. the flux is equal to the four volumes of the cone $V_K = \dfrac{1}{3}\pi \cdot 2^2 \cdot 4$. Thus

$$\oiint\limits_{S} \vec{a} \cdot \mathrm{d}\vec{S} = \frac{4^3}{3}\pi.$$

Example 2. Evaluate the surface integral of the second kind

$$\iint\limits_{\Omega} x^3\mathrm{d}y\mathrm{d}z + y^3\mathrm{d}x\mathrm{d}z + z^3\mathrm{d}x\mathrm{d}y$$

on the sphere $x^2 + y^2 + z^2 = x$ using Gauss-Ostrogradsky formula.

Solution. According to the problem statement we are to calculate the flux of the vector field $\vec{a} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$. The divergence of the field is equal to

$$\operatorname{div}\vec{a} = 3\left(x^2 + y^2 + z^2\right).$$

Applying Gauss – Ostrogradsky formula we obtain

$$\oiint\limits_{\Omega} \vec{a} \cdot \vec{n}\mathrm{d}S = 3\iiint\limits_{G} \left(x^2 + y^2 + z^2\right)\mathrm{d}V.$$

Then we rewrite the implicit equation of the sphere in the form

$$\left(x - \frac{1}{2}\right)^2 + y^2 + z^2 = \frac{1}{4},$$

and introduce the shifted spherical coordinates as below:

$$x - \frac{1}{2} = r\cos\varphi\sin\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\theta,$$

and

$$\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi, \quad 0 \leqslant r \leqslant \frac{1}{2}, \ 0 \leqslant \theta \leqslant \pi, \ 0 \leqslant \varphi \leqslant \pi.$$

As the result we have

$$\oiint\limits_S \vec{a}\cdot\vec{n}\mathrm{d}S = 3\int\limits_0^{2\pi}\mathrm{d}\varphi\int\limits_0^{\pi}\sin\theta\mathrm{d}\theta\int\limits_0^{\frac{1}{2}} r^4\mathrm{d}r = \frac{3\pi}{40}.$$

Example 3. Evaluate the flux of the vector field

$$\vec{a} = \left(\frac{x^2z}{1+z^2} + 6zy^2\right)\vec{i} - \left(\frac{2xy(1+z)}{1+z^2} + 1\right)\vec{j} + 2x\arctan z\,\vec{k}$$

through the outward side of the paraboloid $x^2+z^2 = 4-y,\ y \geqslant 0$.

Solution. In order to apply the Gauss – Ostrogradsky formula we close the given part of the paraboloid by the plane $y = 0$ thus obtaining the closed surface $\Omega+\Omega_1 = \{x^2+z^2 = 4-y,\ y = 0,\ y \geqslant 0\}$ (fig. 4.15). While evaluating the flux we use the additivity property of the surface integral:

$$\Pi = \iint\limits_{\Omega}\vec{a}\cdot\vec{n}\mathrm{d}S = \oint\limits_{\Omega+\Omega_1}\vec{a}\cdot\vec{n}\mathrm{d}S - \iint\limits_{\Omega_1}\vec{a}\cdot\vec{n}\mathrm{d}S.$$

To compute the first integral we find the divergence of the vector field

$$\operatorname{div}\vec{a} = \frac{2xz}{1+z^2} - \frac{2x(1+z)}{1+z^2} + \frac{2x}{1+z^2} = 0.$$

Therefore the flux through the closed surface $\Omega+\Omega_1$ is equal to zero:

$$\oiint\limits_{\Omega+\Omega_1}\vec{a}\cdot\vec{n}\mathrm{d}S = \iiint\limits_V \operatorname{div}\vec{a}\mathrm{d}V = 0$$

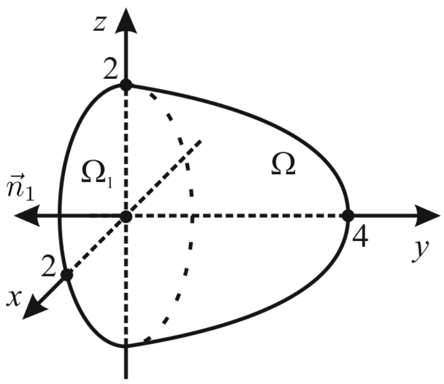

Fig. 4.15

The flux of the vector field $\vec{a}$ through the left side of the disk $\Omega_1 = \{y = 0,\ x^2 + z^2 \leqslant 4\}$ we calculate as follows:

$$\vec{a}\big|_{\Omega_1} = \vec{a}\big|_{y=0} = \frac{x^2 z}{1+z^2}\vec{i} - \vec{j} + 2x \arctan z\vec{k},$$

$\vec{n}_1 = -\vec{j}$, then $\vec{a}\cdot\vec{n}_1 = 1$, and finally

$$\iint\limits_{\Omega_1} \vec{a}\cdot\vec{n}\mathrm{d}S = \iint\limits_{x^2+z^2\leqslant 4} \mathrm{d}S = 4\pi.$$

Combining all previous results we obtain $\Pi = -4\pi$.

Example 4. Evaluate the flux of the vector field $\vec{a} = 4\vec{i} - \vec{j}$ through the upper side of the paraboloid $x^2 + z^2 = y,\ 0 \leqslant y \leqslant 4,\ x \geqslant 0,\ z \geqslant 0$.

Solution. First we close the given surfaces by the planes $\Omega_1 : y = 4$; $\Omega_2 :\ z = 0$; $\Omega_3 :\ x = 0$ with orientations $\Omega_1 :\ \vec{n}_1 = \vec{j}$; $\Omega_2 :\ \vec{n}_2 = -\vec{k}$; $\Omega_3 :\ \vec{n}_3 = -\vec{i}$ (fig. 4.16).

Using Gauss – Ostrogradsky formula and the property of additivity of a surface integral we obtain :

$$\Pi = \iiint\limits_V \operatorname{div}\vec{a}\mathrm{d}V - \iint\limits_{\Omega_1} \vec{a}\cdot\vec{n}_1\mathrm{d}S - \iint\limits_{\Omega_2} \vec{a}\cdot\vec{n}_2\mathrm{d}S - \iint\limits_{\Omega_3} \vec{a}\cdot\vec{n}_3\mathrm{d}S.$$

Consecutive evaluation yields:

$$\operatorname{div}\vec{a} = 0 \quad \Rightarrow \quad \iiint\limits_V \operatorname{div}\vec{a}\mathrm{d}V = 0;$$

$$-\iint\limits_{\Omega_1} \vec{a}\cdot\vec{n}_1\mathrm{d}S = \iint\limits_{\Omega_1} \mathrm{d}S = \iint\limits_{D_{xz}} \mathrm{d}x\mathrm{d}z = \pi;$$

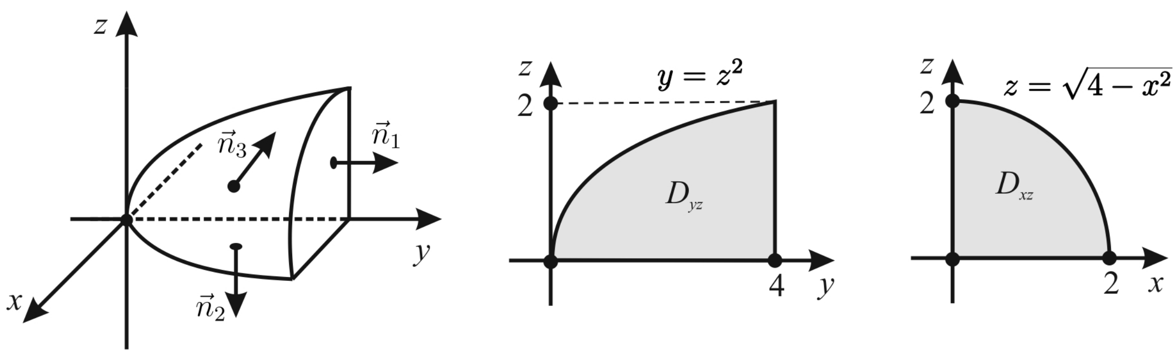

Fig. 4.16

$$-\iint\limits_{\Omega_2} \vec{a}\cdot\vec{n}_2 \mathrm{d}S = 0;$$

$$-\iint\limits_{\Omega_3} \vec{a}\cdot\vec{n}_3 \mathrm{d}S = 4\iint\limits_{\Omega_3} \mathrm{d}S = 4\iint\limits_{D_{yz}} \mathrm{d}y\mathrm{d}z =$$

$$= 4\int\limits_0^4 \mathrm{d}y \int\limits_0^{\sqrt{y}} \mathrm{d}z = 4\int\limits_0^4 \sqrt{y}\mathrm{d}y = 4\left(\frac{2}{3}y^{\frac{3}{2}}\right)\Big|_0^4 = \frac{64}{3}.$$

Finally the flux is

$$\Pi = \pi + \frac{64}{3}.$$

Exercises

Evaluate the given surface integral of the second king applying Gauss – Ostrogradsky formula. Consider all parameters like a, b, p etc. positive.

4.344. $\oiint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: inward side of the surface of the body $x + 2y + 3z \leqslant 1,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0$.

4.345. $\oiint\limits_{\Omega} 3x\mathrm{d}y\mathrm{d}z + (y+z)\mathrm{d}x\mathrm{d}z + (x-z)\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $x + 3y - z \leqslant 6,\ x \geqslant 0,\ y \geqslant 0,\ z \leqslant 0$.

4.346. $\oiint\limits_{\Omega} yz\mathrm{d}y\mathrm{d}z + xz\mathrm{d}x\mathrm{d}z + xy\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $x + y + z \leqslant 2,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0$.

4.347. $\oiint\limits_{\Omega} (3x+z)\mathrm{d}y\mathrm{d}z + (2y+z)\mathrm{d}x\mathrm{d}z - 2y\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $x - y - z \leqslant 1,\ x \geqslant 0,\ y \leqslant 0,\ z \leqslant 0$.

4.348. $\oiint\limits_{\Omega} (y+2z)\mathrm{d}y\mathrm{d}z + (x+2z)\mathrm{d}x\mathrm{d}z + (x-2y)\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $2x + y + 3z \leqslant 8,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0$.

4.349. $\oiint\limits_{\Omega} 2x\mathrm{d}y\mathrm{d}z - y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $x + y + z \leqslant 1,\ x - y + z \leqslant 1,\ x \geqslant 0,\ z \geqslant 0$.

4.350. $\oiint\limits_{\Omega} x^3\mathrm{d}y\mathrm{d}z + y^3\mathrm{d}x\mathrm{d}z + z^3\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $x^2 + y^2 + z^2 \leqslant a^2$.

4.351. $\oiint\limits_{\Omega} 4x\mathrm{d}y\mathrm{d}z - y\mathrm{d}x\mathrm{d}z + z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the torus $x = (a + b\cos u)\cos v$, $y = (a + b\cos u)\sin v$, $z = b\sin u$, $0 \leqslant u \leqslant 2\pi$, $0 \leqslant v \leqslant 2\pi$.

4.352. $\oiint\limits_{\Omega} \left(\dfrac{x^2y}{1+y^2} + 6yz^3\right)\mathrm{d}y\mathrm{d}z - \dfrac{2xz\,(1+y) + 1 + y^2}{1+y^2}\mathrm{d}x\mathrm{d}y + +2x\arctan y\mathrm{d}x\mathrm{d}z$, Ω: outward side of the surface of the body $0 \leqslant z \leqslant \leqslant 1 - x^2 - y^2$.

4.353. $\oiint\limits_{\Omega} x^2\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y$, Ω: outward side of the sphere $(x-2)^2 + (y-3)^2 + (z-4)^2 = a^2$.

4.354. $\oiint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + xz\mathrm{d}x\mathrm{d}z + y\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $x^2 + y^2 \leqslant 4 - z$, $z \geqslant 0$.

4.355. $\oiint\limits_{\Omega} x\mathrm{d}y\mathrm{d}z + 2y\mathrm{d}x\mathrm{d}z - z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $x^2 + y^2 \leqslant z \leqslant \sqrt{x^2 + y^2}$.

4.356. $\oiint\limits_{\Omega} 2x\mathrm{d}y\mathrm{d}z + 3y\mathrm{d}x\mathrm{d}z - z\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $1 \leqslant x \leqslant 5 - y^2 - z^2$.

4.357. $\oiint\limits_{\Omega} x^2\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}x\mathrm{d}z + z^2\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $x^2 + y^2 \leqslant z$, $z \leqslant h$.

4.358. $\oiint\limits_{\Omega} yz^2\mathrm{d}y\mathrm{d}z + zy^2\mathrm{d}x\mathrm{d}z + yx^2\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $0 \leqslant z \leqslant x^2 + y^2$, $x^2 + y^2 \leqslant 1$, $x \geqslant 0$, $y \geqslant 0$.

4.359. $\oiint\limits_{\Omega} (y-x)\mathrm{d}y\mathrm{d}z + (z-y)\mathrm{d}x\mathrm{d}z + (x-z)\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $x^2 + y^2 + z^2 \leqslant 4$, $z \geqslant \dfrac{x^2 + y^2}{3}$.

4.360. $\oiint\limits_{\Omega} xz^2\mathrm{d}y\mathrm{d}z + yx^2\mathrm{d}x\mathrm{d}z + zy^2\mathrm{d}x\mathrm{d}y$, Ω: outward side of the surface of the body $x^2 + y^2 + z^2 \leqslant 2az$, $x^2 + y^2 \geqslant 3z^2$, $x \geqslant y$.

4.361. $\oiint\limits_{\Omega} \left(x^2 + y^2\right)\mathrm{d}y\mathrm{d}z + \left(y^2 + z^2\right)\mathrm{d}x\mathrm{d}z + \left(z^2 + x^2\right)\mathrm{d}x\mathrm{d}y$, Ω: inward side of the surface of the body $x^2 + y^2 + z^2 \leqslant a^2$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$.

4.362. $\displaystyle\oiint\limits_{\Omega} x^{\alpha+1}y^{\beta}z^{\gamma}\left(\frac{x}{\alpha+2}-\frac{1}{3(\alpha+1)}\right)\mathrm{d}y\mathrm{d}z+$

$\displaystyle +x^{\alpha}y^{\beta+1}z^{\gamma}\left(\frac{y}{\beta+2}-\frac{1}{3(\beta+1)}\right)\mathrm{d}x\mathrm{d}z+$

$\displaystyle +x^{\alpha}y^{\beta}z^{\gamma+1}\left(\frac{z}{\gamma+2}-\frac{1}{3(\gamma+1)}\right)\mathrm{d}x\mathrm{d}y,$

Ω: outward side of the surface of the body $x+y+z\leqslant 1$, $x\geqslant 0$, $y\geqslant 0$, $z\geqslant 0$; $\alpha>-1$, $\beta>-1$, $\gamma>-1$.

Evaluate the surface integral of the second king on the given non-closed surface applying Gauss – Ostrogradsky formula.

4.363. $\displaystyle\iint\limits_{\Omega} z^2\mathrm{d}y\mathrm{d}z-y^2\mathrm{d}x\mathrm{d}z+2yx\mathrm{d}x\mathrm{d}y$, Ω: left side of the cone $x^2+z^2=$ $=y^2$, $0\leqslant y\leqslant 1$.

4.364. $\displaystyle\iint\limits_{\Omega} y\mathrm{d}y\mathrm{d}z+z\mathrm{d}x\mathrm{d}z+x\mathrm{d}x\mathrm{d}y$, Ω: outward side of the cylinder $x^2+y^2=a^2$, bounded by the plane $z=x$, $z=0$.

4.365. $\displaystyle\iint\limits_{\Omega} 3yx\mathrm{d}y\mathrm{d}z-z\mathrm{d}x\mathrm{d}z-2x\mathrm{d}x\mathrm{d}y$, Ω: bottom side of the paraboloid $x^2+y^2=z+1$, $0\leqslant z\leqslant 3$.

4.366. $\displaystyle\iint\limits_{\Omega} (xy^2+z^2)\mathrm{d}y\mathrm{d}z+(yz^2+x^2)\mathrm{d}x\mathrm{d}z+(zx^2+y^2)\mathrm{d}x\mathrm{d}y$, Ω: upper side of hemisphere $x^2+y^2+z^2=a^2$, $z\geqslant 0$.

4.367. $\displaystyle\iint\limits_{\Omega} xy\mathrm{d}y\mathrm{d}z+yz\mathrm{d}x\mathrm{d}z+zx\mathrm{d}x\mathrm{d}y$, Ω: upper side of the cone $x^2+y^2=z^2$, $0\leqslant z\leqslant h$.

4.368. $\displaystyle\iint\limits_{\Omega} x^2\mathrm{d}y\mathrm{d}z+y^2\mathrm{d}x\mathrm{d}z+z^2\mathrm{d}x\mathrm{d}y$, Ω: bottom part of the cone $x^2+y^2=z^2$, $0\leqslant z\leqslant h$.

Chapter 5

Basics of the field theory

5.1. Integral characteristics of scalar and vector fields

Example 1. Prove that

$$\operatorname{div}\vec{a}(M) = \lim_{\substack{d(G)\to 0\\ M\in G}} \frac{\oint\limits_{\Omega} \vec{a}\cdot \mathrm{d}\vec{S}}{V(G)},$$

where $V(G)$ is the volume of a simply connected domain G, bounded by the simply smooth surface Ω, $d(G)$ is the diameter of the domain G. The given formula is the *invariant definition of the divergence* of the vector field $\vec{a}$ at the point M.

Solution. Suppose that a vector field $\vec{a}(\vec{r})$ is defined and continuously differentiable in a simply connected domain $G^* \subset E^3$. Let $G \subset G^*$ be a simply connected domain with piecewise smooth boundary Ω with outward orientation and M be a point of the domain G. If we apply Gauss – Ostrogradsky formula to the domain G and then use the mean value formula for the volume integral, we obtain

$$\oiint\limits_{\Omega} \vec{a}\cdot \mathrm{d}\vec{S} = \iiint\limits_{G} \operatorname{div}\vec{a}\mathrm{d}V = \operatorname{div}\vec{a}(M^*)V(G),$$

where M^* is some point in the domain G, $V(G)$ is the volume of G. Then we shrink the domain G to the point M in that manner that the point M^* always belongs to the domain so that the point M^* approaches to M. Taking into account continuality of the divergence we have

$$\operatorname{div}\vec{a}(M) = \lim_{\substack{d(G)\to 0\\ M\in G}} \frac{\oiint\limits_{\Omega} \vec{a}\cdot \mathrm{d}\vec{S}}{V(G)}.$$

Therefore the divergence of a vector field $\vec{a}(\vec{r})$ at a point M is the limit of the ratio of the field flux through an arbitrary piecewise smooth closed surface containing the point M to the volume of the domain bounded by this surface while the volume approaches to zero. The right part of the above formula does not depend on the coordinate system chosen, therefore such definition of the divergence is called the invariant definition.

Example 2. Let $\vec{a}(M)$ be a continuously differentiable vector field in a domain G with piecewise smooth closed boundary Ω. Prove that

$$\oint\!\!\!\oint_{\Omega} (\vec{n} \times \vec{a}) \mathrm{d}S = \iiint_{G} (\vec{\nabla} \times \vec{a}) \mathrm{d}V,$$

where $\vec{n}$ is the outward unit normal vector to the surface Ω.

Solution. The integral in the left part of the equality to prove is the surface integral of the first kind of the vector field. In order to apply Gauss – Ostrogradsky formula we consider the dot product of each part of the equality and a constant vector $\vec{c}$. Taking into account the properties of a scalar triple product we obtain

$$\vec{c} \cdot \oint\!\!\!\oint_{\Omega} (\vec{n} \times \vec{a}) \, \mathrm{d}S = \oint\!\!\!\oint_{\Omega} \vec{c} \cdot (\vec{n} \times \vec{a}) \mathrm{d}S = \oint\!\!\!\oint_{\Omega} (\vec{a} \times \vec{c}) \cdot \vec{n} \mathrm{d}S = \oint\!\!\!\oint_{\Omega} (\vec{a} \times \vec{c}) \cdot \mathrm{d}\vec{S}.$$

Then we apply Gauss – Ostrogradsky formula to the obtained surface integral and use the result of the Exercise 3.85:

$$\oint\!\!\!\oint_{\Omega} (\vec{a} \times \vec{c}) \cdot \mathrm{d}\vec{S} = \iiint_{G} \operatorname{div}(\vec{a} \times \vec{c}) \mathrm{d}V = \iiint_{G} \vec{c} \cdot \operatorname{curl} \vec{a} \mathrm{d}V = \vec{c} \cdot \iiint_{G} (\vec{\nabla} \times \vec{a}) \mathrm{d}V.$$

Thus

$$\vec{c} \cdot \oint_{\Omega} (\vec{n} \times \vec{a}) \mathrm{d}S = \vec{c} \cdot \iiint_{G} (\vec{\nabla} \times \vec{a}) \mathrm{d}V.$$

Since the last equality is true for any vector $\vec{c}$, the latter can be omitted.

Example 3. Suppose that a vector field $\vec{a}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ is continuously differentiable in the neighborhood of a piecewise smooth flat curve $\mathcal{L}$, and $\vec{n}(x, y)$ in unit outward normal vector to the curve at a point (x, y). Convert the line integral of the first kind

$$\int_{\mathcal{L}} \vec{a} \cdot \vec{n} \mathrm{d}l$$

into the line integral of the second kind.

Solution. Consider the unit tangent vector $\vec{\tau}$ to the curve $\mathcal{L}$ at the point (x, y) directed in accordance with the orientation of the curve. If α is the angle between the vector $\vec{\tau}$ and the positive direction of the Ox axis, then (fig. 5.1)

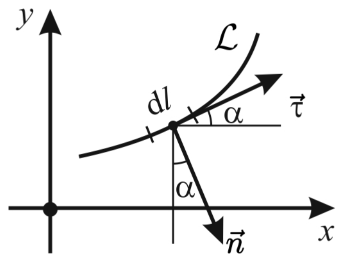

Fig. 5.1

$$\vec{\tau} = \vec{i}\cos\alpha + \vec{j}\sin\alpha.$$

Then it is obvious that

$$\vec{n} = \vec{i}\sin\alpha - \vec{j}\cos\alpha.$$

Taking into account

$$\cos\alpha \mathrm{d}l = \mathrm{d}x, \quad \sin\alpha \mathrm{d}l = \mathrm{d}y,$$

we obtain

$$\int\limits_{\mathcal{L}} (\vec{a}\cdot\vec{n})\mathrm{d}l = \int\limits_{\mathcal{L}} (P\sin\alpha - Q\cos\alpha)\mathrm{d}l = \int\limits_{\mathcal{L}} P\mathrm{d}y - Q\mathrm{d}x.$$

Exercises

5.1. Using Stokes' formula prove that

$$(\operatorname{curl}\vec{a}\cdot\vec{n})(M) = \lim_{\substack{d(\Omega)\to 0\\ M\in\Omega}} \frac{\oint\limits_{\mathcal{L}} \vec{a}\cdot \mathrm{d}\vec{r}}{S(\Omega)},$$

where $S(\Omega)$ is the area of a piecewise smooth surface Ω with the piecewise smooth close boundary $\mathcal{L}$, $\vec{n}$ is the unit normal vector at the point M defining the orientation of Ω , $d(\Omega)$ is the diameter of Ω. The orientation of the boundary corresponds to the orientation of the surface. The given formula is the invariant definition of the projection of the curl of a vector field at the point M onto the vector $\vec{n}$.

5.2. Suppose that $u(M)$ is a continuously differentiable scalar field in a domain G with a piecewise smooth border Ω. Prove that

$$\oiint\limits_{\Omega} \vec{n}u\mathrm{d}S = \iiint\limits_{G} \vec{\nabla}u\mathrm{d}V,$$

where $\vec{n}$ is the outward unit normal vector to Ω.

5.3. Using the result of the Exercise 5.2 prove that

$$\operatorname{grad} u(M) = \lim_{\substack{d(G)\to 0 \\ M\in G}} \frac{\displaystyle\oiint_{\Omega} u\vec{n}\cdot \mathrm{d}S}{V(G)},$$

where $V(G)$ is the volume of a simply connected domain G, bounded by the simply smooth surface Ω, $d(G)$ is the diameter of the domain G. The given formula is the *invariant* definition of the gradient of a scalar field.

5.4. Using the result of the Example 2 prove that

$$\operatorname{curl} \vec{a}(M) = \lim_{\substack{d(G)\to 0 \\ M\in G}} \frac{\displaystyle\oiint_{\Omega} \vec{n}\times\vec{a}\cdot \mathrm{d}S}{V(G)},$$

where $V(G)$ is the volume of a simply connected domain G, bounded by the simply smooth surface Ω, $d(G)$ is the diameter of the domain G. The given formula is the *invariant* definition of the curl of a vector field.

5.5. Let $u(M)$ be a twice continuously differentiable scalar field in a domain G with a piecewise boundary Ω. Prove that

$$\oiint_{\Omega} \frac{\partial u}{\partial n}\mathrm{d}S = \iiint_{G} \Delta u \mathrm{d}V,$$

where $\vec{n}$ is the outward unit normal vector to Ω.

5.6. Using the result of the Exercise 5.5 prove that

$$\Delta u(M) = \lim_{\substack{d(G)\to 0 \\ M\in G}} \frac{\displaystyle\oiint_{\Omega} \frac{\partial u}{\partial n}\mathrm{d}S}{V(G)},$$

where $V(G)$ is the volume of a simply connected domain G, bounded by the simply smooth surface Ω, $d(G)$ is the diameter of the domain G. The given formula is the *invariant* definition of the Laplacian of a scalar field.

5.7. Let $\vec{b}$ be an arbitrary vector and $\vec{a}(M)$ be a continuously differentiable vector field in a domain G with a piecewise boundary Ω. Suppose that a function $f(\vec{b},\vec{a})$ satisfies the condition

$$f(c_1\vec{b}_1 + c_2\vec{b}_2, \vec{a}) = c_1 f(\vec{b}_1, \vec{a}) + c_2 f(\vec{b}_2, \vec{a}),$$

where c_1 and c_2 are arbitrary constants. Prove that

$$\oiint\limits_{\Omega} f(\vec{n}, \vec{a})\mathrm{d}S = \iiint\limits_{G} f(\vec{\nabla}, \vec{a})\mathrm{d}V,$$

where $\vec{n}$ is the outward unit normal vector to Ω. This is generalized Gauss – Ostrogradsky formula.

5.8. Using the result of the Exercise 5.7 prove that

$$f(\vec{\nabla}, \vec{a})(M) = \lim_{\substack{d(G)\to 0\\ M\in G}} \frac{\oiint\limits_{\Omega} f(\vec{n}, \vec{a})\mathrm{d}S}{V(G)},$$

where $V(G)$ is the volume of a simply connected domain G, bounded by the simply smooth surface Ω, $d(G)$ is the diameter of the domain G.

5.9. Prove the following Gauss-Ostrogradsky formula for a tensor field $t_{ij}(x_k)$:

$$\oiint\limits_{\Omega} t_{ij} n_j \mathrm{d}S = \iiint\limits_{G} \frac{\partial t_{ij}}{\partial x_j}\mathrm{d}V,$$

where $\vec{n}$ is the outward unit normal vector to the surface Ω bounding the domain G.

5.10. Let $\vec{a}(M)$ be a twice continuously differentiable vector field in a domain G with a piecewise smooth surface Ω and $\vec{v}$ be a constant vector. Applying the generalized Gauss-Ostrogradsky formula prove that

$$\oiint\limits_{\Omega} (\vec{v}\cdot\vec{n})\vec{a}\mathrm{d}S = \iiint\limits_{G} (\vec{v}\cdot\vec{\nabla})\vec{a}\mathrm{d}V,$$

$$\oiint\limits_{\Omega} \frac{\partial \vec{a}}{\partial n}\mathrm{d}S = \iiint\limits_{G} \Delta\vec{a}\mathrm{d}V.$$

5.11. Using the result of the Exercise 5.10 prove that

$$(\vec{v}\cdot\vec{\nabla})\vec{a}(M) = \lim_{\substack{d(G)\to 0\\ M\in G}} \frac{\oiint\limits_{\Omega} (\vec{v}\cdot\vec{n})\vec{a}\mathrm{d}S}{V(G)},$$

$$\Delta\vec{a}(M) = \lim_{\substack{d(G)\to 0\\ M\in G}} \frac{\oiint\limits_{\Omega} \frac{\partial \vec{a}}{\partial n}\mathrm{d}S}{V(G)},$$

where $V(G)$ is the volume of a simply connected domain G, bounded by the simply smooth surface Ω, $d(G)$ is the diameter of the domain G. The given formulae is the *invariant* definition of the directional derivative and the Laplacian of a vector field respectively.

5.12. Suppose that a simply connected domain $G \subset E^3$ is bounded by a piecewise smooth surface Ω. Show that

$$\oiint\limits_{\Omega} \vec{n} \times (\vec{a} \times \vec{r}) \mathrm{d}S = 2\vec{a}V(G),$$

where $\vec{n}$ is the outward unit normal vector to the surface Ω, $\vec{r}$ is the radius-vector, $\vec{a}$ is an arbitrary constant vector.

5.13. Suppose that a simply connected domain $G \subset E^3$ is bounded by a piecewise smooth surface Ω. Convert the volume integral

$$\iiint\limits_{G} (\operatorname{grad} u \cdot \operatorname{curl} \vec{a}) \mathrm{d}V$$

into the surface integral on Ω.

5.14. Derive the Archimedean law by summing pressure forces exerted to surface elements of a body placed into a liquid.

5.15. Let $u(M)$ and $v(M)$ be twice continuously differentiable scalar fields in a domain G with piecewise smooth boundary Ω. Prove that

$$\oiint\limits_{\Omega} v \frac{\partial u}{\partial n} \mathrm{d}S = \iiint\limits_{G} [v\Delta u + (\vec{\nabla} u) \cdot (\vec{\nabla} v)] \mathrm{d}V$$

(the first Green's formula) and

$$\oiint\limits_{\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \mathrm{d}S = \iiint\limits_{G} (v\Delta u - u\Delta v) \mathrm{d}V$$

(the second Green's formula).

5.16. Suppose that scalar fields $k(M)$, $q(M)$, $u(M)$ and $v(M)$ defined in a domain G with piecewise smooth boundary Ω and are continuously differentiable appropriate times. Consider the operator $L(u)$:

$$L(u) = \operatorname{div}[k(M) \operatorname{grad} u(M)] - q(M)u(M).$$

Prove that

$$\oiint\limits_{\Omega} kv \frac{\partial u}{\partial n} \mathrm{d}S = \iiint\limits_{G} [vL(u) + k(\vec{\nabla} u) \cdot (\vec{\nabla} v) + quv] \mathrm{d}V$$

(the first Green's formula for the operator $L(u)$) and

$$\oiint\limits_{\Omega} k\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)\mathrm{d}S=\iiint\limits_{G}(vL(u)-uL(v))\,\mathrm{d}V$$

(the second Green's formula for the operator $L(u)$).

5.17. Let $u(x,y)$ and $v(x,y)$ be twice continuously differentiable scalar fields in a domain $D\subset E^2$ bounded by a piecewise smooth curve $\mathcal{L}$ and $\vec{n}$ be the outward unit normal vector to the curve at the point (x,y). Prove that

$$\oint\limits_{\mathcal{L}} v\frac{\partial u}{\partial n}\mathrm{d}l=\iint\limits_{D}\left[v\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\right]\mathrm{d}x\mathrm{d}y$$

(the first Green's formula of the plane) and

$$\oint\limits_{\mathcal{L}}\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)\mathrm{d}l=\iint\limits_{D}\left[v\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-u\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right]\mathrm{d}x\mathrm{d}y$$

(the second Green's formula of the plane).

Evaluate the given line integral. Consider $a>0$.

5.18. $\oint\limits_{\mathcal{L}}\dfrac{\partial(x^2+3xy-4y^2)}{\partial n}\mathrm{d}l$, $\mathcal{L}$: circle $4(x+a)^2+(y-2a)^2=4a^2$.

5.19. $\oint\limits_{\mathcal{L}}\dfrac{\partial(x^2+4y^2-xy)}{\partial n}\mathrm{d}l$, $\mathcal{L}$: circle $(x-2)^2+4(y+1)^2=16$.

5.20. $\oint\limits_{\mathcal{L}}\dfrac{\partial(x^2-5xy+3y^2)}{\partial n}\mathrm{d}l$, $\mathcal{L}$: curve formed by the right semicircle $x^2+y^2=2ax,\ x\geqslant a$, and the straight line $x=a$.

5.21. $\oint\limits_{\mathcal{L}}\left(\dfrac{\partial(xy)}{\partial n}\sqrt{x^2+4y^2}-\dfrac{\partial\sqrt{x^2+4y^2}}{\partial n}xy\right)\mathrm{d}l$, $\mathcal{L}$: curve formed by the upper semicircle $x^2+y^2=2y,\ y\geqslant 1$, and the straight line $y=1$.

5.22. Suppose that $\mathcal{L}$ is a closed piecewise smooth curve, $\vec{n}$ is the outward unit normal vector at the point (x,y) and $\vec{c}$ is a constant vector. Prove that

$$\oint\limits_{\mathcal{L}}\cos(\widehat{\vec{c},\vec{n}})\mathrm{d}l=0.$$

5.23. Evaluate the integral

$$\oint\limits_{\mathcal{L}}[x\cos(\widehat{\vec{n},\vec{i}})+y\cos(\widehat{\vec{n},\vec{j}})]\mathrm{d}l,$$

where $\mathcal{L}$ is a simple closed piecewise smooth curve, $\vec{n}$ is the outward unit normal vector at the point (x, y).

5.24. Evaluate Gauss' integral

$$u(x_0, y_0) = \oint\limits_{\mathcal{L}} \frac{\cos(\widehat{\vec{n}, \vec{R}})}{R} \mathrm{d}l,$$

where $\mathcal{L}$ is a simple closed piecewise smooth curve, $\vec{R} = (x - x_0)\vec{i} + (y - y_0)\vec{j}$, $R = |\vec{R}|$, $\vec{n}$ is the outward unit normal vector at the point (x, y).

5.25. Prove if $u(x, y)$ is a harmonic function in a domain D bounded by a simple closed piecewise smooth curve $\mathcal{L}$, then at the point $(x_0, y_0) \in D$

$$u(x_0, y_0) = \frac{1}{2\pi} \oint\limits_{\mathcal{L}} \left(u \frac{\partial \ln R}{\partial n} - \ln R \frac{\partial u}{\partial n} \right) \mathrm{d}l,$$

where $R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$, $\vec{n}$ is the outward unit normal vector at the point (x, y).

5.26. Prove the mean-value theorem for a harmonic scalar field $u(x, y)$:

$$u(x_0, y_0) = \frac{1}{2\pi R} \oint\limits_{C_R} u(x, y) \mathrm{d}l,$$

where C_R is the circle of radius R and center at the point (x_0, y_0).

5.27. Evaluate the Gauss' integral

$$u(x_0, y_0, z_0) = \oiint\limits_{\Omega} \frac{\cos(\widehat{\vec{n}, \vec{R}})}{R^2} \mathrm{d}S,$$

where Ω is a simple smooth closed surface, $\vec{R} = (x - x_0)\vec{i}+, +(y - y_0)\vec{j} + +(z - z_0)\vec{k}$, $R = |\vec{R}|$, $\vec{n}$ is the outward unit normal vector to the surface at the point (x, y, z). Consider two cases: a) the surface Ω does not surround the point (x_0, y_0, z_0); b) the surface Ω surrounds the point (x_0, y_0, z_0).

5.28. Prove if $u(x, y, z)$ is a harmonic scalar field in a domain G, then at the point $(x_0, y_0, z_0) \in G$

$$u(x_0, y_0, z_0) = \frac{1}{4\pi} \oiint\limits_{\Omega} \left(u \frac{\cos(\widehat{\vec{n}, \vec{R}})}{R^2} + \frac{1}{R} \frac{\partial u}{\partial n} \right) \mathrm{d}S,$$

where Ω is the boundary of the domain G, $\vec{n}$ is the outward unit normal vector to the surface at the point (x, y, z),

$$R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

5.29. Prove the mean-value theorem for a harmonic scalar field $u(x, y, z)$:

$$u(x_0, y_0, z_0) = \frac{1}{4\pi R^2} \oint\limits_{S_R} u(x, y, z)\mathrm{d}S,$$

where S_R is the sphere of the radius R and the center at the point (x_0, y_0, z_0).

5.2. Potential vector fields

A vector field $\vec{a}(M)$ is said to be the *potential* field in a domain $G \subset E^3$, if there exists a scalar field $u(M)$, such that $\vec{a} = \operatorname{grad} u$ for each point $M \in G$.

In cartesian coordinates a vector field $\vec{a}(M) = P\vec{i} + Q\vec{j} + R\vec{k}$ is potential in a domain $G \subset E^3$ if there exists a scalar field $u(M)$, such that for all $M \in G$

$$P = \frac{\partial u}{\partial x}, \quad Q = \frac{\partial u}{\partial y}, \quad R = \frac{\partial u}{\partial z}.$$

A domain $G \subset E^3$ is called surface-wise simply connected domain if for each closed piecewise smooth curve $\gamma \in G$ there exists a two-sided piecewise smooth surface $\Omega \in G$ bounded by the curve γ.

Criterion of potentiality.

The following conditions are equivalent for a continuously differentiable vector field $\vec{a}(M)$ on a domain G:

1. Vector field $\vec{a}(M)$ is the potential field on the domain G.

2. The circulation of the vector field $\vec{a}(M)$ is equal to zero for any closed piecewise smooth simple curve.

3. For any points A and B in the domain G and for any piecewise smooth simple curve $\mathcal{L}_{AB} \in G$, connecting these points,

$$\int\limits_{\mathcal{L}_{AB}} \vec{a} \cdot \mathrm{d}\vec{r} = u(B) - u(A),$$

or the line integral depends only on endpoints of the curve.

4. If G is surface-wise simply connected domain, then the vector field $\vec{a}$ is irrotational, i. e. $\operatorname{curl}\vec{a} = \vec{0}$.

In cartesian coordinates the fourth condition takes the form

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

Let M_0 be a particular fixed point and M be an arbitrary point in a domain G. Then the potential $u(M)$ of a vector field $\vec{a}(M)$ can be evaluated as

$$u\left(M\right) = \int\limits_{\mathcal{L}_{MM_0}} \vec{a}\cdot \mathrm{d}\vec{r} = \int\limits_{\mathcal{L}_{MM_0}} P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z,$$

where $\mathcal{L}_{MM_0}$ is an arbitrary curve, connecting the points M and M_0. The potential is determined to within a constant that is specified by the point M_0. Commonly the point M_0 is chosen in such a way that the potential at this point equals to zero. The potential evaluation procedure can be simplified by the appropriate selection of the curve. For example, if we take as the curve $\mathcal{L}_{MM_0}$ a broken line which sides are parallel to the coordinate axes, then we evaluate the potential by the formula

$$u(x,y,z) = \int\limits_{x_0}^{x} P(x,y_0,z_0)\mathrm{d}x + \int\limits_{y_0}^{y} Q(x,y,z_0)\mathrm{d}y + \int\limits_{z_0}^{z} R(x,y,z)\mathrm{d}z. \quad (5.1)$$

Example 1. Show that the curl of the given vector field

$$\vec{a} = -\vec{i}\frac{y}{x^2+y^2} + \vec{j}\frac{x}{x^2+y^2}$$

is equal to zero, but the circulation of this field around the circle $x^2+y^2=1$ is not zero.

Solution. Expanding the determinant, we find

$$\operatorname{curl}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} =$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right]\vec{k} =$$

$$= \left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right]\vec{k} = \vec{0}.$$

The circle $x^2 + y^2 = 1$ we can parameterize as

$$x = \cos t, \quad y = \sin t, \quad t \in [0, 2\pi].$$

Substituting the parametrization into the line integral we obtain the circulation

$$\oint\limits_{\mathcal{L}} -\frac{y\mathrm{d}x}{x^2+y^2} + \frac{x\mathrm{d}y}{x^2+y^2} = \int\limits_0^{2\pi} (\sin^2 t + \cos^2 t)\mathrm{d}t = 2\pi \neq 0.$$

Thus, the field $\vec{a}$ is irrotational, but the circulation around the curve $x^2+y^2 = 1$ is not zero. The reason is that the field $\vec{a}$ is defined and continuous in not surface-wise simply connected domain $G = E^3 \setminus \{(x, y, z):\ x^2 + +y^2 = 0\}$. That means that there is no surface belonging to G and bounded by the curve $x^2 + y^2 = 1$.

Example 2. Ascertain whether the vector field

$$\vec{a}(x, y, z) = x(y^2 + z^2)\vec{i} + y(x^2 + z^2)\vec{j} + z(x^2 + y^2)\vec{k}$$

is potential and find its potential $u(x, y, z)$ if it is possible.

Solution. First we evaluate $\operatorname{curl}\vec{a}$:

$$\operatorname{curl}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x(y^2+z^2) & y(x^2+z^2) & z(x^2+y^2) \end{vmatrix} =$$

$$= \vec{i}(2yz - 2yz) - \vec{j}(2xz - 2xz) + (2xy - 2xy)\vec{k} = \vec{0}.$$

Since the vector field is continuously differentiable in E^3, being a simply connected domain, then it is potential.

M e t h o d 1. To find the potential we use the formula (5.1). First we place the point M_0 into the origin of coordinates. Then

$$u\,(x, y, z) = \int\limits_0^x x(y^2+z^2)\Big|_{\substack{y=0\\ z=0}} \mathrm{d}x + \int\limits_0^y y(x^2+z^2)\Big|_{z=0} \mathrm{d}y +$$

$$+ \int\limits_0^z z(x^2+y^2)\mathrm{d}z = \frac{x^2y^2 + y^2z^2 + z^2x^2}{2} + C.$$

M e t h o d 2. The potential $u(x, y, z)$ satisfies the system of differential equations $\vec{a} = \operatorname{grad} u$ or

$$\frac{\partial u}{\partial x} = P(x, y, z) = x(y^2 + z^2),$$

$$\frac{\partial u}{\partial y} = Q(x, y, z) = y(x^2 + z^2),$$

$$\frac{\partial u}{\partial z} = R(x, y, z) = z(x^2 + y^2).$$

We integrate the first equation:

$$u(x, y, z) = \int x(y^2 + z^2)\mathrm{d}x = \frac{x^2}{2}(y^2 + z^2) + C_1(y, z).$$

Then substituting obtained relation into the second equation, we have

$$x^2 y + \frac{\partial}{\partial y} C_1(y, z) = y(x^2 + z^2) \quad \text{or} \quad \frac{\partial}{\partial y} C_1(y, z) = yz^2,$$

whence it follows

$$C_1(y, z) = \int y^2 z \mathrm{d}y = \frac{y^2 z^2}{2} + C_2(z).$$

Then we substitute the potential $u(x, y, z) = \dfrac{x^2}{2}(y^2 + z^2) + \dfrac{y^2 z^2}{2} + C_2(z)$ into the third equation and find

$$zx^2 + zy^2 + \frac{\partial}{\partial z} C_2(z) = z(x^2 + y^2) \quad \text{or} \quad \frac{\partial}{\partial z} C_2(z) = 0.$$

Thus $C_2(z) = C$. Finally we obtain

$$u(x, y, z) = \frac{x^2 y^2 + y^2 z^2 + z^2 x^2}{2} + C.$$

Example 3. Evaluate the line integral

$$\int\limits_{\mathcal{L}_{AB}} x(y^2 + z^2)\mathrm{d}x + y(x^2 + z^2)\mathrm{d}y + z(x^2 + y^2)\mathrm{d}z,$$

where $A(-2, -1, 2)$ and $B(2, 3, -2)$.

Solution. This line integral is one of potential vector field, therefore it does not depends on the integrating path. In this case we can connect points A and B by the simples appropriate curve and directly evaluate the line integral.

From the other hand the integral is

$$\int\limits_{\mathcal{L}_{AB}} \vec{a} \cdot \mathrm{d}\vec{r} = u(B) - u(A).$$

In the Example 1, we have found that the potential is

$$u(x,y,z) = \frac{x^2y^2 + y^2z^2 + z^2x^2}{2} + C.$$

Then we can easily evaluate the integral:

$$\int\limits_{\mathcal{L}_{AB}} \vec{a} \cdot \mathrm{d}\vec{r} = u(2,3,-2) - u\,(-2,-1,2) = \frac{2^2 3^2 + 3^2(-2)^2 + (-2)^2 2^2}{2} -$$

$$-\frac{(-2)^2(-1)^2 + (-1)^2 2^2 + 2^2(-2)^2}{2} = 36.$$

Exercises

Ascertain whether the given vector field is potential and find its potential $u(x,y,z)$ if it is possible.

5.30. $\vec{a} = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$.

5.31. $\vec{a} = (yz+1)\vec{i} + xz\vec{j} + xy\vec{k}$.

5.32. $\vec{a} = (2xy+z)\vec{i} + (x^2-2y)\vec{j} + x\vec{k}$.

5.33. $\vec{a} = (yz\vec{i} + xz\vec{j} + xy\vec{k})(1+x^2y^2z^2)^{-1}$.

5.34. $\vec{a} = (\vec{i} + \vec{j} + \vec{k})(x+y+z)^{-1}$.

5.35. $\vec{a} = y\vec{i} + x\vec{j} + e^z\vec{k}$.

5.36. $\vec{a} = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$.

5.37. $\vec{a} = yz(2x+y+z)\vec{i} + xz(x+2y+z)\vec{j} + xy(x+y+2z)\vec{k}$.

5.38. $\vec{a} = e^x \sin y\,\vec{i} + e^x \cos y\,\vec{j} + \vec{k}$.

5.39. $\vec{a} = yz\cos xy\,\vec{i} + xz\cos xy\,\vec{j} + \sin xy\,\vec{k}$.

5.40. a) $\vec{a} = r^{-1}\vec{r}$; b) $\vec{a} = r^{-2}\vec{r}$; c) $\vec{a} = r\vec{r}$.

Evaluate the given line integral by proving that the integrand is a full differential.

5.41. $\int\limits_{\mathcal{L}_{AB}} (2x-y)\mathrm{d}x + (3x-y)\mathrm{d}y$, where $A(-1,-2)$, $B(1,0)$.

5.42. $\int\limits_{\mathcal{L}_{AB}} (3x^2-2xy+y^2)\mathrm{d}x - (x^2-2xy)\mathrm{d}y$, where $A(0,1)$, $B(1,0)$.

5.43. $\int\limits_{\mathcal{L}_{AB}} 2x(y^2-2)\mathrm{d}x + 2y(x^2+1)\mathrm{d}y$, where $A(1,1)$, $B(2,3)$.

5.44. $\int\limits_{\mathcal{L}_{AB}} x(1+6y^2)\mathrm{d}x + y(1+6x^2)\mathrm{d}y$, where $A(0,0)$, $B(1,1)$.

5.45. $\int\limits_{\mathcal{L}_{AB}} (x+y)\mathrm{d}x + (x-y)\mathrm{d}y$, where $A(0,1)$, $B(2,3)$.

5.46. $\int\limits_{\mathcal{L}_{AB}} (x-y)(\mathrm{d}x-\mathrm{d}y)$, where $A(1,-1)$, $B(1,1)$.

5.47. $\int\limits_{\mathcal{L}_{AB}} f(x+y)(\mathrm{d}x+\mathrm{d}y)$, where $A(0,0)$, $B(a,b)$, $f(u)$ is a continuous function.

Evaluate the given line integral by choosing an appropriate path.

5.48. $\int\limits_{\mathcal{L}_{AB}} \left(1-\frac{y^2}{x^2}\cos\frac{y}{x}\right)\mathrm{d}x + \left(\sin\frac{y}{x}+\frac{y}{x}\cos\frac{y}{x}\right)\mathrm{d}y$, where $A(1,\pi)$, $B(2,\pi)$.

5.49. $\int\limits_{\mathcal{L}_{AB}} \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2}$, where $A(-1,-2)$, $B(-2,-3)$.

5.50. $\int\limits_{\mathcal{L}_{AB}} \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2}$, where $A(-1,5)$, $B(2,2)$.

5.51. $\int\limits_{\mathcal{L}_{AB}} \frac{x\mathrm{d}x + y\mathrm{d}y}{\sqrt{x^2+y^2}}$, where $A(1,0)$, $B(6,8)$.

5.52. $\int\limits_{\mathcal{L}_{AB}} \frac{x\mathrm{d}y - y\mathrm{d}x}{(x-y)^2}$, where $A(0,-1)$, $B(1,0)$.

5.53. $\int\limits_{\mathcal{L}_{AB}} (2xy + y^2 + yz^2)\mathrm{d}x + (x^2 + 2xy + xz^2)\mathrm{d}y + 2xyz\mathrm{d}z$, where $A(1,1,1)$, $B(2,3,4)$.

5.54. $\int\limits_{\mathcal{L}_{AB}} yzx^{yz-1}\mathrm{d}x + zx^{yz}\ln x\mathrm{d}y + yx^{yz}\ln x\mathrm{d}z$, where $A(1,1,1)$, $B(2,2,2)$.

5.55. $\int\limits_{\mathcal{L}_{AB}} \frac{2\mathrm{d}x}{(y+z)^{1/2}} - \frac{x\mathrm{d}y}{(y+z)^{3/2}} - \frac{x\mathrm{d}z}{(y+z)^{3/2}}$, where $A(1,1,3)$, $B(2,4,5)$.

5.56. $\int\limits_{\mathcal{L}_{AB}} \frac{xz\mathrm{d}y + xy\mathrm{d}z - yz\mathrm{d}x}{(x-yz)^2}$, where $A(7,2,3)$, $B(5,3,1)$.

5.57. $\int\limits_{\mathcal{L}_{AB}} \frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{\sqrt{x^2 + y^2 + +z^2}}$, where A belongs to the sphere $x^2 + y^2 + +z^2 = a^2$, and B to the sphere $x^2 + y^2 + z^2 = b^2$, $0 < a < b$.

5.58. Convert the line integral $\int\limits_{\mathcal{L}_{AB}} f(\sqrt{x^2+y^2+z^2})(x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z)$ into the definite integral, if $A(0,0,0)$, $B(a,b,c)$, $f(u)$ is continuous function.

Find the field $u(x,y,z)$ by its differential.

5.59. $\mathrm{d}u = (2x\cos y - y^2\sin x)\mathrm{d}x + (2y\cos x - x^2\sin y)\mathrm{d}y$.

5.60. $\mathrm{d}u = \left(1 + e^{\frac{x}{y}}\right)\mathrm{d}x + \left(1 + \frac{x}{y}\right)e^{\frac{x}{y}}\mathrm{d}y$.

5.61. $\mathrm{d}u = \frac{x\mathrm{d}x + y\mathrm{d}y}{\sqrt{x^2+y^2}} + \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2}$.

5.62. $$\mathrm{d}u = \left(\sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} - \frac{y}{x^2+y^2}\right)\mathrm{d}x + \\ + \left(\sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} + \frac{x}{x^2+y^2} + \frac{1}{y}\right)\mathrm{d}y.$$

5.63. $\mathrm{d}u = \left(2xy + z^2 + yz\right)\mathrm{d}x + \left(x^2 + 2yz + xz\right)\mathrm{d}y + \left(y^2 + 2xz + xy\right)\mathrm{d}z$.

5.64. $\mathrm{d}u = \left(2xyz + \frac{1}{z}\right)\mathrm{d}x + \left(x^2z - \frac{1}{z^2}\right)\mathrm{d}y + \left(x^2y - \frac{x}{z^2} + \frac{2y}{z^3}\right)\mathrm{d}z$.

5.3. Divergenceless vector fields

A vector field $\vec{a}(M)$ is said to be *divergenceless* or solenoidal in a domain $G \subset E^3$, if $\operatorname{div} \vec{a} = 0$ for each point $M \in G$.

In cartesian coordinates a vector field $\vec{a}(M) = P\vec{i} + Q\vec{j} + R\vec{k}$ is *divergenceless* in a domain $G \subset E^3$, if

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$$

for each point $M \in G$.

Criterion of divergenceless field: In order to a continuously differentiable vector field $\vec{a}$ be divergence-free in a domain G it is sufficient and in simply connected domain G is necessary that the flux of the field through any closed piecewise smooth simple surface is zero.

A vector field $\vec{A}(M)$ such that

$$\vec{a} = \operatorname{curl} \vec{A}.$$

is called the vector potential of a divergenceless vector field $\vec{a}(M)$.

A vector potential is determined to within the gradient of an arbitrary scalar field, since the vector potential B

$$\vec{B}(M) = \vec{A}(M) + \operatorname{grad} f(M)$$

determines the same divergenceless vector field $\vec{a}(M)$ as the vector potential $\vec{A}(M)$, because $\operatorname{curl} \operatorname{grad} f(M) = \vec{0}$.

Vector potential can be calculated as follows:

$$\begin{aligned}
A_x =& 0, \\
A_y =& \int R(x, y, z) dx, \\
A_z =& \int \left[\frac{\partial}{\partial y} \int Q(x, y, z) \mathrm{d}x + \frac{\partial}{\partial z} \int R(x, y, z) \mathrm{d}x + P\,(x, y, z) \right] \mathrm{d}y - \\
&- \int Q(x, y, z) \mathrm{d}x.
\end{aligned}$$

Example 1. Show that the divergence of the vector field $\vec{a} = k\dfrac{\vec{r}}{r^3}$, $k = \text{const} \neq 0$, is zero while the flux through the sphere $x^2 + y^2 + z^2 = 1$ is not zero.

Solution. The divergence in cartesian coordinate system is

$$\operatorname{div}\left(k\frac{\vec{r}}{r^3}\right) = k\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = k\left(\frac{1}{r^3}\frac{\partial x_i}{\partial x_i} - 3\frac{x_i}{r^4}\frac{x_i}{r}\right) = k\left(\frac{3}{r^3} - 3\frac{r^2}{r^5}\right) = 0.$$

But the flux is

$$\oiint\limits_{x^2+y^2+z^2=1} \vec{a}\cdot \mathrm{d}\vec{S} = \oint\limits_{x^2+y^2+z^2=1} \frac{k}{r^3}\vec{r}\cdot\vec{r}\mathrm{d}\vec{S} = \oint\limits_{x^2+y^2+z^2=1} \left.\frac{k}{r}\right|_{r=1} \mathrm{d}\vec{S} = 4\pi k \neq 0.$$

That is because the domain $G = E\backslash\{(0,0,0)\}$ where the vector field $\vec{a}$ is continuously differentiable is not simply connected and divergenceless criterion is not fulfilled.

Example 2. Ascertain whether the vector field

$$\vec{a} = 3y^2\vec{i} - 3x^2\vec{j} - (y^2 + 2x)\vec{k}$$

is divergenceless and find its vector potential $\vec{A}(M)$ if possible.

Solution. It is obvious that $\operatorname{div}\vec{a} = 0$ in simply connected domain E^3. Therefore the field is divergenceless.

The vector potential $\vec{A}(M)$ is defined by the expression $\operatorname{curl}\vec{A} = \vec{a}$ or by the system of differential equations

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = P(x,y,z) = 3y^2,$$

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = Q(x,y,z) = -3x^2,$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = R(x,y,z) = -(y^2 + 2x).$$

We are to find a particular solution of this system. Since the vector potential is defined to within the gradient of a scalar field, then we can select such a scalar field that $A_x(x,y,z) \equiv 0$. Then the system of differential equations is simplified to

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 3y^2, \frac{\partial A_z}{\partial x} = 3x^2, \frac{\partial A_y}{\partial x} = -(y^2 + 2x).$$

Integrating the second and the third equations, we obtain

$$A_z = \int 3x^2\mathrm{d}x = x^3 + C_1(y,z),$$

$$A_y = -\int (y^2 + 2x)\mathrm{d}x = -y^2 x - x^2 + C_2(y, z).$$

Suppose that $C_2(y, z) \equiv 0$. Then substituting A_y and A_z into the first equation, we find

$$\frac{\partial}{\partial y} C_1(y, z) = 2y^2,$$

Therefore

$$C_1(y, z) = \int 3y^2 dy = y^3 + C_3(z).$$

Assuming $C_3(z) \equiv 0$, we finally obtain

$$A_x = 0, \quad A_y = -y^2 x - x^2, \quad A_z = x^3 + y^3.$$

Exercises

Ascertain whether the given vector field is divergenceless and find its vector potential if possible.

5.65. $\vec{a} = \vec{i} + \vec{j} + \vec{k}$.

5.66. $\vec{a} = 2y\vec{i} + 2z\vec{j}$.

5.67. $\vec{a} = (e^x - e^y)\vec{k}$.

5.68. $\vec{a} = 2\cos xz\vec{j}$.

5.69. $\vec{a} = 5x^2 y\vec{i} - 10xyz\vec{k}$.

5.70. $\vec{a} = 6x\vec{i} - 15y\vec{j} + 9z\vec{k}$.

5.71. $\vec{a} = \dfrac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$, $x^2 + y^2 \neq 0$.

5.72. $\vec{a} = (y + z)\vec{i} + (x + z)\vec{j} + (x + y)\vec{k}$.

5.73. $\vec{a} = 2y\vec{i} - z\vec{j} + 2x\vec{k}$.

5.74. $\vec{a} = x(z^2 - y^2)\vec{i} + y(x^2 - z^2)\vec{j} + z(y^2 - x^2)\vec{k}$.

5.75. $\vec{a} = y^2\vec{i} - (x^2 + y^3)\vec{j} + 3zy^2\vec{k}$.

5.76. $\vec{a} = (1 + 2xy)\vec{i} - y^2 z\vec{j} + (z^2 y - 2zy + 1)\vec{k}$.

5.77. $\vec{a} = 6y^2\vec{i} + 6z\vec{j} + 6x\vec{k}$.

5.78. $\vec{a} = ye^{x^2}\vec{i} + 2yz\vec{j} - (2xyze^{x^2} + z^2)\vec{k}$.

5.4. Orthogonal curvilinear coordinate systems

A curvilinear coordinate system in E^3 associates to each point (x, y, z) ordered real number triple (q^1, q^2, q^3). Curvilinear coordinates q^1, q^2, q^3 of the point (x, y, z) are connected with its cartesian coordinates x, y, z by formulas

$$q^1 = q^1(x, y, z), \quad q^2 = q^2(x, y, z), \quad q^3 = q^3(x, y, z),$$

where $q^i(x, y, z)$, $i = 1, 2, 3$, are single valued continuously differentiable in E^3 functions and the Jacobian determinant $\dfrac{\partial(x, y, z)}{\partial(q^1, q^2, q^3)} \neq 0$.

The condition $q^i = q^i(x, y, z) = \text{const}$ defines for a fixed value of the index i a family of nonintersecting surfaces called *coordinate surfaces* of coordinate q^i. The curve of intersection of two coordinate surfaces related to different coordinates q^i and q^j $(i \neq j)$ is called the *coordinate curve* of the third coordinate q^k $(i \neq j,\ i \neq k,\ j \neq k)$.

Three coordinate surfaces related to three different coordinates intersect in on point as well as three coordinate curves of different coordinates.

In each point M it is possible to construct the *natural basis*, vectors of which are tangents to corresponding coordinate curves:

$$\frac{\partial \vec{r}}{\partial q^i} = \frac{\partial x^k}{\partial q^i} \vec{e}_k, \quad i, k = 1, 2, 3.$$

A curvilinear coordinate system is called *orthogonal*, if the natural basis at every point is orthogonal. Normalized natural basis[1]

$$\vec{e}_\alpha = \frac{1}{H_\alpha} \frac{\partial \vec{r}}{\partial q^\alpha},$$

where

$$H_k = \left| \frac{\partial \vec{r}}{\partial q^k} \right| = \sqrt{\left(\frac{\partial x}{\partial q^k}\right)^2 + \left(\frac{\partial y}{\partial q^k}\right)^2 + \left(\frac{\partial z}{\partial q^k}\right)^2},$$

is called the *physical basis*. Coefficients H_k, $k = 1, 2, 3$, are called *Lamé coefficients*.

The differential of a radius-vector can be written as

$$\mathrm{d}\vec{r} = \frac{\partial \vec{r}}{\partial q^k} \mathrm{d}q^k = H_1 \mathrm{d}q^1 \vec{e}_1 + H_2 \mathrm{d}q^2 \vec{e}_2 + H_3 \mathrm{d}q^3 \vec{e}_3.$$

[1]There is no summation over the index α here.

Element of the coordinate curve length:

$$\mathrm{d}l_\alpha = H_\alpha \mathrm{d}q^\alpha$$

Element of the coordinate surface area:

$$\mathrm{d}S_1 = H_2H_3\mathrm{d}q^2\mathrm{d}q^3, \quad \mathrm{d}S_2 = H_1H_3\mathrm{d}q^1\mathrm{d}q^3, \quad \mathrm{d}S_3 = H_1H_2\mathrm{d}q^1\mathrm{d}q^2.$$

Element of the volume:

$$\mathrm{d}V = H_1H_2H_3\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Cylindrical coordinates

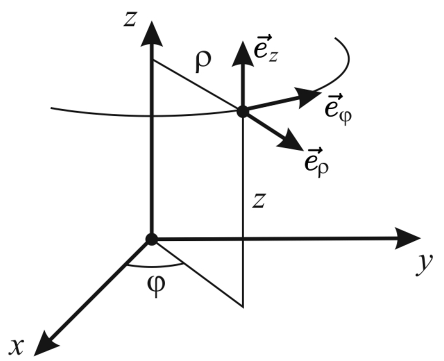

Fig. 5.2

In the cylindrical coordinate system the position of a point is defined by coordinates (fig. 5.2):

$$q^1 = \rho, \quad 0 \leqslant \rho < +\infty,$$
$$q^2 = \varphi, \quad 0 \leqslant \varphi \leqslant 2\pi,$$
$$q^3 = z, \quad -\infty < z < +\infty,$$

where ρ is the distance from the point to the Oz axis, φ is the angle between positive half of the Ox axis and the straight line connecting the origin and the projection of the point onto the xOy plane, z is the third cartesian coordinate.

Cartesian coordinates are related to cylindrical by formulas

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z,$$

and inversely

$$\rho = \sqrt{x^2 + y^2}, \quad \tan\varphi = \frac{y}{x}, \quad z = z.$$

Lamé coefficients, physical basis, elements of the length and area in cylindrical coordinates:

$$\begin{array}{llll} H_1 = 1, & \vec{e}_\rho = \vec{i}\cos\varphi + \vec{j}\sin\varphi, & \mathrm{d}l_1 = \mathrm{d}\rho & \mathrm{d}S_1 = \rho\mathrm{d}\varphi\mathrm{d}z; \\ H_2 = \rho, & \vec{e}_\varphi = -\vec{i}\sin\varphi + \vec{j}\cos\varphi, & \mathrm{d}l_1 = \rho\mathrm{d}\varphi, & \mathrm{d}S_2 = \mathrm{d}\rho\mathrm{d}z; \\ H_3 = 1, & \vec{e}_z = \vec{k}, & \mathrm{d}l_1 = \mathrm{d}z, & \mathrm{d}S_3 = \rho\mathrm{d}\rho\mathrm{d}\varphi. \end{array}$$

The volume element is $\mathrm{d}V = \rho\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z$.

Spherical coordinates

In the spherical coordinate system the position of a point is defined by coordinates (fig. 5.3):

$$q^1 = r, \quad 0 \leqslant r < +\infty,$$
$$q^2 = \theta, \quad 0 \leqslant \theta \leqslant \pi,$$
$$q^3 = \varphi, \quad 0 \leqslant \varphi \leqslant 2\pi,$$

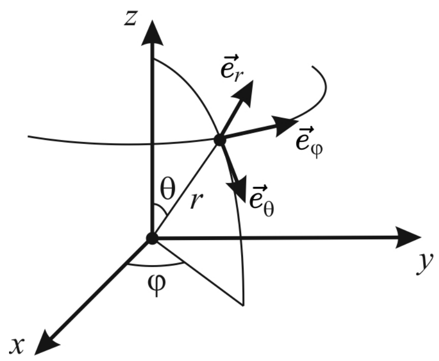

Fig. 5.3

where r is the distance from the point to the origin, θ is the angle between positive half of the Oz axis and the straight line connecting the origin and the point, φ is the angle between positive half of the Ox axis and the straight line connecting the origin and the projection of the point onto the xOy.

Cartesian coordinates are related to spherical by formulas

$$x = r\cos\varphi\sin\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\theta.$$

and inversely

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arccos\frac{z}{r}, \quad \tan\varphi = \frac{y}{x}.$$

Lamé coefficients, physical basis, elements of the length and area in cylindrical coordinates:

$$\begin{aligned} &H_1 = 1, && \vec{e}_r = \left(\vec{i}\cos\varphi + \vec{j}\sin\varphi\right)\sin\theta + \vec{k}\cos\theta, && \mathrm{d}l_1 = \mathrm{d}r; \\ &H_2 = r, && \vec{e}_\theta = \left(\vec{i}\cos\varphi + \vec{j}\sin\varphi\right)\cos\theta - \vec{k}\sin\theta, && \mathrm{d}l_2 = r\mathrm{d}\theta; \\ &H_3 = r\sin\theta, && \vec{e}_\varphi = -\vec{i}\sin\varphi + \vec{j}\cos\varphi, && \mathrm{d}l_2 = r\sin\theta\mathrm{d}\varphi; \end{aligned}$$

$$\mathrm{d}S_1 = r^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi, \quad \mathrm{d}S_2 = r\sin\theta\mathrm{d}r\mathrm{d}\varphi, \quad \mathrm{d}S_3 = r\mathrm{d}r\mathrm{d}\theta.$$

The volume element is $\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$.

Example 1. Find coordinate surfaces of the cylindrical and spherical coordinate systems.

Solution. To find coordinate surfaces of the cylindrical coordinate system we use relations

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z. \tag{5.2}$$

If we eliminate the coordinate φ form the system (5.2), we obtain

$$x^2 + y^2 = \rho^2, \quad z = z.$$

Thus the equation $\rho = \text{const}$ defines the family of coaxial cylinders of radius ρ and axis Oz (fig. 5.4).

Then to obtain a family $\varphi = \text{const}$ we eliminate the coordinate ρ from (5.2):

$$x = y \tan \varphi, \quad z = z.$$

Thus the coordinate surfaces related to the coordinate φ are a set of half-planes adjoining the axis Oz.

And finally the family $z = \text{const}$ is a set of planes perpendicular to the axis Oz.

Similarly to find coordinate surfaces of the spherical coordinate system we are to use relations

$$x = r \cos \varphi \sin \theta, \quad y = r \sin \varphi \sin \theta, \quad z = r \cos \theta. \tag{5.3}$$

Squaring and then summing each equation of (5.3) we obtain

$$x^2 + y^2 + z^2 = r^2.$$

Thus the family $r = \text{const}$, is a set of concentric spheres of radius r having the center at the origin (fig. 5.5).

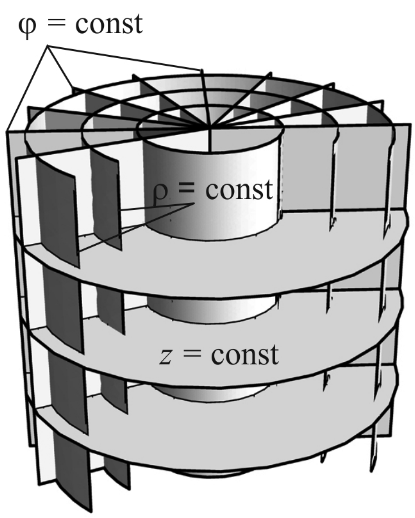

Fig. 5.4

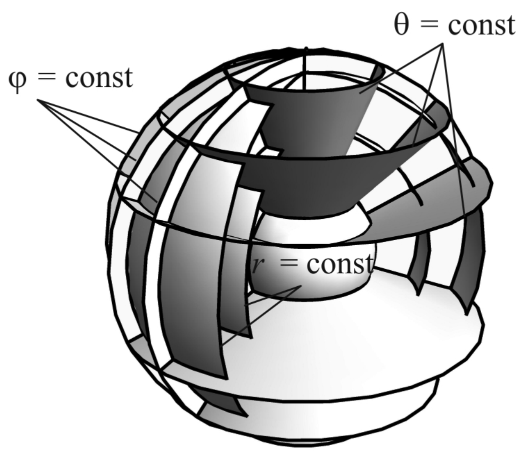

Fig. 5.5

To find the coordinate surfaces of $\theta = \text{const}$ we transform the equations (5.3) to the form

$$x^2 + y^2 = r^2 \sin^2\theta, \quad r = \frac{z}{\cos\theta},$$

thus

$$x^2 + y^2 = z^2 \tan^2\theta.$$

This equation defines a set of the cones with axes Oz and apex angle θ.

As in the cylindrical coordinate system the family $\varphi = \text{const}$ is a set of half-planes adjoining the axis Oz.

Exercises

5.79. Let $(\vec{e}_\rho, \vec{e}_\varphi, \vec{e}_z)$ be the physical basis of the cylindrical coordinate system. Suppose that there are given three points $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$.

a) Draw the basis $(\vec{e}_\rho, \vec{e}_\varphi, \vec{e}_z)$ at the given points.

b) Find angles between vectors 1) $\vec{e}_\rho(A)$ and $\vec{e}_\rho(B)$, 2) $\vec{e}_\varphi(A)$ and $\vec{e}_\rho(B)$, 3) $\vec{e}_\rho(A)$ and $\vec{e}_\varphi(B)$, 4) $\vec{e}_z(A)$ and $\vec{e}_z(B)$.

c) Find components of 1) vectors $\vec{i}, \vec{j}, \vec{k}$ in the basis $(\vec{e}_\rho, \vec{e}_\varphi, \vec{e}_z)$ at the points A and B; 2) vectors $\vec{e}_\rho, \vec{e}_\varphi, \vec{e}_z$ in the basis $(\vec{i}, \vec{j}, \vec{k})$ at the points A and B.

5.80. Let $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ be the physical basis of the spherical coordinate system. Suppose that there are given three points $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$.

a) Draw the basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ at the given points.

b) Find angles between vectors 1) $\vec{e}_r(A)$ and $\vec{e}_\theta(B)$, 2) $\vec{e}_\varphi(A)$ and $\vec{e}_r(B)$, 3) $\vec{e}_r(A)$ and $\vec{e}_\varphi(B)$, 4) $\vec{e}_\theta(A)$ and $\vec{e}_\theta(B)$.

c) Find components of 1) vectors $\vec{i}, \vec{j}, \vec{k}$ in the basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ at the points A and B; 2) vectors $\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi$ in the basis $(\vec{i}, \vec{j}, \vec{k})$ at the points A and B.

5.81. The elliptic cylindrical coordinates u, v, z are related to the Cartesian coordinates by formulas:

$$x = a\cosh u \cos v, \quad y = a \sinh u \sin v, \quad z = z,$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v \leqslant 2\pi, \quad -\infty < z < +\infty.$$

Find the coordinate surfaces and prove that the given coordinate system is orthogonal. Evaluate the Lamé coefficients.

5.82. The parabolic cylindrical coordinates ξ, η, z are related to the Cartesian coordinates by formulas:

$$x = \xi\eta, \quad y = \frac{1}{2}(\eta^2 - \xi^2), \quad z = z,$$

$$-\infty < \xi < +\infty, \quad 0 \leqslant \eta < +\infty, \quad -\infty < z < +\infty.$$

Find the coordinate surfaces and prove that the given coordinate system is orthogonal. Evaluate the Lamé coefficients.

5.83. The bipolar coordinates ξ, η, z are related to the Cartesian coordinates by formulas:

$$x = \frac{a \sinh \eta}{\cosh \eta - \cos \xi}, \quad y = \frac{a \sin \xi}{\cosh \eta - \cos \xi}, \quad z = z,$$

$$0 \leqslant \xi \leqslant 2\pi, \quad -\infty < \eta < +\infty, \quad -\infty < z < +\infty.$$

Find the coordinate surfaces and prove that the given coordinate system is orthogonal. Evaluate the Lamé coefficients.

5.84. The equations

$$\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} = 1, \qquad \xi \geqslant -c^2,$$

$$\frac{x^2}{a^2+\eta} + \frac{y^2}{b^2+\eta} + \frac{z^2}{c^2+\eta} = 1, \qquad -b^2 \leqslant \eta \leqslant -c^2,$$

$$\frac{x^2}{a^2+\zeta} + \frac{y^2}{b^2+\zeta} + \frac{z^2}{c^2+\zeta} = 1, \qquad -a^2 \leqslant \zeta \leqslant -b^2,$$

describe correspondingly an ellipsoid, one-sheeted and two-sheeted hyperboloids confocal with the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad c < b < a.$$

The real numbers ξ, η, ζ are called the ellipsoidal coordinates. Establish the relation between the ellipsoidal and the Cartesian coordinates. Prove that the given coordinate system is orthogonal and evaluate the Lamé coefficients.

5.85. The prolate ellipsoidal coordinate system is the particular case of the ellipsoidal coordinate system when $a > b = c$. In this case the coordinate η becomes constant and we are to introduce the azimuthal angle φ in the plane yOz counted off the axis Oy. Making the substitution $\xi = \cosh u$, $\zeta = \cos v$, we obtain the relation between the prolate ellipsoidal coordinates u, v, φ and the Cartesian coordinates:

$$x = a \sinh u \sin v \cos\varphi, \quad y = a \sinh u \sin v \sin\varphi, \quad z = a \cosh u \cos v,$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v \leqslant \pi, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

Find the coordinate surfaces and prove that the given coordinate system is orthogonal. Evaluate the Lamé coefficients.

5.86. The oblate ellipsoidal coordinate system is the particular case of the ellipsoidal coordinate system when $a = b > c$. In this case the coordinate ζ becomes constant and we are to introduce the azimuthal angle φ in the plane xOy counted off the axis Ox. Making the substitution $\xi = \sinh u$, $\eta = \cos v$, we obtain the relation between the prolate ellipsoidal coordinates u, v, φ and the Cartesian coordinates:

$$x = a \cosh u \sin v \cos\varphi, \quad y = a \cosh u \sin v \sin\varphi, \quad z = a \sinh u \cos v,$$

$$0 \leqslant u < +\infty, \quad 0 \leqslant v \leqslant \pi, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

Find the coordinate surfaces and prove that the given coordinate system is orthogonal. Evaluate the Lamé coefficients.

5.87. The parabolic coordinates ξ, η, φ are related to the Cartesian coordinates by formulas:

$$x = \xi\eta \cos\varphi, \quad y = \xi\eta \sin\varphi, \quad z = \frac{1}{2}\left(\eta^2 - \xi^2\right),$$

$$0 \leqslant \xi < +\infty, \quad 0 \leqslant \eta < +\infty, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

Find the coordinate surfaces and prove that the given coordinate system is orthogonal. Evaluate the Lamé coefficients.

5.88. The toroidal coordinates ξ, η, φ are related to the Cartesian coordinates by formulas:

$$x = \frac{a \sinh\eta \cos\varphi}{\cosh\eta - \cos\xi}, \quad y = \frac{a \sinh\eta \sin\varphi}{\cosh\eta - \cos\xi}, \quad z = \frac{a \sin\xi}{\cosh\eta - \cos\xi},$$

$$-\pi \leqslant \xi \leqslant \pi, \quad 0 \leqslant \eta < +\infty, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

Find the coordinate surfaces and prove that the given coordinate system is orthogonal. Evaluate the Lamé coefficients.

5.5. Differential operators and integrals in curvilinear coordinates

A vector field in curvilinear coordinates $\vec{a}$:

$$\vec{a} = a_{q^1}(q^1, q^2, q^3)\vec{e}_{q^1} + a_{q^2}(q^1, q^2, q^3)\vec{e}_{q^2} + a_{q^3}(q^1, q^2, q^3)\vec{e}_{q^3}.$$

Note that the basis vectors $\vec{e}_{q^i}$ are also depend on the curvilinear coordinates q^k.

The equation of vector lines in curvilinear coordinates:

$$\frac{H_1 \mathrm{d}q^1}{a_{q^1}} = \frac{H_2 \mathrm{d}q^2}{a_{q^2}} = \frac{H_3 \mathrm{d}q^3}{a_{q^3}},$$

in the cylindrical coordinates:

$$\frac{\mathrm{d}\rho}{a_\rho} = \frac{\rho \mathrm{d}\varphi}{a_\varphi} = \frac{\mathrm{d}z}{a_z},$$

in the spherical coordinates:

$$\frac{\mathrm{d}r}{a_r} = \frac{r \mathrm{d}\theta}{a_\theta} = \frac{r \sin\theta \mathrm{d}\varphi}{a_\varphi}.$$

The gradient in curvilinear coordinates:

$$\operatorname{grad} u = \frac{1}{H_1}\frac{\partial u}{\partial q^1}\vec{e}_{q^1} + \frac{1}{H_2}\frac{\partial u}{\partial q^2}\vec{e}_{q^2} + \frac{1}{H_3}\frac{\partial u}{\partial q^3}\vec{e}_{q^3},$$

in the cylindrical coordinates:

$$\operatorname{grad} u = \frac{\partial u}{\partial \rho}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \varphi}\vec{e}_\varphi + \frac{\partial u}{\partial z}\vec{e}_z,$$

in the spherical coordinates:

$$\operatorname{grad} u = \frac{\partial u}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\vec{e}_\varphi.$$

The divergence in curvilinear coordinates:

$$\operatorname{div}\vec{a} = \frac{1}{H_1 H_2 H_3}\left(\frac{\partial(a_{q^1} H_2 H_3)}{\partial q^1} + \frac{\partial(a_{q^2} H_1 H_3)}{\partial q^2} + \frac{\partial(a_{q^3} H_1 H_2)}{\partial q^3}\right),$$

in the cylindrical coordinates:

$$\operatorname{div}\vec{a} = \frac{1}{\rho}\frac{\partial(\rho a_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial a_\varphi}{\partial\varphi} + \frac{\partial a_z}{\partial z},$$

in the spherical coordinates:

$$\operatorname{div}\vec{a} = \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta a_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial\varphi}.$$

The curl in curvilinear coordinates:

$$\operatorname{curl}\vec{a} = \frac{1}{H_2H_3}\left(\frac{\partial(a_{q^3}H_3)}{\partial q^2} - \frac{\partial(a_{q^2}H_2)}{\partial q^3}\right)\vec{e}_{q^1} +$$

$$+\frac{1}{H_3H_1}\left(\frac{\partial(a_{q^1}H_1)}{\partial q^3} - \frac{\partial(a_{q^3}H_3)}{\partial q^1}\right)\vec{e}_{q^2} +$$

$$+\frac{1}{H_1H_2}\left(\frac{\partial(a_{q^2}H_2)}{\partial q^1} - \frac{\partial(a_{q^1}H_1)}{\partial q^2}\right)\vec{e}_{q^3}.$$

The curl also can be written in the form of the formal determinant:

$$\operatorname{curl}\vec{a} = \begin{vmatrix} \frac{1}{H_2H_3}\vec{e}_{q^1} & \frac{1}{H_1H_3}\vec{e}_{q^2} & \frac{1}{H_1H_2}\vec{e}_{q^3} \\ \frac{\partial}{\partial q^1} & \frac{\partial}{\partial q^2} & \frac{\partial}{\partial q^3} \\ H_1 a_{q^1} & H_2 a_{q^2} & H_3 a_{q^3} \end{vmatrix}.$$

The curl in the cylindrical coordinates:

$$\operatorname{curl}\vec{a} = \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\varphi} - \frac{\partial a_\varphi}{\partial z}\right)\vec{e}_\rho + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\vec{e}_\varphi + \left(\frac{1}{\rho}\frac{\partial(\rho a_\varphi)}{\partial\rho} - \frac{\partial a_\rho}{\partial\varphi}\right)\vec{e}_z,$$

in the spherical coordinates:

$$\operatorname{curl}\vec{a} = \left(\frac{1}{r\sin\theta}\frac{\partial(a_\varphi\sin\theta)}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial\varphi}\right)\vec{e}_r +$$

$$+\left(\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial\varphi} - \frac{1}{r}\frac{\partial(ra_\varphi)}{\partial r}\right)\vec{e}_\theta +$$

$$+\left(\frac{1}{r}\frac{\partial(ra_\theta)}{\partial r} - \frac{1}{r}\frac{\partial a_r}{\partial\theta}\right)\vec{e}_\varphi.$$

The Laplacian in curvilinear coordinates:

$$\Delta u = \frac{1}{H_1H_2H_3}\left[\frac{\partial}{\partial q^1}\left(\frac{H_2H_3}{H_1}\frac{\partial u}{\partial q^1}\right)+\right.$$

$$\left.+\frac{\partial}{\partial q^2}\left(\frac{H_1H_3}{H_2}\frac{\partial u}{\partial q^2}\right)+\frac{\partial}{\partial q^3}\left(\frac{H_1H_2}{H_3}\frac{\partial u}{\partial q^3}\right)\right],$$

in the cylindrical coordinates:

$$\Delta u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2},$$

in the spherical coordinates:

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}.$$

The line integral of the second kind of a vector field $\vec{a}$ along a curve $\mathcal{L}$:

$$\int\limits_{\mathcal{L}}\vec{a}\cdot\mathrm{d}\vec{r}=\int\limits_{\mathcal{L}}a_{q^1}H_1\mathrm{d}q^1+a_{q^2}H_2\mathrm{d}q^2+a_{q^3}H_3\mathrm{d}q^3.$$

The surface integral of the second kind of a vector field $\vec{a}$ on the coordinate surface $\Omega_1:\ q^1=C,\ q^2\in[\alpha_2,\beta_2],\ q^3\in[\alpha_3,\beta_3]$ can be evaluated as follows. Since the unit normal vector to the coordinate surface is the basis vector $\pm\vec{e}_{q^1}$, and the area element is $\mathrm{d}S_1=H_2H_3\mathrm{d}q^2\mathrm{d}q^3$, then

$$\iint\limits_{\Omega_1}\vec{a}\cdot\mathrm{d}\vec{S}=\pm\iint\limits_{\Omega_1}(\vec{a}\cdot\vec{e}_{q^1})\mathrm{d}S=$$

$$=\pm\int\limits_{\alpha_2}^{\beta_2}\mathrm{d}q^2\int\limits_{\alpha_3}^{\beta_3}a_{q^1}(C,q^2,q^3)H_2(C,q^2,q^3)H_3(C,q^2,q^3)\mathrm{d}q^3.$$

The surface integral of the second kind of a vector field $\vec{a}$ on other coordinate surfaces can by evaluated similarly.

Example 1. Find vector lines of the field

$$\vec{a}=r^2\vec{e}_r-\cos^2\theta\,\vec{e}_\theta+r\sin\theta\,\vec{e}_\varphi.$$

Solution. In the spherical coordinates the system of differential equation for vector lines of the given field is

$$\frac{\mathrm{d}r}{r^2}=\frac{r\mathrm{d}\theta}{-\cos^2\theta}=\frac{r\sin\theta\mathrm{d}\varphi}{r\sin\theta}.$$

This system is equivalent to

$$\frac{dr}{r^3} = \frac{d\theta}{-\cos^2\theta}, \quad \frac{dr}{r^2} = d\varphi.$$

Integrating each equation, we obtain

$$r^2 = \frac{1}{\tan\theta + C_1}, \quad r = -\frac{1}{\varphi + C_2}.$$

Example 2. Find the gradient of the following scalar field in the cylindrical coordinates:

$$u = \rho\cos\varphi + z^2\sin^2\varphi - 3^\rho.$$

Solution.

$$\operatorname{grad} u = \frac{\partial u}{\partial\rho}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial\varphi}\vec{e}_\varphi + \frac{\partial u}{\partial z}\vec{e}_z =$$

$$= (\cos\varphi - 3^\rho\ln 3)\vec{e}_\rho + \frac{1}{\rho}(-\rho\sin\varphi + 2z^2\sin\varphi\cos\varphi)\vec{e}_\varphi + 2z\sin^2\varphi\vec{e}_z =$$

$$= (\cos\varphi - 3^\rho\ln 3)\vec{e}_\rho + (-\sin\varphi + \frac{z^2}{\rho}\sin 2\varphi)\vec{e}_\varphi + 2z\sin^2\varphi\vec{e}_z.$$

Example 3. Find the gradient of the following scalar field in the spherical coordinates:

$$u = \alpha\frac{\cos\theta}{r^3}, \quad \alpha = \text{const}.$$

Solution.

$$\operatorname{grad} u = \frac{\partial u}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial u}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi}\vec{e}_\varphi =$$

$$= -\frac{3\alpha\cos\theta}{r^4}\vec{e}_r - \frac{1}{r}\frac{\alpha\sin\theta}{r^3}\vec{e}_\theta + \frac{1}{r\sin\theta}\cdot 0\cdot\vec{e}_\varphi = -\frac{3\alpha\cos\theta}{r^4}\vec{e}_r - \frac{\alpha\sin\theta}{r^4}\vec{e}_\theta.$$

Example 4. Find the divergence of the following vector field in the cylindrical coordinates:

$$\vec{a} = \rho\,\vec{e}_\rho + z\sin\varphi\,\vec{e}_\varphi + e^\varphi\cos z\,\vec{e}_z.$$

Solution.

$$\operatorname{div}\vec{a} = \frac{1}{\rho}\frac{\partial(\rho a_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial a_\varphi}{\partial\varphi} + \frac{\partial a_z}{\partial z} =$$

$$= \frac{1}{\rho}\frac{\partial(\rho^2)}{\partial\rho} + \frac{1}{\rho}\frac{\partial(z\sin\varphi)}{\partial\varphi} + \frac{\partial(e^{\varphi}\cos z)}{\partial z} = 2 + \frac{z\cos\varphi}{\rho} - e^{\varphi}\sin z.$$

Example 5. Find the divergence of the following vector field in the spherical coordinates:

$$\vec{a} = r^2\vec{e}_r - 2\cos^2\varphi\vec{e}_\theta + \frac{\varphi}{r^2+1}\vec{e}_\varphi.$$

Solution.

$$\operatorname{div}\vec{a} = \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta a_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial\varphi} =$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta(-2\cos^2\varphi)) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\varphi}{r^2+1}\right) =$$

$$= 4r - \frac{2}{r}\cos^2\varphi\cot\theta + \frac{1}{r(r^2+1)\sin\theta}.$$

Example 6. Find the curl of the following vector field in the cylindrical coordinates:

$$\vec{a} = \cos\varphi\,\vec{e}_\rho - \frac{\sin\varphi}{\rho}\,\vec{e}_\varphi + \rho^2\,\vec{e}_z.$$

Solution.

$$\operatorname{curl}\vec{a} = \begin{vmatrix} \frac{1}{\rho}\vec{e}_\rho & \vec{e}_\varphi & \frac{1}{\rho}\vec{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ a_\rho & \rho a_\varphi & a_z \end{vmatrix} = \begin{vmatrix} \frac{1}{\rho}\vec{e}_\rho & \vec{e}_\varphi & \frac{1}{\rho}\vec{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ \cos\varphi & -\sin\varphi & \rho^2 \end{vmatrix} =$$

$$= \frac{1}{\rho}\vec{e}_\rho(0+0) - \vec{e}_\varphi(2\rho - 0) + \vec{e}_z\left(\frac{\sin\varphi}{\rho}\right) = -2\rho\vec{e}_\varphi + \frac{\sin\varphi}{\rho}\vec{e}_z.$$

Example 7. Find the curl of the following vector field in the spherical coordinates:

$$\vec{a} = r\,\vec{e}_r + r\cos\theta\,\vec{e}_\theta + r\sin\varphi\,\vec{e}_\varphi.$$

Solution.

$$\operatorname{curl}\vec{a} = \begin{vmatrix} \frac{1}{r^2\sin\theta}\vec{e}_r & \frac{1}{r\sin\theta}\vec{e}_\theta & \frac{1}{r}\vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ a_r & r a_\theta & r\sin\theta a_\varphi \end{vmatrix} =$$

$$= \begin{vmatrix} \dfrac{1}{r^2\sin\theta}\vec{e}_r & \dfrac{1}{r\sin\theta}\vec{e}_\theta & \dfrac{1}{r}\vec{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\varphi} \\ r & r^2\cos\theta & r^2\sin\theta\sin\varphi \end{vmatrix} =$$

$$= \frac{1}{r^2\sin\theta}(r^2\sin\varphi\cos\theta - 0)\vec{e}_r - \frac{1}{r\sin\theta}(2r\sin\theta\sin\varphi - 0)\vec{e}_\theta +$$
$$+\frac{1}{r}(2r\cos\theta - 0)\vec{e}_\varphi = \cot\theta\sin\varphi\,\vec{e}_r - 2\sin\varphi\,\vec{e}_\theta + 2\cos\theta\,\vec{e}_\varphi.$$

Example 8. Evaluate the flux of the vector field, defined in cylindrical coordinates as $\vec{a} = \rho\vec{e}_\rho + z\vec{e}_\varphi$, through the outward side of the outward side of the closed surface formed by the planes $z = 0$, $z = 1$ and the cylinder $\rho = 1$.

Solution. Method 1. The given surface consists of the parts of coordinate surfaces of the cylindrical coordinate system (fig. 5.6):

$$F = \oint\!\!\!\oint_{\Omega} \vec{a}\cdot\vec{n}\mathrm{d}S = \iint_{\Omega_\rho} \vec{a}\cdot\vec{n}\mathrm{d}S + \iint_{\Omega_1} \vec{a}\cdot\vec{n}\mathrm{d}S + \iint_{\Omega_2} \vec{a}\cdot\vec{n}\mathrm{d}S,$$

where

$$\begin{array}{llll} \Omega_\rho: \ \rho = 1, & \vec{n} = \vec{e}_\rho, & \vec{a}\cdot\vec{n} = \rho, & \mathrm{d}S = H_2H_3\mathrm{d}\varphi\mathrm{d}z = \rho\mathrm{d}\varphi\mathrm{d}z; \\ \Omega_1: \ z = 0, & \vec{n} = -\vec{e}_z, & \vec{a}\cdot\vec{n} = 0, & \mathrm{d}S = H_1H_2\mathrm{d}\rho\mathrm{d}\varphi = \rho\mathrm{d}\rho\mathrm{d}\varphi; \\ \Omega_2: \ z = 1, & \vec{n} = \vec{e}_z, & \vec{a}\cdot\vec{n} = 0, & \mathrm{d}S = H_1H_2\mathrm{d}\rho\mathrm{d}\varphi = \rho\mathrm{d}\rho\mathrm{d}\varphi. \end{array}$$

Thus we are to evaluate only the flux through Ω_ρ. Taking into account that for Ω_1 $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant z \leqslant 1$, we obtain

$$F = \iint_{\Omega_\rho} \vec{a}\cdot\vec{n}\mathrm{d}S = \int_0^{2\pi}\mathrm{d}\varphi\int_0^1 \rho^2\big|_{\rho=1}\mathrm{d}z = 2\pi.$$

Method 2. Since the given surface is closed, we can use Gauss – Ostrogradsky formula. To apply the formula we first find the divergence:

$$\operatorname{div}\vec{a} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2) + \frac{1}{\rho}\frac{\partial z}{\partial\varphi} = 2.$$

Thus we obtain

$$F = \oint\!\!\!\oint_{\Omega} \vec{a}\cdot\vec{n}\mathrm{d}S = \iiint_G \operatorname{div}\vec{a}\mathrm{d}V = 2\iiint_G \mathrm{d}V = 2V_\mathrm{C} = 2\pi,$$

where V_C is the cylinder volume equal to π.

`Method 3`. If we consider the cartesian coordinate system formed by coordinates (ρ, φ, z), then the initial cylinder takes the form of the parallelepiped shown in fig. 5.7. The total flux is the sum of the fluxes through the faces of this parallelepiped that can be easily evaluated as follows:

1) $\rho = 0, \quad \vec{n} = -\vec{e}_\rho, \quad \mathrm{d}S = \rho \mathrm{d}\varphi \mathrm{d}z = 0, \quad \vec{a}\cdot\vec{n} = -\rho = 0,$

$$F_0 = 0;$$

2) $\rho = 1, \quad \vec{n} = \vec{e}_\rho, \quad \mathrm{d}S = \mathrm{d}\varphi \mathrm{d}z, \quad \vec{a}\cdot\vec{n} = \rho = 1,$

$$F_1 = \int_0^1 \mathrm{d}z \int_0^2 \pi \mathrm{d}\varphi = 2\pi;$$

3) $\varphi = 0, \quad \vec{n} = -\vec{e}_\varphi, \quad \mathrm{d}S = \mathrm{d}\rho \mathrm{d}z, \quad \vec{a}\cdot\vec{n} = -z,$

$$F_2 = -\int_0^1 \mathrm{d}\rho \int_0^1 z \mathrm{d}z = -\frac{1}{2};$$

4) $\varphi = 2\pi, \quad \vec{n} = \vec{e}_\varphi, \quad \mathrm{d}S = \mathrm{d}\rho \mathrm{d}z, \quad \vec{a}\cdot\vec{n} = z,$

$$F_3 = \int_0^1 \mathrm{d}\rho \int_0^1 z \mathrm{d}z = \frac{1}{2};$$

5) $z = 0, \quad \vec{n} = -\vec{e}_z, \quad \mathrm{d}S = \rho \mathrm{d}\rho \mathrm{d}\varphi, \quad \vec{a}\cdot\vec{n} = 0,$

$$F_4 = 0;$$

6) $z = 1, \quad \vec{n} = \vec{e}_z, \quad \mathrm{d}S = \rho \mathrm{d}\rho \mathrm{d}\varphi, \quad \vec{a}\cdot\vec{n} = 0,$

$$F_5 = 0;$$

and finally $F = \sum_{i=0}^{5} F_i = 2\pi$.

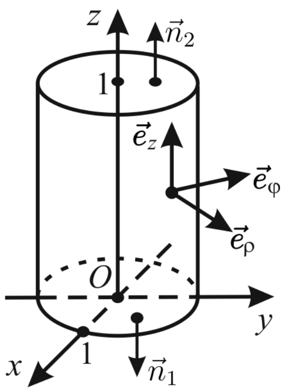

Fig. 5.6

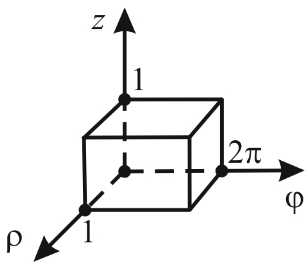

Fig. 5.7

Example 9. Evaluate the flux of the vector field, defined in the cylindrical coordinate system as $\vec{a} = \rho\vec{e}_\rho + \rho\varphi\vec{e}_\varphi - 2z\vec{e}_z$, through the outward side of the closed surface formed by the half-planes $\varphi = 0$, $\varphi = \dfrac{\pi}{2}$, the planes $z = -1$, $z = 1$, and the cylinder $\rho = 1$.

Solution. The given surface consist of the five smooth portions (fig. 5.8). Thus the flux is the sum

$$F = \sum_{i=1}^{5} \iint\limits_{\Omega_i} \vec{a} \cdot \vec{n}\,\mathrm{d}S,$$

where

1) $\Omega_1:\ \varphi = 0, \quad \vec{n} = -\vec{e}_\varphi,\ \ \vec{a}\cdot\vec{n} = -\rho\varphi = 0, \quad \mathrm{d}S = \mathrm{d}\rho\mathrm{d}z,$

$$F_1 = 0;$$

2) $\Omega_2:\ \varphi = \dfrac{\pi}{2}, \quad \vec{n} = \vec{e}_\varphi, \quad \vec{a}\cdot\vec{n} = \rho\varphi = \dfrac{\pi}{2}\rho, \quad \mathrm{d}S = \mathrm{d}\rho\mathrm{d}z,$

$$F_2 = \int\limits_{-1}^{1} \mathrm{d}z \int\limits_{0}^{1} \frac{\pi}{2}\rho\mathrm{d}\rho = \frac{\pi}{2};$$

3) $\Omega_3:\ z = 1, \quad \vec{n} = \vec{e}_z, \quad \vec{a}\cdot\vec{n} = -2z = -2, \quad \mathrm{d}S = \rho\mathrm{d}\rho\mathrm{d}\varphi,$

$$F_3 = -\int\limits_{0}^{\pi/2} \mathrm{d}\varphi \int\limits_{0}^{1} 2\rho\mathrm{d}\rho = -\frac{\pi}{2};$$

4) $\Omega_4:\ z = -1, \quad \vec{n} = -\vec{e}_z, \quad \vec{a}\cdot\vec{n} = 2z = -2, \quad \mathrm{d}S = \rho\mathrm{d}\rho\mathrm{d}\varphi,$

$$F_4 = -\int\limits_{0}^{\pi/2} \mathrm{d}\varphi \int\limits_{0}^{1} 2\rho\mathrm{d}\rho = -\frac{\pi}{2};$$

5) $\Omega_5:\ \rho = 1, \quad \vec{n} = \vec{e}_\rho, \quad \vec{a}\cdot\vec{n} = \rho = 1, \quad \mathrm{d}S = \rho\mathrm{d}\varphi\mathrm{d}z,$

$$F_5 = \int\limits_{0}^{\pi/2} \mathrm{d}\varphi \int\limits_{-1}^{1} \mathrm{d}z = \pi.$$

Finally the flux is

$$F = \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} + \pi = \frac{\pi}{2}.$$

Solve this problem by the second and third methods as in Example 8, using for the latter the fig. 5.9.

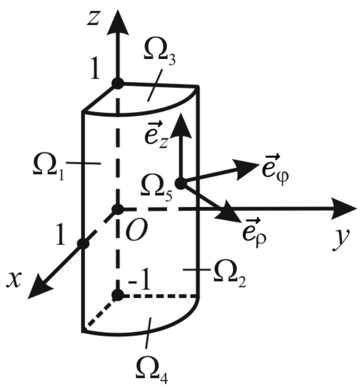

Fig. 5.8

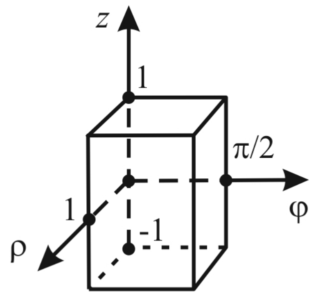

Fig. 5.9

Example 10. Evaluate the flux of the vector field, given in the spherical coordinates as $\vec{a} = r^2\vec{e}_r + R\,r\sin\theta\cos\varphi\,\vec{e}_\varphi$, through the outward side of the closed surface formed by the coordinate surfaces $r = R$, $\varphi = 0$, $\varphi = \dfrac{\pi}{2}$ and $\theta = \dfrac{\pi}{2}$ (fig. 5.10).

Solution. Consider the partial flux through each coordinate surface:

$$F = \sum_{i=1}^{4} \iint_{\Omega_i} \vec{a}\cdot\vec{n}\,\mathrm{d}S,$$

where

$$1)\quad \Omega_1:\quad \varphi = 0,\quad \vec{n} = -\vec{e}_\varphi,\quad \vec{a}\cdot\vec{n} = -Rr\sin\theta,\quad \mathrm{d}S = r\mathrm{d}r\mathrm{d}\theta,$$

$$F_1 = -\int_0^R r^2\mathrm{d}r\int_0^{\pi/2} R\sin\theta\mathrm{d}\theta = -\frac{R^4}{3};$$

$$2)\quad \Omega_2:\quad \varphi = \frac{\pi}{2},\quad \vec{n} = \vec{e}_\varphi,\quad \vec{a}\cdot\vec{n} = 0,\quad \mathrm{d}S = r\mathrm{d}r\mathrm{d}\theta,$$

$$F_2 = 0;$$

$$3)\quad \Omega_3:\quad \theta = \frac{\pi}{2},\quad \vec{n} = \vec{e}_\theta,\quad \vec{a}\cdot\vec{n} = 0,\quad \mathrm{d}S = r\mathrm{d}r\mathrm{d}\varphi,$$

$$F_3 = 0;$$

$$4)\quad \Omega_4:\quad r = R,\quad \vec{n} = \vec{e}_r,\quad \vec{a}\cdot\vec{n} = R^2,\quad \mathrm{d}S = R^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi;$$

$$F_4 = \int_0^{\pi/2}\mathrm{d}\varphi\int_0^{\pi/2} R^4\sin\theta\mathrm{d}\theta = \frac{\pi}{2}R^4.$$

Thus

$$F = -\frac{R^4}{3} + \frac{R^4\pi}{2} = R^4\left(\frac{\pi}{2} - \frac{1}{3}\right).$$

Example 11. Evaluate the circulation of the vector field, defined in the cylindrical coordinates as $\vec{a} = \rho \sin\varphi\, \vec{e}_\rho - \rho^2 z\, \vec{e}_\varphi + \rho^2 \vec{e}_z$, around the circle $\mathcal{L}:\ \rho = R,\ z = R$, positively oriented of the upper side of the plane.

Solution. M e t h o d 1 . The circulation C is

$$C = \oint\limits_{\mathcal{L}} \vec{a} \cdot \mathrm{d}\vec{r} = \oint\limits_{\mathcal{L}} \vec{a} \cdot \vec{\tau}\, \mathrm{d}l.$$

The closed curve $\mathcal{L}$ is the circle $\rho = R$, for which $\vec{\tau} = \vec{e}_\varphi$, (fig. 5.11), $\vec{a} \cdot \vec{\tau} = -\rho^2 z = -R^3$, $\mathrm{d}l = H_2 \mathrm{d}\varphi = R\mathrm{d}\varphi$. Thus,

$$C = -\int\limits_0^{2\pi} R^4 \mathrm{d}\varphi = -2\pi R^4.$$

M e t h o d 2. Applying Stokes' formula we find in first the curl

$$\operatorname{curl} \vec{a} = \begin{vmatrix} \frac{1}{\rho}\vec{e}_\rho & \vec{e}_\varphi & \frac{1}{\rho}\vec{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ \rho \sin\varphi & -\rho^3 z & \rho^2 \end{vmatrix} = -\rho^2 \vec{e}_\rho - 2\rho \vec{e}_\varphi + (-3\rho z - \cos\varphi)\vec{e}_z.$$

Then we chose the upper side of the plane $z = R$ bounded by the circle $\rho = R$ as the surface through which we will evaluate the flux. Then $\vec{n} = \vec{e}_z$, $\vec{n} \cdot \operatorname{curl} \vec{a} = -3\rho z - \cos\varphi$, $\mathrm{d}S = \rho \mathrm{d}\rho \mathrm{d}\varphi$ and

$$C = \oint\limits_{\mathcal{L}} \vec{a} \cdot \mathrm{d}\vec{r} = \iint\limits_{z=R, \rho \leqslant R} (-3\rho z - \cos\varphi)\mathrm{d}S =$$

$$= -\int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^R (3\rho R + \cos\varphi)\rho \mathrm{d}\rho = -2\pi R^4.$$

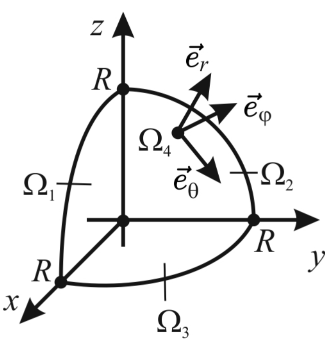

Fig. 5.10

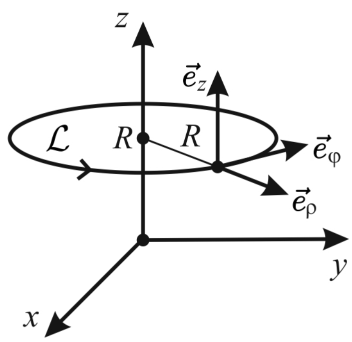

Fig. 5.11

Example 12. Evaluate the line integral of the second kind of the vector field, defined in the cylindrical coordinates as

$$\vec{a} = 2\rho\cos\varphi\vec{e}_\rho + z\vec{e}_\varphi + (3\rho + \varphi)\vec{e}_z,$$

along the straight line $\mathcal{L}: \ \varphi = \dfrac{\pi}{4}, \ z = 0$, from the point $A\left(0, \dfrac{\pi}{4}, 0\right)$ to $B\left(1, \dfrac{\pi}{4}, 0\right)$.

Solution. Taking into account that in the cylindrical coordinates $\mathrm{d}\vec{r} = \vec{e}_\rho \mathrm{d}\rho + \vec{e}_\varphi \rho \mathrm{d}\varphi + \vec{e}_z \mathrm{d}z$, the line integral takes the form

$$\int\limits_{\mathcal{L}} \vec{a}\cdot \mathrm{d}\vec{r} = \int\limits_{\mathcal{L}} 2\rho\cos\varphi \mathrm{d}\rho + \rho z \mathrm{d}\varphi + (3\rho + \varphi)\mathrm{d}z.$$

If we write the parametric equation of the line $\mathcal{L}$ as

$$\rho = \rho, \quad \varphi = \frac{\pi}{4}, \quad z = 0, \quad 0 \leqslant \rho \leqslant 1,$$

then $\mathrm{d}\rho = \mathrm{d}\rho, \ \mathrm{d}\varphi = 0, \ \mathrm{d}z = 0$, and

$$\int\limits_{\mathcal{L}} \vec{a}\cdot \mathrm{d}\vec{r} = \int\limits_0^1 2\rho\frac{\sqrt{2}}{2}\mathrm{d}\rho = \frac{\sqrt{2}}{2}.$$

Example 13. Evaluate the line integral of the second kind of the vector field, defined in the spherical coordinates as

$$\vec{a} = r\sin\theta\,\vec{e}_r + 6\theta^2\sin\varphi\,\vec{e}_\theta + e^r\varphi\,\vec{e}_\varphi,$$

along the circle $\mathcal{L}: \ r = 1, \ \varphi = \dfrac{\pi}{2}, 0 \leqslant \theta \leqslant \dfrac{\pi}{2}$, from the point $A\left(1, 0, \dfrac{\pi}{2}\right)$ to $B\left(1, \dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Solution. Since in the cylindrical coordinates $\mathrm{d}\vec{r} = \vec{e}_r \mathrm{d}r + +\vec{e}_\theta r\mathrm{d}\theta + \vec{e}_\varphi r\sin\theta \mathrm{d}\varphi$, then the line integral takes the form

$$\int\limits_{\mathcal{L}} \vec{a}\cdot \mathrm{d}\vec{r} = \int\limits_{\mathcal{L}} r\sin\theta \mathrm{d}r + 6r\theta^2\sin\varphi \mathrm{d}\theta + e^r\varphi\, r\sin\theta \mathrm{d}\varphi.$$

The curve $\mathcal{L}$ is the circle arc with the center in the origin and unit radius. We can write its parametric equation as

$$\theta = \theta, \quad r = 1, \quad \varphi = \frac{\pi}{2}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

Then $\mathrm{d}r = 0$, $\mathrm{d}\theta = \mathrm{d}\theta$, $\mathrm{d}\varphi = 0$, and

$$\int\limits_{\mathcal{L}} \vec{a}\cdot \mathrm{d}\vec{r} = \int\limits_{0}^{\pi/2} 6\theta^2 \mathrm{d}\theta = \left.\frac{6\theta^3}{3}\right|_0^{\frac{\pi}{2}} = 2\left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{4}.$$

Example 14. Ascertain whether the vector field $\vec{a} = \rho\vec{e}_\rho + \dfrac{\varphi}{\rho}\vec{e}_\varphi + z\vec{e}_z$ is potential and find its potential if possible.

Solution. To establish the potentiality in the domain $\rho > 0$ we evaluate the curl

$$\operatorname{curl}\vec{a} = \begin{vmatrix} \frac{1}{\rho}\vec{e}_\rho & \vec{e}_\varphi & \frac{1}{\rho}\vec{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ \rho & \varphi & z \end{vmatrix} = \vec{0}.$$

Method 1. To evaluate the potential we connect the point $M_0(0,0,0)$ with the point $M(\rho\cos\varphi, \rho\sin\varphi, z)$ (the given coordinates are Cartesian coordinates x, y, z) by the curve compose from to segments and the circle arc as shown in fig. 5.12. Image of the given curve in Cartesian coordinates (ρ, φ, z) is the curve, formed by segments parallel to the coordinate axes (fig. 5.13). Then

$$u = \int\limits_{\mathcal{L}} \vec{a}\cdot \mathrm{d}\vec{r} = \int\limits_{\mathcal{L}} \rho \mathrm{d}\rho + \varphi \mathrm{d}\varphi + z dz = \int\limits_0^{\rho} \tilde{\rho}\mathrm{d}\tilde{\rho} + \int\limits_0^{\varphi} \tilde{\varphi}\mathrm{d}\tilde{\varphi} + \int\limits_0^{z} \tilde{z}\mathrm{d}\tilde{z} =$$

$$= \frac{\rho^2}{2} + \frac{\varphi^2}{2} + \frac{z^2}{2} + C.$$

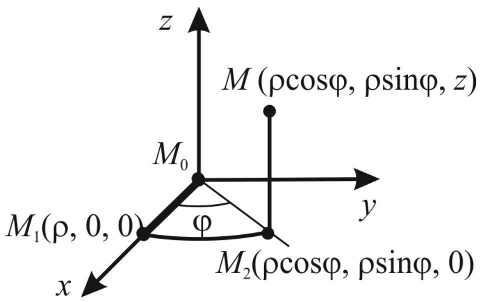

Fig. 5.12

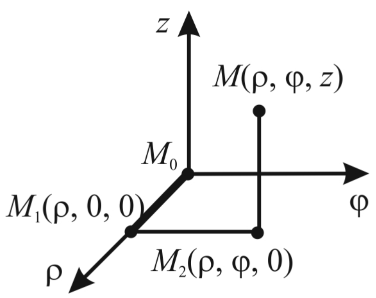

Fig. 5.13

M e t h o d 2. To find the potential u we compose the system of differential equations:

$$\frac{\partial u}{\partial \rho} = \rho, \quad \frac{1}{\rho}\frac{\partial u}{\partial \varphi} = \frac{\varphi}{\rho}, \quad \frac{\partial u}{\partial z} = z.$$

Integrating the first equation we obtain $u = \dfrac{\rho^2}{2} + C_1(\varphi, z)$. Substituting the given relation into the second equation, we have $\dfrac{\partial C_1}{\partial \varphi} = \varphi$. Thus $C_1(\varphi, z) = \dfrac{\varphi^2}{2} + C_2(z)$. Finally substituting the potential $u = \dfrac{\rho^2}{2} + \dfrac{\varphi^2}{2} + {}$ $+ C_2(z)$ into the last equation, we find that $C_2(z) = \dfrac{z^2}{2} + C$. Therefore the potential is

$$u = \frac{1}{2}(\rho^2 + \varphi^2 + z^2) + C.$$

Example 15. Ascertain whether the vector field

$$\vec{a} = e^r \sin\theta\, \vec{e}_r + \frac{1}{r} e^r \cos\theta\, \vec{e}_\theta + \frac{2\varphi}{(1+\varphi^2) r \sin\theta}\, \vec{e}_\varphi$$

is potential and find its potential if possible.

Solution. Computing the curl we make certain that $\operatorname{curl} \vec{a} = \vec{0}$, therefore the vector field is potential in the domain $r > 0$.

To find the potential u we compose the system of differential equations:

$$\frac{\partial u}{\partial r} = e^r \sin\theta, \quad \frac{\partial u}{\partial \theta} = e^r \cos\theta, \quad \frac{\partial u}{\partial \varphi} = \frac{2\varphi}{1+\varphi^2}.$$

Integrating each equation, we obtain

$$u = e^r \sin\theta + C_1(\theta, \varphi),$$

$$u = e^r \sin\theta + C_2(r, \varphi),$$

$$u = \ln(1+\varphi^2) + C_3(r, \theta).$$

Right parts of the equalities coincide only if

$$C_1(\theta, \varphi) = \ln(1+\varphi^2) + C, \quad C_2(r, \varphi) = \ln(1+\varphi^2) + C,$$

$$C_3(\rho, \varphi) = e^r \sin\theta + C.$$

Therefore

$$u = e^r \sin\theta + \ln\left(1+\varphi^2\right) + C.$$

Exercises

Find the vector lines of the given scalar field.

5.89. $\vec{a} = \vec{e}_\rho + \varphi\, \vec{e}_\varphi$.

5.90. $\vec{a} = \dfrac{2\cos\theta}{r^3}\vec{e}_r + \dfrac{\sin\theta}{r^3}\vec{e}_\theta$.

Find the gradient of the given vector field in the appropriate coordinate system.

5.91. $u = \rho + z\cos\varphi$.

5.92. $u = \rho^2 + 2\rho\cos\varphi - e^z\sin\varphi$.

5.93. $u = \rho\cos\varphi + z\sin^2\varphi$.

5.94. $u = \rho^2\sin\varphi + z\cos^2\varphi - 3z\rho$.

5.95. $u = r^2\cos\theta$.

5.96. $u = 3r\cos\theta + e^r\sin\varphi - 2r$.

5.97. $u = r^2\sin\theta + \dfrac{\cos\theta}{r}$.

5.98. $u = \dfrac{\cos\theta}{r^2}$.

Find the divergence and the curl of the given vector field in the appropriate coordinate system.

5.99. $\vec{a} = \rho^2\vec{e}_\rho + z\cos\varphi\,\vec{e}_\varphi + e^\varphi\sin z\,\vec{e}_z$.

5.100. $\vec{a} = \varphi\ arctg\rho\,\vec{e}_\rho + 2\vec{e}_\varphi - z^2e^z\vec{e}_z$.

5.101. $\vec{a} = \sin\varphi\,\vec{e}_\rho + \dfrac{\cos\varphi}{\rho}\vec{e}_\varphi - \rho z\vec{e}_z$.

5.102. $\vec{a} = \rho\,\vec{e}_\rho + \rho\,\varphi\,\vec{e}_\varphi - 2z\,\vec{e}_z$.

5.103. $\vec{a} = r^2\vec{e}_r - 2\cos^2\varphi\,\vec{e}_\theta + \dfrac{\varphi}{r^2+1}\vec{e}_\varphi$.

5.104. $\vec{a} = (2r + \cos\varphi)\vec{e}_r - \sin\theta\vec{e}_\theta + r\cos\theta\vec{e}_\varphi$.

5.105. $\vec{a} = r^2\vec{e}_r + 2\cos\theta\,\vec{e}_\theta - \varphi\,\vec{e}_\varphi$.

5.106. $\vec{a} = \dfrac{2\cos\theta}{r^3}\vec{e}_r + \dfrac{\sin\theta}{r^3}\vec{e}_\theta$.

5.107. Show that the vector field $\vec{a} = f(r)\vec{e}_r$ is potential. Here $f(r)$ is a continuously differentiable function.

Find the Laplacian of the given scalar field in the appropriate coordinate system.

5.108. $u = \rho^2 \cos 2\varphi$.

5.109. $u = \rho^2\varphi + z^2\varphi^3 - \rho\varphi z$.

5.110. $u = r \cos 2\,\theta$.

5.111. $u = r^2\varphi\,\theta + r^3\varphi^2 + \varphi + \theta^2$.

5.112. Evaluate the flux of the vector field $\vec{a} = \rho\,\vec{e}_\rho - z\vec{e}_\varphi + \cos\varphi\,\vec{e}_z$ through the outward side of the closed surface formed by the cylinder $\rho = 2$ and planes $z = 0$ and $z = 2$.

5.113. Evaluate the flux of the vector field $\vec{a} = \rho\,\vec{e}_\rho + \varphi\,\vec{e}_\varphi - z\,\vec{e}_z$ through the outward side of the closed surface formed by the cylinder $\rho = 1$, half-planes $\varphi = 0$, $\varphi = \dfrac{\pi}{2}$ and planes $z = 1$, $z = -1$.

5.114. Evaluate the flux of the vector field $\vec{a} = r^2\theta\,\vec{e}_r + re^\theta\,\vec{e}_\theta$, through the outward side of the upper hemisphere of radius R and center in the origin.

5.115. Evaluate the flux of the vector field $\vec{a} = \dfrac{2\cos\theta}{r^3}\vec{e}_r + \dfrac{\sin\theta}{r^3}\vec{e}_\theta$ through the outward side of the sphere of radius R and center in the origin

5.116. Evaluate the flux of the vector field $\vec{a} = r\,\vec{e}_r + r\sin\theta\,\vec{e}_\theta - 3r\varphi\sin\theta\,\vec{e}_\varphi$ through the outward side of the closed surface formed by upper hemisphere $r = R$ and plane $\theta = \dfrac{\pi}{2}$.

5.117. Evaluate the flux of the vector field $\vec{a} = r^2\,\vec{e}_r + 2\cos\theta\,\vec{e}_\theta - \varphi\,\vec{e}_\varphi$ through the outward side of the closed surface formed by the coordinates planes $r = R$, $\varphi = 0$, $\varphi = \dfrac{\pi}{2}$, $\theta = \dfrac{\pi}{2}$.

5.118. Evaluate the flux of the vector field $\vec{a} = r^2\,\vec{e}_r$ through the outward side of the closed surface formed by the upper hemisphere $r = R$ and plane $\theta = \dfrac{\pi}{2}$.

Evaluate the line integral of the second kind of the given vector field $\vec{a}$ along the given curve $\mathcal{L}$.

5.119. $\vec{a} = 2\rho\cos\varphi\,\vec{e}_\rho + \rho\varphi\,\vec{e}_\varphi + \rho\varphi\,\vec{e}_z$, $\mathcal{L}$: $\varphi = \dfrac{\pi}{4}$, $z = 0$ from the point $A\left(0, \dfrac{\pi}{4}, 0\right)$ to $B\left(2, \dfrac{\pi}{4}, 0\right)$.

5.120. $\vec{a} = \rho\sin\varphi\,\vec{e}_\rho + \alpha\cos\varphi\,\vec{e}_\varphi$, $\mathcal{L}$: $\rho = \alpha\varphi$, $z = 0$, $0 \leqslant \varphi \leqslant 2\pi$.

5.121. $\vec{a} = \ln\rho\sin\varphi\,\vec{e}_\rho + \rho^2\sin\varphi\,\vec{e}_\varphi + \rho^2\,\vec{e}_z$, $\mathcal{L}$: $\rho = R$, $z = \varphi$, $0 \leqslant \varphi \leqslant 2\pi$.

5.122. $\vec{a} = r\sin\theta\,\vec{e}_r + 3\theta^2\,\vec{e}_\theta + \varphi r\,\vec{e}_\varphi$, $\mathcal{L}$: $r = 1$, $\varphi = 0$, $0 \leqslant \theta \leqslant \pi$.

5.123. $\vec{a} = 3r^2\tan\dfrac{\varphi}{4}\,\vec{e}_r + \theta\varphi\,\vec{e}_\theta + \sin^2\varphi\,\vec{e}_\varphi$, $\mathcal{L}$: $\varphi = \pi$, $\theta = \dfrac{\pi}{4}$, $0 \leqslant r \leqslant 1$.

5.124. $\vec{a} = \sin^2\theta\,\vec{e}_r + \sin\theta\,\vec{e}_\theta + r^2\theta\,\vec{e}_\varphi$, $\mathcal{L}$: $r = \dfrac{1}{\sin\theta}$, $\varphi = \dfrac{\pi}{2}$, $\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{2}$.

Evaluate the circulation of the given vector field $\vec{a}$ around the closed curve $\mathcal{L}$ directly and applying Stokes' formula.

5.125. $\vec{a} = \rho\sin\varphi\,\vec{e}_\rho + 2\rho\varphi\,\vec{e}_\varphi + z\varphi\,\vec{e}_z$, $\mathcal{L}$: $\rho = 1$, $z = 0$, $\varphi = 0$, $\varphi = \dfrac{\pi}{2}$.

5.126. $\vec{a} = \rho\varphi\,\vec{e}_\rho + \rho z^2\cos\varphi\,\vec{e}_\varphi + \rho^2\cos z\,\vec{e}_z$, $\mathcal{L}$: $\rho = R$, $z = 1$.

5.127. $\vec{a} = \rho\sin\varphi\,\vec{e}_\rho + \rho\cos z\,\vec{e}_\varphi + \rho\varphi\,\vec{e}_z$, $\mathcal{L}$: $\rho = \sin\varphi$, $z = 0$, $0 \leqslant \varphi \leqslant \pi$.

5.128. $\vec{a} = e^r\sin\theta\,\vec{e}_r + r^2\sin\theta\;\vec{e}_\varphi$, $\mathcal{L}$: $r = 1$, $\theta = \dfrac{\pi}{4}$.

5.129. $\vec{a} = r\cos\varphi\sin\theta\,\vec{e}_r + (R + r)\,\vec{e}_\varphi$, $\mathcal{L}$: $r = R$, $\theta = \dfrac{\pi}{2}$.

5.130. $\vec{a} = r\sin\theta\,\vec{e}_r + e^\varphi\,\vec{e}_\theta$, $\mathcal{L}$: $r = \sin\theta$, $\varphi = 0$, $0 \leqslant \theta \leqslant \pi$.

Ascertain whether the given vector field is potential and find its potential if possible.

5.131. $\vec{a} = \vec{e}_\rho + \dfrac{1}{\rho}\vec{e}_\varphi + \vec{e}_z$.

5.132. $\vec{a} = \rho\vec{e}_\rho + \dfrac{z}{\rho}\vec{e}_\varphi + \varphi\,\vec{e}_z$.

5.133. $\vec{a} = \varphi z\vec{e}_\rho + z\vec{e}_\varphi + \rho\varphi\,\vec{e}_z$.

5.134. $\vec{a} = e^\rho\sin\varphi\,\vec{e}_\rho + \dfrac{e^\rho}{\rho}\cos\varphi\,\vec{e}_\varphi + 2z\vec{e}_z$.

5.135. $\vec{a} = \varphi\sin z\,\vec{e}_\rho + \sin z\,\vec{e}_\varphi + \rho\varphi\cos z\,\vec{e}_z$.

5.136. $\vec{a} = \left(\dfrac{\arctan z}{\rho} - \sin\varphi\right)\vec{e}_\rho - \cos\varphi\,\vec{e}_\varphi + \dfrac{\ln\rho}{1 + z^2}\vec{e}_z$.

5.137. $\vec{a} = \dfrac{\sin\varphi}{\rho}\vec{e}_\rho + \dfrac{1}{\rho}\ln\rho\cdot\cos\varphi\,\vec{e}_\varphi + 2z\,\vec{e}_z.$

5.138. $\vec{a} = \theta\vec{e}_r + \vec{e}_\theta.$

5.139. $\vec{a} = e^r\theta\,\vec{e}_r + \dfrac{e^r}{r}\vec{e}_\theta.$

5.140. $\vec{a} = \varphi\cos\theta\,\vec{e}_r - \varphi\sin\theta\,\vec{e}_\theta + \cot\theta\,\vec{e}_\varphi.$

5.141. $\vec{a} = 2r\vec{e}_r + \dfrac{1}{r}\vec{e}_\theta + \dfrac{1}{r\sin\theta}\vec{e}_\varphi.$

5.142. $\vec{a} = \dfrac{\varphi^2}{2}\vec{e}_r + \dfrac{\theta}{r}\vec{e}_\theta + \dfrac{\varphi}{\sin\theta}\vec{e}_\varphi.$

5.143. $\vec{a} = \cos\varphi\sin\theta\,\vec{e}_r + \cos\varphi\cos\theta\,\vec{e}_\theta - \sin\varphi\,\vec{e}_\varphi.$

5.144. $\vec{a} = e^r\sin\theta\vec{e}_r + \dfrac{e^r}{r}\cos\theta\vec{e}_\theta + \dfrac{2\varphi}{(1+\varphi^2)r\sin\theta}\vec{e}_\varphi.$

5.145. $\vec{a} = \dfrac{1}{r}e^{\theta\varphi}\vec{e}_r + \dfrac{\varphi\ln r}{r}e^{\theta\varphi}\vec{e}_\theta + \dfrac{\theta\ln r}{r\sin\theta}e^{\theta\varphi}\vec{e}_\varphi.$

5.146. Let $(\vec{e}_{q^1}, \vec{e}_{q^2}, \vec{e}_{q^3})$ be the physical basis of a curvilinear coordinate system. Find $\operatorname{div}\vec{e}_{q^i}, i = 1, 2, 3$. Write obtained results in the cylindrical and spherical coordinates.

5.147. Let $(\vec{e}_{q^1}, \vec{e}_{q^2}, \vec{e}_{q^3})$ be the physical basis of a curvilinear coordinate system. Prove that $\operatorname{curl}\vec{e}_{q^1} = \dfrac{1}{H_1}(\operatorname{grad} H_1 \times \vec{e}_{q^1})$. Derive analogous formulas for $\operatorname{curl}\vec{e}_{q^2}$ and $\operatorname{curl}\vec{e}_{q^3}$. Write obtained results in the cylindrical and spherical coordinates.

5.148. Find the general solution of the Laplace's equation $\Delta u = 0$ in the cylindrical coordinates that a) depends only on ρ; b) depends only on φ; c) depends only on z.

5.149. Find the general solution of the Laplace's equation $\Delta u = 0$ in the spherical coordinates that a) depends only on r; b) depends only on θ; c) depends only on φ.

5.150. Find the general solution of the Poissons' equation $\Delta u(r) = r^{n-1}$ in the spherical coordinate system.

5.151. Write the expression for $\Delta\Phi$ in the orthogonal coordinate systems described in Exercises 5.81–5.88.

Keys

Chapter 1

1.1. $a_1 = b_1{}^1c_1 + b_1{}^2c_2 + b_1{}^3c_3$, $a_2 = b_2{}^1c_1 + b_2{}^2c_2 + b_2{}^3c_3$, $a_3 = b_3{}^1c_1 + {} + b_3{}^2c_2 + b_3{}^3c_3$. **1.2.** $d = a_{11}b^1c^1 + a_{12}b^1c^2 + a_{21}b^2c^1 + a_{22}b^2c^2$. **1.3.** $d = {} = a^1{}_1 + a^2{}_2 + a^3{}_3$. **1.4.** $c^1{}_1{}^1 + c^1{}_2{}^2 + c^1{}_3{}^3 = a^1$, $c^2{}_1{}^1 + c^2{}_2{}^2 + c^2{}_3{}^3 = a^2$, $c^3{}_1{}^1 + {} + c^3{}_2{}^2 + c^3{}_3{}^3 = a^3$. **1.5.** $c^1{}_{11}a^1 + c^1{}_{12}a^2 = a^1b_1$, $c^2{}_{11}a^1 + c^2{}_{12}a^2 = a^2b_1$, $c^1{}_{21}a^1 + {} + c^1{}_{22}a^2 = a^1b_2$, $c^2{}_{21}a^1 + c^2{}_{22}a^2 = a^2b_2$. **1.6.** $a^k = b^k$. **1.7.** 0. **1.8.** $a^{kl} = {} = b^kc^l$. **1.9.** $(a^i{}_k - \delta^i{}_k)b^k$. **1.10.** $(b_ic^k - \delta_i{}^k)a_k{}^i$. **1.11.** $a^i{}_{rs}b^r(\delta_k{}^s - b^sc_k)$. **1.12.** $a^l{}_{jk}(\delta_l{}^ib^jb^k - c_lb^id^{jk})$. **1.14.** $\delta_m{}^i$. **1.15.** n. **1.16.** $\underset{\to}{e}^1 = \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$, $\underset{\to}{e}^2 = \left(-\frac{1}{3}, -\frac{5}{6}, -\frac{2}{3}\right)$, $\underset{\to}{e}^3 = (-1,-1,-1)$, $\vec{x} = -3\vec{e}_1 - 4\vec{e}_2 - 6\vec{e}_3$. **1.17.** $\underset{\to}{e}^1 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, $\underset{\to}{e}^2 = \left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}\right)$, $\underset{\to}{e}^3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$, $\vec{x} = 2\vec{e}_1 + \vec{e}_2$. **1.18.** $\underset{\to}{e}^1 = \left(\frac{1}{3}, -\frac{5}{3}, -\frac{2}{3}\right)$, $\underset{\to}{e}^2 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\underset{\to}{e}^3 = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$, $\vec{x} = -2\vec{e}_1 + {} + \vec{e}_2 + \vec{e}_3$. **1.19.** $\underset{\to}{e}^1 = (0,0,1)$, $\underset{\to}{e}^2 = (1,0,-1)$, $\underset{\to}{e}^3 = (0,1,-1)$, $\vec{x} = -\vec{e}_1 + 2\vec{e}_3$. **1.21.** $\underset{\to}{e}^1 = (3,-2)$, $\underset{\to}{e}^2 = (-4,3)$, $\vec{x} = \vec{e}_1 + \vec{e}_2$, $\underset{\to}{y} = 5\underset{\to}{e}^1 + 4\underset{\to}{e}^2$, $\langle\vec{x}, \underset{\to}{y}\rangle = 9$. **1.22.** $\underset{\to}{e}^1 = \left(-\frac{1}{3}, \frac{2}{3}\right)$, $\underset{\to}{e}^2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\vec{x} = \vec{e}_1 + \vec{e}_2$, $\underset{\to}{y} = -3\underset{\to}{e}^1 + 6\underset{\to}{e}^2$, $\langle\vec{x}, \underset{\to}{y}\rangle = 3$. **1.23.** $\underset{\to}{e}^1 = \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\underset{\to}{e}^2 = (-3,2)$, $\vec{x} = -\vec{e}_1 - 3\vec{e}_2$, $\underset{\to}{y} = {} = -8\underset{\to}{e}^1 + \underset{\to}{e}^2$, $\langle\vec{x}, \underset{\to}{y}\rangle = 5$. **1.24.** $\underset{\to}{e}^1 = \left(\frac{1}{2}, \frac{5}{2}\right)$, $\underset{\to}{e}^2 = (0,1)$, $\vec{x} = -4\vec{e}_1 - \vec{e}_2$, $\underset{\to}{y} = 2\underset{\to}{e}^1 - 8\underset{\to}{e}^2$, $\langle\vec{x}, \underset{\to}{y}\rangle = 0$. **1.27.** $(x^{i'}) = (-5,-2)$, $(f_{i'}) = (7,-19)$. **1.28.** $(x^{i'}) = (1,2)$, $(f_{i'}) = (6,-4)$. **1.29.** $(x^{i'}) = (-32,-24)$, $(f_{i'}) = {} = (-3,4)$. **1.30.** $(x^{i'}) = (11,9)$, $(f_{i'}) = (17,-18)$. **1.32.** $A^k{}_{k'} = \dfrac{\partial x^k}{\partial x^{k'}}$. **1.33.** $a^{i'j'k'} = A^{i'}{}_iA^{j'}{}_jA^{k'}{}_ka^{ijk}$, $a^{i'j'}{}_{k'} = A^{i'}{}_iA^{j'}{}_jA_{k'}{}^ka^{ij}{}_k$, $a^{i'}{}_{j'k'} = A^{i'}{}_iA_{j'}{}^j \times {} \times A_{k'}{}^ka^i{}_{jk}$, $a_{i'j'k'} = A_{i'}{}^iA_{j'}{}^jA_{k'}{}^ka_{ijk}$. **1.34.** a) $\dfrac{n(n+1)}{2}$; b) $\dfrac{n(n-1)}{2}$.

1.39. No. **1.40.** Consider specific tensors. **1.42.** $(a^i b^k) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$.
1.43. $(a^i b_k) = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$. **1.44.** $(a_i b_{jk}) = \left(\begin{array}{cc|cc} 12 & 20 & 20 & 36 \\ -6 & -10 & -10 & -18 \end{array}\right)$.

1.45. $(a_{ijk} b_l) = \left(\begin{array}{cc|cc} -3 & -4 & 3 & 4 \\ -5 & -7 & 5 & 7 \\ \hline -2 & -5 & 2 & 5 \\ -1 & -3 & 1 & 3 \end{array}\right)$.

1.46. $(a_i b_{jkl}) = \left(\begin{array}{cc|cc} -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ \hline -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{array}\right)$.

1.47. a) $(a_{ij}x^j) = (10, -21, -38)$; b) $(a_{ij}x^i) = (-6, 6, -24)$; c) $(a_{ij}x^i y^j) = 30$; d) $(a_{ij}x^j y^i) = -5$. **1.48.** a) $(a^i_j x^j) = (4, 6, 8)$; b) $(a^i_j y_i) = (5, 9, 2)$; c) $a^i_j x^j y_i = 14$; d) $a^i_i = 6$. **1.49.** a) no; b) yes; c) no. **1.50.** $a^i{}_i, b^j{}_j, a^i{}_j b^j{}_i$.
1.51. Tensor of valence 5: $a_{ij}b^{klm}$, tensors of valence 3: $a_{ij}b^{ilm}$, $a_{ij}b^{kim}$, $a_{ij}b^{kli}$, $a_{ij}b^{jlm}$, $a_{ij}b^{kjm}$, $a_{ij}b^{klj}$, tensors of valence 1: $a_{ij}b^{ijm}$, $a_{ij}b^{jim}$, $a_{ij}b^{ilj}$, $a_{ij}b^{jli}$, $a_{ij}b^{kij}$, $a_{ij}b^{kji}$. **1.52.** a) $(a^{ij}_i) = (5, 13)$; $(a^{ij}_j) = (7, 11)$; b) $(a^{ij}_i) = (3, 0)$; $(a^{ij}_j) = (5, 3)$. **1.53.** a) $\begin{pmatrix} -4 & 4 \\ -7 & 7 \end{pmatrix}$; b) $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$; c) $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$; d) $\begin{pmatrix} -8 & 8 \\ -8 & 8 \end{pmatrix}$; e) 3; f) 0. **1.54.** a) $n!$; b) $\left(\begin{array}{cc|cc} 1 & 5 & 3 & 7 \\ 2 & 6 & 4 & 8 \end{array}\right)$; $\left(\begin{array}{cc|cc} 1 & 3 & 2 & 4 \\ 5 & 7 & 6 & 8 \end{array}\right)$; $\left(\begin{array}{cc|cc} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{array}\right)$; $\left(\begin{array}{cc|cc} 1 & 5 & 2 & 6 \\ 3 & 7 & 4 & 8 \end{array}\right)$; $\left(\begin{array}{cc|cc} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{array}\right)$. **1.55.** $\begin{pmatrix} x^1y^1 & x^1y^2 \\ x^2y^1 & x^2y^2 \end{pmatrix}$.

1.56. $\begin{pmatrix} x^1y^1 & \frac{1}{2}(x^1y^2 + x^2y^1) \\ \frac{1}{2}(x^1y^2 + x^2y^1) & x^2y^2 \end{pmatrix}$.

1.57. $\begin{pmatrix} 0 & \frac{1}{2}(x^1y^2 - x^2y^1) \\ \frac{1}{2}(x^2y^1 - x^1y^2) & 0 \end{pmatrix}$.

1.58. $\left(\begin{array}{cc|cc} x^1a_{11} & x^1a_{21} & x^1a_{12} & x^1a_{22} \\ x^2a_{11} & x^2a_{21} & x^2a_{12} & x^2a_{22} \end{array}\right)$. **1.59.** $(x^1a_{11}+x^2a_{21}, x^1a_{12}+x^2a_{22})$.
1.60. $\big(x^1(a^1{}_1 + a^2{}_2), x^2(a^1{}_1 + a^2{}_2)\big)$. **1.61.** $\big(x^1a^1{}_1 + \frac{1}{2}(x^1a^2{}_2 + x^2a^1{}_2)$, $x^2a^2{}_2 + \frac{1}{2}(x^2a^1{}_1 + x^1a^2{}_1)\big)$. **1.62.** $\left(\frac{1}{2}(x^1a^2{}_2 - x^2a^1{}_2), \frac{1}{2}(x^2a^1{}_1 - x^1a^2{}_1)\right)$.

1.63. $(a^1{}_1 + a^2{}_2)^2$. **1.64.** $(a^1{}_1)^2 + (a^2{}_2)^2 + a^1{}_1 a^2{}_2 + a^1{}_2 a^2{}_1$. **1.65.** $a^1{}_1 a^2{}_2 - a^1{}_2 a^2{}_1$. **1.66.** $\begin{pmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & a^2{}_2 \end{pmatrix}$. **1.67.** $\begin{pmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & a^2{}_2 \end{pmatrix}$. **1.68.** $a^1{}_1 + a^2{}_2$.

1.69. $\left(\begin{array}{cc|cc} a^1{}_1 & 0 & a^1{}_2 & 0 \\ 0 & a^1{}_1 & 0 & a^1{}_2 \\ \hline a^2{}_1 & 0 & a^2{}_2 & 0 \\ 0 & a^2{}_1 & 0 & a^2{}_2 \end{array}\right)$. **1.70.** $\left(\begin{array}{cc|cc} a^1{}_1 & a^1{}_2 & 0 & 0 \\ a^2{}_1 & a^2{}_2 & 0 & 0 \\ \hline 0 & 0 & a^1{}_1 & a^1{}_2 \\ 0 & 0 & a^2{}_1 & a^2{}_2 \end{array}\right)$.

1.71. $(a_{(ij)}) = \begin{pmatrix} 1 & 3 & 6 \\ 3 & 0 & 2 \\ 6 & 2 & 6 \end{pmatrix}$, $(a_{[ij]}) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$.

1.72. $(a_{(ij)}) = \begin{pmatrix} 4 & 3 & 1 \\ 3 & 3 & \frac{7}{2} \\ 1 & \frac{7}{2} & 3 \end{pmatrix}$, $(a_{[ij]}) = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -\frac{5}{2} \\ -1 & \frac{5}{2} & 0 \end{pmatrix}$.

1.73. $(a_{(ij)}) = \begin{pmatrix} 1 & 2 & \frac{1}{2} \\ 2 & 3 & \frac{5}{2} \\ \frac{1}{2} & \frac{5}{2} & 1 \end{pmatrix}$, $(a_{[ij]}) = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{5}{2} \\ \frac{1}{2} & -\frac{5}{2} & 0 \end{pmatrix}$.

1.74. $(a_{(ij)}) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & -1 \\ 1 & -1 & 6 \end{pmatrix}$, $(a_{[ij]}) = \begin{pmatrix} 0 & -3 & 4 \\ 3 & 0 & -3 \\ -4 & 3 & 0 \end{pmatrix}$.

1.75. a) $\left(\begin{array}{cc|cc} 3 & \frac{9}{2} & 2 & 3 \\ \frac{9}{2} & 7 & 3 & 3 \end{array}\right)$; b) $\left(\begin{array}{cc|cc} 3 & 4 & \frac{7}{2} & 6 \\ \frac{7}{2} & 6 & 1 & 3 \end{array}\right)$.

1.76. $(a^{[ij]}{}_{kl}) = \left(\begin{array}{cc|cc} 0 & 3 & 0 & 3 \\ -3 & 0 & -3 & 0 \\ \hline 0 & -4 & 0 & -4 \\ 4 & 0 & 4 & 0 \end{array}\right)$;

$(a^{ij}{}_{[kl]}) = \left(\begin{array}{cc|cc} 0 & 0 & \frac{7}{2} & \frac{7}{2} \\ 0 & 0 & -\frac{7}{2} & -\frac{7}{2} \\ \hline -\frac{7}{2} & -\frac{7}{2} & 0 & 0 \\ \frac{7}{2} & \frac{7}{2} & 0 & 0 \end{array}\right)$; $(a^{[ij]}{}_{[kl]}) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & \frac{7}{2} \\ 0 & 0 & -\frac{7}{2} & 0 \\ \hline 0 & -\frac{7}{2} & 0 & 0 \\ \frac{7}{2} & 0 & 0 & 0 \end{array}\right)$.

1.77. $(a_{(ijk)}) = \left(\begin{array}{ccc|ccc|ccc} 1 & \frac{10}{3} & \frac{19}{3} & \frac{10}{3} & 3 & \frac{19}{3} & \frac{19}{3} & \frac{19}{3} & \frac{19}{3} \\ \frac{10}{3} & 3 & \frac{19}{3} & 3 & 5 & 7 & \frac{19}{3} & 7 & 4 \\ \frac{19}{3} & \frac{19}{3} & \frac{19}{3} & \frac{19}{3} & 7 & 4 & \frac{19}{3} & 4 & 1 \end{array}\right),$

$(a_{[ijk]}) = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$

1.78. $(a_{(ijk)}) = \left(\begin{array}{ccc|ccc|ccc} 2 & \frac{14}{3} & \frac{16}{3} & \frac{14}{3} & \frac{10}{3} & 6 & \frac{16}{3} & 6 & \frac{8}{3} \\ \frac{14}{3} & \frac{10}{3} & 6 & \frac{10}{3} & 6 & 4 & 6 & 4 & \frac{10}{3} \\ \frac{16}{3} & 6 & \frac{8}{3} & 6 & 4 & \frac{10}{3} & \frac{8}{3} & \frac{10}{3} & 6 \end{array}\right),$

$(a_{[ijk]}) = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$

1.79. The tensor is antisymmetric with respect to all indices. **1.80.** The tensor is antisymmetric with respect to the first and the third indices. **1.81.** The tensor is antisymmetric with respect to the first and the second indices. **1.82.** $a^k{}_k = 6$, $a^i{}_{[i}a^k{}_{k]} = 1$, $a^i{}_{[i}a^j{}_j a^k{}_{k]} = 0$. **1.83.** $a^k{}_k = 11$, $a^i{}_{[i}a^k{}_{k]} = 27$, $a^i{}_{[i}a^j{}_j a^k{}_{k]} = 1$. **1.102.** a) (2,0); b) no; c) no; d) no; e) (1,1); f) no; g) (2,0); h) (0,2). **1.103.** a) $\begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$; b) $\begin{pmatrix} -1 & 1 \\ -5 & 7 \end{pmatrix}$; c) $\begin{pmatrix} -1 & 27 \\ 1 & -17 \end{pmatrix}$; d) $\begin{pmatrix} -164 & 102 \\ 248 & -156 \end{pmatrix}$. **1.104.** $(x^i y_k) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $(x^{i'} y_{k'}) = \begin{pmatrix} 42 & 105 & -42 \\ -20 & 50 & -20 \\ -8 & -20 & 8 \end{pmatrix}$. **1.105.** $a = (\vec{e}_1 - 2\vec{e}_2) \otimes (2\vec{e}_1 - \vec{e}_2)$. **1.106.** $a = (\vec{e}_1 + \vec{e}_2) \otimes (\vec{e}_1 - \vec{e}_2) \otimes (\underset{\to}{e}^1 + 2\underset{\to}{e}^2)$. **1.107.** Euclidean, $\vec{x} \cdot \vec{y} = -1$, $\|\vec{x}\|^2 = 2$, $\|\vec{y}\|^2 = 5$, $(w_{ij}) = \begin{pmatrix} 0 & \frac{3}{2} \\ -\frac{3}{2} & 0 \end{pmatrix}$, $(v^i{}_k) = \begin{pmatrix} -5 & 6 \\ -1 & 3 \end{pmatrix}$.

1.108. Pseudo–Euclidean, $\vec{x}\cdot\vec{y} = -1$, $\|\vec{x}\|^2 = -3$, $\|\vec{y}\|^2 = 0$, $(w_{ij}) =$ $= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $(v^i{}_k) = \begin{pmatrix} \frac{7}{2} & -\frac{27}{2} \\ \frac{3}{2} & -\frac{11}{2} \end{pmatrix}$. **1.109.** Euclidean, $\vec{x}\cdot\vec{y} = 1$, $\|\vec{x}\|^2 = 1$, $\|\vec{y}\|^2 = 3$, $(w_{ij}) = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $(v^i{}_k) = \begin{pmatrix} -1 & -1 & 2 \\ 1 & 1 & -2 \\ -1 & -2 & 2 \end{pmatrix}$.

1.110. Pseudo–Euclidean, $\vec{x}\cdot\vec{y} = 0$, $\|\vec{x}\|^2 = 2$, $\|\vec{y}\|^2 = 0$,

$$(w_{ij}) = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}, (v^i{}_k) = \begin{pmatrix} 2 & 2 & -1 & -1 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

1.111. $(a^i{}_j) = \begin{pmatrix} 60 & -34 \\ -37 & 21 \end{pmatrix}$, $(a_i{}^j) = \begin{pmatrix} 60 & -37 \\ -34 & 21 \end{pmatrix}$, $(a^{ij}) = \begin{pmatrix} 402 & -248 \\ -248 & 153 \end{pmatrix}$. **1.112.** $(a^i{}_j) = \begin{pmatrix} -2 & 6 \\ 1 & -2 \end{pmatrix}$, $(a_i{}^j) = \begin{pmatrix} 3 & -1 \\ 19 & -7 \end{pmatrix}$, $(a^{ij}) = \begin{pmatrix} -56 & 22 \\ 23 & -9 \end{pmatrix}$. **1.113.** $(a^i{}_j) = \begin{pmatrix} 4 & 7 & 13 \\ 4 & 7 & 17 \\ 11 & 19 & 25 \end{pmatrix}$, $(a_i{}^j) = \begin{pmatrix} 2 & -1 & 13 \\ 8 & 9 & 19 \\ 13 & 17 & 25 \end{pmatrix}$, $(a^{ij}) = \begin{pmatrix} 18 & 17 & 51 \\ 18 & 9 & 71 \\ 49 & 67 & 87 \end{pmatrix}$.

1.114. $$(a_{ijkl}) = \left(\begin{array}{cc|cc} 39 & 101 & 39 & 101 \\ 100 & 259 & 100 & 259 \\ \hline 39 & 101 & 78 & 202 \\ 100 & 259 & 200 & 518 \end{array}\right),$$

$$(a^{ijkl}) = \left(\begin{array}{cc|cc} 89 & 89 & -34 & -34 \\ 0 & 89 & 0 & -34 \\ \hline -34 & -34 & 13 & 13 \\ 0 & -34 & 0 & 13 \end{array}\right).$$

1.115. $(a_{ijkl}) = \left(\begin{array}{cc|cc} -4 & -5 & 4 & 7 \\ -11 & -15 & 1 & 3 \\ \hline 4 & 7 & 12 & 19 \\ 15 & 24 & 27 & 42 \end{array}\right)$,

$(a^{ijkl}) = \left(\begin{array}{cc|cc} 4 & 5 & 2 & 2 \\ -15 & -14 & 2 & 2 \\ \hline -5 & -5 & 0 & 0 \\ 9 & 9 & 0 & 0 \end{array}\right)$. **1.116.** $g^i{}_k = \delta^i{}_k$. **1.119.** $\underset{\to}{e}^1 = -\vec{\imath} - \vec{\jmath}$, $\underset{\to}{e}^2 = -2\vec{\imath} - \vec{\jmath}$. **1.120.** $\underset{\to}{e}^1 = -\frac{3}{5}\vec{\imath} + \frac{2}{5}\vec{\jmath}$, $\underset{\to}{e}^2 = \frac{1}{5}\vec{\imath} + \frac{1}{5}\vec{\jmath}$. **1.121.** $\underset{\to}{e}^1 = -\frac{1}{4}\vec{\jmath} + \frac{1}{4}\vec{k}$, $\underset{\to}{e}^2 = -\frac{1}{3}\vec{\imath} - \frac{1}{4}\vec{\jmath} + \frac{7}{12}\vec{k}$, $\underset{\to}{e}^3 = \frac{2}{3}\vec{\imath} + \frac{1}{2}\vec{\jmath} - \frac{1}{6}\vec{k}$. **1.122**. $\underset{\to}{e}^1 = -\frac{3}{5}\vec{\imath} + \frac{1}{5}\vec{\jmath} - 2\vec{k}$, $\underset{\to}{e}^2 = -\frac{1}{5}\vec{\imath} + \frac{2}{5}\vec{\jmath} - \vec{k}$, $\underset{\to}{e}^3 = -\frac{4}{5}\vec{\imath} + \frac{3}{5}\vec{\jmath} - 2\vec{k}$. **1.123.** $\begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix}$.
1.129. $\vec{b}(\vec{a}\cdot\vec{c}) - \vec{a}(\vec{b}\cdot\vec{c})$. **1.130.** $(\vec{a}\times\vec{b})^2 = \vec{a}^{\,2}\vec{b}^{\,2} - (\vec{a}\cdot\vec{b})^2$. **1.135.** a) $\vec{a}^2(\vec{b}\cdot\vec{c}) + (\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{c})$; b) $[(\vec{a}\times\vec{b})\times\vec{c}]\cdot[(\vec{a}'\times\vec{b}')\times\vec{c}']$. **1.136.** Rotation by the angle $\frac{\pi}{6}$ around the x^3 axis corresponds to the matrix $A_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and rotation by the angle $\frac{\pi}{2}$ around the $x^{1'}$ axis in such a way that the axis $x^{2'}$ coincides with the axis x^3 corresponds to the matrix $A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Then the transformation matrix is $A = A_2A_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}$. Inverse matrix $A^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \end{pmatrix}$. **1.137.** The change of basis can

be realized by three consecutive rotations: around the z axis by an angle φ — $A_1 = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$; then around the $\vec{u}$ by an angle θ — $A_2 =$ $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$; and finally around the z' axis by an angle ψ — $A_3 = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The transformation matrix is $A = A_3A_2A_1 =$

$$= \begin{pmatrix} \cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi & \cos\psi\sin\varphi + \cos\theta\sin\psi\cos\varphi & \sin\theta\sin\psi \\ -\sin\psi\cos\varphi - \cos\theta\cos\psi\sin\varphi & -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi & \sin\theta\cos\psi \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix}.$$

Chapter 2

2.1. $\vec{i} + \vec{k}$. **2.2.** $-\vec{i} + \vec{j}$. **2.3.** $e\vec{i} - \vec{j}$. **2.4.** $\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} + \vec{k}$. **2.5.** $-a\sin t\vec{i} + b\cos t\vec{j} + c\vec{k}$. **2.6.** $a\sinh t\vec{i} + b\cosh t\vec{j} + 2ct\vec{k}$. **2.7.** $e^t(\cos t - \sin t)\vec{i} + e^t(\sin t + \cos t)\vec{j} + e^t\vec{k}$. **2.8.** $(1-\cos t)\vec{i} + \sin t\,\vec{j} + 2\cos\frac{t}{2}\vec{k}$. **2.9.** $(t-1)e^t\vec{i} + \frac{2t - \sin 2t}{4}\vec{j} - \arctan t\vec{k} + \vec{c}$. **2.10.** $\frac{1}{2}\ln(1+t^2)\vec{i} + \frac{1}{2}e^{t^2}\vec{j} + \sin t\vec{k} + \vec{c}$. **2.11.** $\left(2 - \frac{3}{\sqrt{e}}\right)\vec{i} + (\sqrt{e} - 1)\vec{j} + (e-1)\vec{k}$. **2.12.** $\pi^2(\vec{i} + \vec{j} + \vec{k})$. **2.13.** The straight line $\frac{x-2}{0} = \frac{y}{1} = \frac{z}{-1}$. **2.14.** The straight line $x+y=1$. **2.15.** The circle $x^2+y^2=1,\ z=1$. **2.16.** $y = \frac{x^2}{3},\ z = \frac{x^3}{9}$. **2.17.** $2\vec{r}\cdot\vec{r}'$. **2.18.** $2\vec{r}'\cdot\vec{r}''$. **2.19.** $\vec{r}' \times \vec{r}'''$. **2.20.** $\vec{r}'\vec{r}''\vec{r}^{(4)}$. **2.21.** $(\vec{r}' \times \vec{r}''') \times \vec{r}''' + (\vec{r}' \times \vec{r}'') \times \vec{r}^{(4)}$. **2.22.** $\frac{\vec{r}\cdot\vec{r}'}{\sqrt{\vec{r}^2}}$. **2.23.** $\vec{r}'^2(\vec{r}' \times \vec{a})^2,\ -(\vec{r}'\cdot\vec{a})(\vec{r}' \times \vec{a})^2$. **2.26.** $\vec{v}^2 = \rho^2 + \rho'^2 + z'^2$. **2.27.** $\vec{v}^2 = r'^2 + r^2\theta'^2 + r^2\sin^2\theta\varphi'^2$. **2.28.** $\vec{r}'' = \vec{r}_0'' + \vec{\omega}' \times \vec{\rho} + 2\vec{\omega} \times \vec{v}_r + \vec{\omega} \times (\vec{\omega} \times \vec{\rho})$, where $\vec{v}_r = x'(t)\vec{e}_x(t) + y'(t)\vec{e}_y(t) + z'(t)\vec{e}_z(t)$ is a velocity with respect to the the moving frame of reference. **2.29.** If we introduce cartesian coordinates so that $\vec{e} = \vec{k}$, then $x = c_1e^t$, $y = c_2e^t$, $z = c_3$, where c_1, c_2, c_3 are arbitrary constants. **2.30.** If we introduce cartesian coordinates so that $\vec{e} = \vec{k}$, then $a\vec{e} + \vec{e} \times \vec{r} = -y\vec{i} + x\vec{j} + a\vec{k}$ and the differential equation is $x' = -y,\ y' = x,\ z' = a$. From the equations $x' = -y$, $y' = x$

we obtain that $x^2 + y^2 = c_1$. Then we find: $\dfrac{\mathrm{d}x}{\mathrm{d}z} = -\dfrac{y}{a}, \dfrac{\mathrm{d}y}{\mathrm{d}z} = \dfrac{x}{a}$, hence $\mathrm{d}z = \dfrac{a}{1+\frac{y^2}{x^2}}\mathrm{d}\Big(\dfrac{y}{x}\Big)$ and $\tan(z+c_2) = a\dfrac{y}{x}$. At last $z = at + c_3$. Finally it is easy to express coordinates x, y, z as functions of t from obtained relations. **2.31.** The circles that contact at the origin the axis Oz parallel to the vector $\vec{e}$. **2.32.** The helix whose axis is parallel to the vector $\vec{a}$. **2.33.** a) The equality is true only if $\vec{r}\,' = \lambda\vec{r}, \lambda \geqslant 0$; ?satisfies $\forall\vec{r}$. **2.35.** $\vec{r}\,' = \dot{\vec{r}}\,l'$, $\vec{r}\,'' = \ddot{\vec{r}}\,l'^2 + \dot{\vec{r}}\,l''$, $\vec{r}\,''' = \dddot{\vec{r}}\,l'^3 + 3\ddot{\vec{r}}\,l'l'' + \dot{\vec{r}}\,l'''$, $\vec{r}\,'^2 = \dot{\vec{r}}^{\,2}\,l'^2$, $\vec{r}\,' \times \vec{r}\,'' = (\dot{\vec{r}} \times \ddot{\vec{r}})l'^3$, $\vec{r}\,'\vec{r}\,''\vec{r}\,''' = (\dot{\vec{r}}\ddot{\vec{r}}\dddot{\vec{r}})l'^{\,6}$. **2.37.** The inverse proposition is not always true, for example for the function $\vec{r}(t) = \begin{cases} -\cos t\vec{i} + \sin t\vec{j}, & \text{if } t \leqslant 0; \\ \cos t\vec{i} + \sin t\vec{j}, & \text{if } t > 0. \end{cases}$ **2.38.** True. **2.39.** True. **2.40.** The necessity is obvious. To prove the sufficiency we are to represent the function $\vec{r}(t)$ in the form $\vec{r}(t) = \varphi(t)\vec{e}(t)$, $|\vec{e}(t)| = 1$ and show that $\vec{e}\,'(t) = \vec{0}$. **2.41.** The given condition is not sufficient. For example, for the function $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $-\infty < t < +\infty$, where $x(t) = \begin{cases} e^{1/t}, & \text{if } t < 0, \\ 0, & \text{if } t \geqslant 0; \end{cases}$ $y(t) = t$; $z(t) = \begin{cases} e^{-1/t}, & \text{if } t > 0, \\ 0, & \text{if } t \leqslant 0, \end{cases}$ the condition $(\vec{r} \times \vec{r}\,') \cdot \vec{r}\,'' = 0$ is true but for $t \in (-\infty, 0)$ vectors $\vec{r}(t)$ are parallel to the plane xOy, and for $t \in (0, +\infty)$ vectors $\vec{r}(t)$ are parallel to the plane yOz. If for each $t \in (a, b)$ $\vec{r} \times \vec{r}\,' \neq 0$, then the formulated condition is sufficient. Prove that. **2.42.** $x = a\cos t$, $y = a\sin t$, $z = bt$. The quantity a is called the radius of a helix and $2\pi b$ is called a pitch. Projections: 1) $x^2 + y^2 = a^2$; 2) $y = a\sin(z/b)$; 3) $x = a\cos(z/b)$. **2.43.** $x = a\cos t$, $y = a\sin t$, $z = be^{kt}$. **2.44.** $x = at\cos t$, $y = at\sin t$, $z = bt$. **2.45.** $x = ae^{kt}\cos t$, $y = ae^{kt}\sin t$, $z = be^{kt}$. **2.46.** $x = at - d\sin t$, $y = a - d\cos t$. **2.47.** $x = (r+R)\cos\dfrac{rt}{R} - r\cos\dfrac{(r+R)t}{R}$, $y = (r+R)\sin\dfrac{rt}{R} - r\sin\dfrac{(r+R)t}{R}$. **2.48.** $x = (R-r)\cos\dfrac{rt}{R} + r\cos\dfrac{(R-r)t}{R}$, $y = (R-r) \times \sin\dfrac{rt}{R} - r\sin\dfrac{(R-r)t}{R}$. **2.49.** $x = a\cos t$, $y = \pm\sqrt{b^2 - a^2\sin^2 t}$, $z = a\sin t$. If $a = b$, the curve consists of two separated ellipses. **2.50.** $x = t$, $y = \pm\sqrt{2at - t^2}$, $z = \pm\sqrt{4a^2 - 2at}$, $t \in [0, 2a]$, or $x = 2a\cos u$, $y = 2a\cos u\sin u$, $z = \pm 2a\sin u$, $u \in [0, 2\pi]$. **2.51.** $x = 1 + 3t$, $y = -2 - t$, $t \in [0, 1]$. **2.52.** $x = t$, $y = 2t^2$, $t \in [-1, 2]$. **2.53.** $x = t$, $y = \sqrt{3 - t^3 - 2t^2}$, $t \in (-\infty, 1]$. **2.54.** $x = e^{t - \sin t}$, $y = t$, $t \in (-\infty, +\infty)$. **2.55.** $x = a\cos t$, $y = b\sin t$, $t \in [0, 2\pi]$. **2.56.** Right branch: $x = a\cosh t$, $y = b\sinh t$, $t \in (-\infty, +\infty)$, left branch $x = -a\cosh t$,

$y = b\sinh t,\ t \in (-\infty, +\infty)$. **2.57.** $x = a\cos^4 t,\ y = b\sin^4 t,\ t \in [0, \pi/2]$. **2.58.** $x = a\cos^3 t,\ y = a\sin^3 t,\ t \in [0, 2\pi]$. **2.59.** $x = \frac{a}{2}(\cosh^3 t + \sinh^3 t), y = \frac{a}{2}(\cosh^3 t - \sinh^3 t),\ t \in (-\infty, +\infty)$. **2.60.** $x = \frac{\sqrt{t}}{1+t}$, $y = \frac{-t\sqrt{t}}{1+t},\ t \geqslant 0$. **2.61.** $x = t^2 + t,\ y = t^2 - t,\ t \in (-\infty, +\infty)$. **2.62.** $x = a\cos t,\ y = a(\cos t + \sin t),\ t \in [0, 2\pi]$. **2.63.** $x = a(t^2 + t^3), y = a(t^3 + t^4),\ t \in (-\infty, +\infty)$. **2.64.** $x = a\cos t,\ y = a\cos t\,\mathrm{sgn}(\sin t)\sqrt{|\sin t|}$, $t \in [0, 2\pi]$. **2.65.** $x = (a^2\cos^{4/3} t + b^2\sin^{4/3} t)^{1/2}\cos^{1/3} t,\ y = (a^2\cos^{4/3} t + b^2\sin^{4/3} t)^{1/2}\sin^{1/3} t,\ t \in [0, 2\pi]$. **2.66.** $x = a\cos 2t\cot t,\ y = a\cos 2t$, $t \in (0, \pi)$, or $x = \frac{a(t^2-1)}{1+t^2},\ y = \frac{at(t^2-1)}{1+t^2},\ t \in (-\infty, +\infty)$. **2.67.** $x = a\sqrt{\sin 2t}\cos t,\ y = a\sqrt{\sin 2t}\sin t,\ t \in \left[0, \frac{\pi}{2}\right] \bigcup \left[\pi, \frac{3\pi}{2}\right]$. **2.68.** $x = \sqrt{\frac{t}{t^4-1}},\ y = t\sqrt{\frac{t}{t^4-1}},\ t \in [2, +\infty)$. **2.69.** $x = 1 + t,\ y = 3$, $z = -1 + t,\ t \in [0, 1]$. **2.70.** $x = R\cos t,\ y = R\sin t,\ z = h,\ t \in [0, 2\pi]$. **2.71.** $x = R\cos t,\ y = R\sin t,\ z = R(\cos t + \sin t),\ t \in [0, 2\pi]$. **2.72.** $x = \frac{t^2}{2p},\ y = t,\ z = t + \frac{t^2}{2p},\ t \in (-\infty, +\infty)$. **2.73.** $x = \frac{R}{\sqrt{2}}\cos t,\ y = \frac{R}{\sqrt{2}}\sin t,\ z = \frac{R}{\sqrt{2}},\ t \in [0, 2\pi]$. **2.74.** $x = \frac{a}{2}(1 + \cos t),\ y = \frac{a}{2}\sin t,\ z = \pm a\sqrt{\frac{1+\cos t}{2}},\ t \in [0, 2\pi]$. **2.75.** $x = \frac{R}{\sqrt{2}}\left(\frac{\sin t}{\sqrt{3}} + \cos t\right),\ y = \frac{R}{\sqrt{2}} \times \left(\frac{\sin t}{\sqrt{3}} - \cos t\right),\ z = -\frac{2}{\sqrt{6}}R\sin t,\ t \in [0, 2\pi]$. **2.76.** $x = t, y = \pm t, z = \pm 1,\ t \in (-\infty, +\infty)$. **2.77.** Two straight lines $x = t,\ y = -t,\ z = 0,\ t \in (-\infty, +\infty)$, and $x = t,\ y = t - 1,\ z = 2t - 1,\ t \in (-\infty, +\infty)$. **2.78.** $x = \frac{1}{4}\left(\sqrt[3]{9at^2} + \frac{(3t)^{4/3}}{2a^{1/3}}\right),\ y = \frac{1}{4}\left(\frac{(3t)^{4/3}}{2a^{1/3}} - \sqrt[3]{9at^2}\right),\ z = t,\ t \in [0, z_0]$. **2.79.** $x = \frac{a\cos t}{\cosh t},\ y = \frac{a\sin t}{\cosh t},\ z = a\tanh t,\ t \in [0, t_0]$. **2.86.** C^∞. **2.87.** Equivalent. **2.90.** $2a\sinh 1$. **2.91.** $2\pi^2 a$. **2.92.** $16a$. **2.93.** $\frac{a}{2}\ln 2$. **2.94.** $2\pi a$. **2.95.** $8a$. **2.96.** $\frac{a}{8}(4\pi - 3\sqrt{3})$. **2.97.** $\frac{a}{2}\big(2\pi\sqrt{1+4\pi^2} + \ln(2\pi + \sqrt{1+4\pi^2})\big)$. **2.98.** $a\big(\pi - \tanh(\pi/2)\big)$. **2.99.** $a\big(\sqrt{2} + \ln(1+\sqrt{2})\big)$.

2.100. 2. **2.101.** $\frac{31}{12}$. **2.102.** $x = a\cos\frac{l}{\sqrt{a^2+h^2}}$, $y = a\sin\frac{l}{\sqrt{a^2+h^2}}$, $z = h\frac{l}{\sqrt{a^2+h^2}}$. **2.103.** $x = \left(1-\frac{2l}{5}\right)^{3/2}$, $y = \left(\frac{2l}{5}\right)^{3/2}$, $z = 1-\frac{4l}{5}$. **2.104.** $x = a\sqrt{1+\frac{l^2}{2a^2}}$, $y = \frac{l}{\sqrt{2}}$, $z = a\ln\left(\frac{l}{a\sqrt{2}}+\sqrt{1+\frac{l^2}{2a^2}}\right)$. **2.105.** $x = \frac{l}{2}-\sin\frac{l}{2}$, $y = 1-\cos\frac{l}{2}$, $z = 4\sin\frac{l}{4}$. **2.106.** $x = \frac{l}{2} + \sqrt{1+\frac{l^2}{4}}$, $y = \left(\frac{l}{2}+\sqrt{1+\frac{l^2}{4}}\right)^{-1}$, $z = \sqrt{2}\ln\left(\frac{l}{2}+\sqrt{1+\frac{l^2}{4}}\right)$. **2.107.** Normal plane: $(x-1)+3(y-1)+2(z-5)=0$; osculating plane: $-3(x-1)-(y-1)+3(z-5)=0$; rectifying plane: $11(x-1)-9(y-1)+8(z-5)=0$; $k = \frac{1}{7}\sqrt{\frac{19}{14}}$, $\varkappa = -\frac{3}{19}$. **2.108.** Normal plane: $(x-1)+\left(y-\frac{1}{3}\right)+\frac{1}{2}\left(z+\frac{1}{2}\right)=0$; osculating plane: $-2(x-1)+\left(y-\frac{1}{3}\right)+2\left(z+\frac{1}{2}\right)=0$; rectifying plane: $\frac{1}{2}(x-1)-\left(y-\frac{1}{3}\right)+\left(z+\frac{1}{2}\right)=0$; $k=\frac{8}{9}$, $\varkappa=\frac{8}{9}$. **2.109.** Normal plane: $\sqrt{6}\left(x-\sqrt{\frac{3}{2}}\right)-(y-1)+3(z-1)=0$; osculating plane: $-\sqrt{6}\left(x-\sqrt{\frac{3}{2}}\right)-3(y-1)+(z-1)=0$; rectifying plane: $\sqrt{2}\left(x-\sqrt{\frac{3}{2}}\right)-\sqrt{3}(y-1)-\sqrt{3}(z-1)=0$; $k=\frac{\sqrt{6}}{16}$, $\varkappa=\frac{\sqrt{6}}{16}$. **2.110.** Normal plane: $\left(x-\frac{a}{2}\right)-\frac{1}{\sqrt{2}}\left(z-\frac{a}{\sqrt{2}}\right)=0$; osculating plane: $-2\left(x-\frac{a}{2}\right)+\left(y-\frac{a}{2}\right)-2\sqrt{2}\left(z-\frac{a}{\sqrt{2}}\right)=0$; rectifying plane: $\left(x-\frac{a}{2}\right)+6\left(y-\frac{a}{2}\right)+\sqrt{2}\left(z-\frac{a}{\sqrt{2}}\right)=0$; $k=\frac{2}{3a}\sqrt{\frac{13}{3}}$, $\varkappa=\frac{6\sqrt{2}}{13a}$. **2.111.** Normal plane: $3\left(x-\frac{1}{2\sqrt{2}}\right)-3\left(y-\frac{1}{2\sqrt{2}}\right)+4\sqrt{2}z=0$; osculating plane: $\frac{3}{\sqrt{2}}\left(x-\frac{1}{2\sqrt{2}}\right)-\frac{3}{\sqrt{2}}\left(y-\frac{1}{2\sqrt{2}}\right)-\frac{9}{4}z=0$; rectifying plane: $\left(x-\frac{1}{2\sqrt{2}}\right)+$

$+\left(y-\dfrac{1}{2\sqrt{2}}\right)=0$; $k=\dfrac{6}{25}$, $\varkappa=\dfrac{8}{25}$. **2.112.** Normal plane: $ay+hz=0$; osculating plane: $hy-az=0$; rectifying plane: $x=a$; $k=\dfrac{a}{a^2+h^2}$, $\varkappa=\dfrac{h}{a^2+h^2}$. **2.113.** Normal plane: $-\left(x+\dfrac{\ln 2}{2}\right)+\left(y+\dfrac{\ln 2}{2}\right)+$ $+\sqrt{2}\left(z-\dfrac{\pi}{2\sqrt{2}}\right)=0$; osculating plane: $\left(x+\dfrac{\ln 2}{2}\right)-\left(y+\dfrac{\ln 2}{2}\right)+$ $+\sqrt{2}\left(z-\dfrac{\pi}{2\sqrt{2}}\right)=0$; rectifying plane: $\left(x+\dfrac{\ln 2}{2}\right)+\left(y+\dfrac{\ln 2}{2}\right)=0$; $k=\dfrac{1}{\sqrt{2}}$, $\varkappa=-\dfrac{1}{\sqrt{2}}$. **2.114.** Normal plane: $\left(x+1-\dfrac{\pi}{2}\right)+(y-1)+$ $+\sqrt{2}\left(z-2\sqrt{2}\right)=0$; osculating plane: $\left(x+1-\dfrac{\pi}{2}\right)-3(y-1)+$ $+\sqrt{2}\left(z-2\sqrt{2}\right)=0$; rectifying plane: $\sqrt{2}\left(x+1-\dfrac{\pi}{2}\right)-\left(z-2\sqrt{2}\right)=0$; $k=\dfrac{1}{4}\sqrt{\dfrac{3}{2}}$, $\varkappa=-\dfrac{5}{12\sqrt{2}}$. **2.115.** Normal plane: $(x-1)+y+(z-1)=0$; osculating plane: $(x-1)+y-2(z-1)=0$; rectifying plane: $(x-1)-y=0$; $k=\dfrac{\sqrt{2}}{3}$, $\varkappa=\dfrac{1}{3}$. **2.116.** Normal plane: $\left(x-\dfrac{1}{\sqrt{2}}\right)-\left(y-\dfrac{1}{\sqrt{2}}\right)+$ $+2\sqrt{2}(z-1)=0$; osculating plane: $\sqrt{2}\left(x-\dfrac{1}{\sqrt{2}}\right)+3\sqrt{2}\left(y-\dfrac{1}{\sqrt{2}}\right)+$ $+(z-1)=0$; rectifying plane: $13\left(x-\dfrac{1}{\sqrt{2}}\right)-3\left(y-\dfrac{1}{\sqrt{2}}\right)-4\sqrt{2}(z-1)=$ $=0$, $k=\dfrac{1}{5}\sqrt{\dfrac{21}{5}}$, $\varkappa=-\dfrac{6}{7}$. **2.117.** Normal plane: $x+az=0$; osculating plane: $z-ax=0$; rectifying plane: $y=0$; $k=\dfrac{2}{a^2+1}$, $\varkappa=\dfrac{3a}{2(a^2+1)}$. **2.118.** Normal plane: $(x-1)-(y-1)=0$; osculating plane: $z=0$; rectifying plane: $(x-1)+(y-1)=0$; $k=\dfrac{1}{\sqrt{2}}$, $\varkappa=0$. **2.119.** Normal plane: $(x-1)-\dfrac{1}{3}(y-3)+\dfrac{1}{4}(z-4)=0$; osculating plane: $5(x-1)-81(y-3)-128(z-4)=0$; rectifying plane: $755(x-1)+1551(y-3)-952(z-4)=0$; $k=\dfrac{5\sqrt{22970}}{2197}$,

$\varkappa = \dfrac{540}{2297}$. **2.120.** Normal plane: $\sqrt{2}\left(x-\dfrac{1}{2}\right)-\left(z-\dfrac{1}{\sqrt{2}}\right)=0$; osculating plane: $2\left(x-\dfrac{1}{2}\right)-\left(y-\dfrac{1}{2}\right)+2\sqrt{2}\left(z-\dfrac{1}{\sqrt{2}}\right)=0$; rectifying plane: $\left(x-\dfrac{1}{2}\right)+6\left(y-\dfrac{1}{2}\right)+\sqrt{2}\left(z-\dfrac{1}{\sqrt{2}}\right)=0$; $k=\dfrac{2}{3}\sqrt{\dfrac{13}{3}}$, $\varkappa=\dfrac{6\sqrt{2}}{13}$.
2.121. Normal plane: $5(x-3)+5(y-4)+7(z-5)=0$; osculating plane: $(x-3)-(y-4)=0$; rectifying plane: $7(x-3)+7(y-4)-10(z-5)=0$; $k=\dfrac{1}{297}\sqrt{\dfrac{2}{11}}$, $\varkappa=0$. **2.122.** Normal plane: $2(x-1)+(y-1)+4(z-1)=0$; osculating plane: $6(x-1)-8(y-1)-(z-1)=0$; rectifying plane: $31(x-1)+26(y-1)-22(z-1)=0$; $k=\dfrac{2}{21}\sqrt{\dfrac{101}{21}}$, $\varkappa=-\dfrac{12}{101}$. **2.123.** Normal plane: $(x-1)-(y-1)=0$; osculating plane: $(x-1)+(y-1)=0$; rectifying plane: $z-1=0$; $k=1$, $\varkappa=0$. **2.124.** Normal plane: $(x-1)+(y-1)=0$; osculating plane: $(x-1)-(y-1)+(z-1)=0$; rectifying plane: $(x-1)-(y-1)-2(z-1)=0$; $k=\dfrac{1}{\sqrt{6}}$, $\varkappa=1$.
2.125. Normal plane: $x+y+z=0$; osculating plane: $x-y=0$; rectifying plane: $x+y-2z=0$; $k=\dfrac{1}{3}\sqrt{\dfrac{2}{3}}$, $\varkappa=\dfrac{1}{2}$. **2.128.** $\dfrac{x+2}{-3}=\dfrac{y-12}{4}=$ $=\dfrac{z-14}{5}$ and $\dfrac{x+2}{0}=\dfrac{y-3}{-1}=\dfrac{z+4}{1}$. **2.129.** $\dfrac{x-4}{2}=\dfrac{y-1}{1}=\dfrac{z-e}{e}$.
2.130. $3x+3y+z+1=0$, $3x-3y+z-1=0$, $108x-18y+z-216=$ $=0$. **2.132.** $M(1,\ln 2,-4)$. **2.134.** Apply Rolle's theorem to the function $\vec{a}\cdot(\vec{r}(t)-\vec{r}(t_0))$. **2.139.** Darboux vector $\vec{\omega}=\varkappa\vec{\tau}+k\vec{\beta}$ is instantaneous angular velocity of Frenet trihedron while the point speed along a curve is unit. **2.143.** $k=\dfrac{|f''|}{(1+f'^2)^{3/2}}$. **2.144.** $k=\dfrac{|\rho^2+2\rho'^2-\rho\rho''|}{(\rho^2+\rho'^2)^{3/2}}$.
2.150. $(\vec{r}-\vec{c})\,\vec{a}\,\vec{b}=0$, where $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$, $\vec{c}=(c_1,c_2,c_3)$.
2.151. $f(t)=C_1+C_2\sin t+C_3\cos t$. **2.152.** Equating to zero the torsion we obtain the differential equation $t(f'''-f')+3f''-f=0$, which can be solved by the substitution $f(t)=u(t)/t$: $f(t)=t^{-1}(C_1+C_2e^t+$ $+C_3e^{-t})$. **2.154.** $x=(a+b\cos u)\cos v$, $y=(a+b\cos u)\sin v$, $z=$ $=b\sin u$. **2.155.** $x=a\cosh(u/a)\cos v$, $y=a\cosh(u/a)\sin v$, $z=u$.
2.156. $x=a\cos v\sin u$, $y=a\sin v\sin u$, $z=a\cos u+a\ln\tan(u/a)$.
2.157. $x=a(u+v)$, $y=b(u-v)$, $z=2uv$; $x=u$, $y=v$, $z=puv$
2.158. $x=\varphi(u)$, $y=\psi(u)$, $z=v$. **2.159.** $\vec{r}=\vec{\rho}(u)+v\vec{e}$. **2.160.** $x=$

$= u + v,\ y = u^2 + 2v,\ z = u^3 + 3v$. **2.162.** $(2x - z)^2 + (2y + 3z)^2 = 4$. **2.163.** $(nx - lz)^2 + (ny - mz)^2 = a(ny - mz)$. **2.164.** $x - a = v(\varphi(u) - a)$, $y - b = v(\psi(u) - b),\ z - c = v(\chi(u) - c)$. **2.165.** $(bz - cy)^2 = 2p(z - c) \times \times (az - cx)$. **2.166.** $(x + 1)^2 = 2y^2 + z^2$. **2.167.** a) $x = a\cos(u + v),\ y = = a\sin(u + v),\ z = bu$; b) $x = a\cos u,\ y = a\sin u,\ z = bu + v$; c) $x = = a\cos(u + v),\ y = a\sin(u + v),\ z = b(u - v)$. **2.168.** $\vec{r} = \vec{\rho}(u) + v\vec{\rho}\,'(u)$. **2.169.** $x = a(\cos u - v\sin u),\ y = a(\sin u + v\cos u),\ z = b(u + v)$. Obtained figure is not a surface. Excluding points of the original helix we obtain a surface. **2.170.** $x = u\cos v,\ y = u\sin v,\ z = f(u) + av$. **2.171.** Right helicoid: $x = u\cos v,\ y = y\sin v,\ z = av$; oblique helicoid: $x = u\cos v,\ y = y\sin v$, $z = mu + av$. **2.172.** $x = a(1 - u)\cos v,\ y = a(1 - u)\sin v,\ z = bv$. **2.173.** $x = u\cos v,\ y = u\sin v,\ z = f(v)$. **2.175.** $\vec{n} = \cos v\sin u\,\vec{i} + + \sin v\sin u\,\vec{j} + \cos u\,\vec{k}$. **2.176.** $\vec{n} = \cos u\cos v\,\vec{i} + \cos u\sin v\,\vec{j} + \sin u\,\vec{k}$.

2.177. $\vec{n} = \dfrac{a\sqrt{u^2 + a^2}\cos v\,\vec{i} + a\sqrt{u^2 + a^2}\sin v\,\vec{j} - u\,\vec{k}}{\sqrt{a^2(u^2 + a^2) + u^2}}$.

2.178. $\vec{n} = \dfrac{\cos v\,\vec{i} + \sin v\,\vec{j} - u\,\vec{k}}{\sqrt{u^2 + 1}}$. **2.179.** $\vec{n} = \dfrac{\cos v\,\vec{i} + \sin v\,\vec{j} - \sinh u\,\vec{k}}{\cosh u}$.

2.180. $\vec{n} = \dfrac{(u^2 + v^2)\vec{i} + (v^2 - u^2)\vec{j} - 4uv\,\vec{k}}{\sqrt{2}\sqrt{u^4 + 8u^2v^2 + v^4}}$.

2.181. $\vec{n} = \dfrac{\sin v\,\vec{i} - \cos v\,\vec{j} + u\,\vec{k}}{\sqrt{u^2 + 1}}$.

2.182. $\vec{n} = \dfrac{(u\cos v - \sin v)\vec{i} + (\cos v + u\sin v)\vec{j} - u\,\vec{k}}{\sqrt{2u^2 + 1}}$. **2.183.** $y = z$. **2.184.** $18x + 3y - 4z - 41 = 0$. **2.185.** $3x - y - 2z - 4 = 0$. **2.186.** $6x + + 3y - 2z - 7 = 0$. **2.187.** $2x + 2y - z - 2 = 0$. **2.188.** $3x + 12y - z - 18 = 0$. **2.189.** $x + y + z - 19 = 0$. **2.190.** $\dfrac{x_0}{a^2}(x - x_0) + \dfrac{y_0}{b^2}(y - y_0) + \dfrac{z_0}{c^2}(z - z_0) = 0$. **2.191.** $x + y + z - 3 = 0$. **2.195.** $\tan u = \pm C/\sqrt{A^2 + B^2},\ \tan v = B/A$. **2.196.** $\cos u\cos(v - \pi/4) + \sin u = 0$. **2.197.** A torus can be obtained by rotation of a circle γ that does not cross an axis of rotation axis. Let P_1 and P_2 are contact points of the straight line passing through the center and the circle γ. While rotating γ the points P_1 and P_2 circumscribe circles γ_1 and γ_2. At these circles $\vec{n} \cdot \vec{r} = 0$, at the "outer" part of the torus $\vec{n} \cdot \vec{r} > 0$, at the "inner" part $\vec{n} \cdot \vec{r} < 0$. **2.208.** $E = = c^2\sin^2 u + \cos^2 u(a^2\cos^2 v + b^2\sin^2 v),\ F = (b^2 - a^2)\cos u\sin u\cos v\sin v$, $G = \sin^2 u(b^2\cos^2 v + a^2\sin^2 v)$. **2.209.** $E = \big(a^4 + (a^2 + 1)u^2\big)/(a^2 + u^2)$, $F = 0,\ G = a^2 + u^2$. **2.210.** $E = (u^2 + 1)/(a^2 + u^2),\ F = 0,\ G =$

$= a^2 + u^2$. **2.211.** $E = 8u^2 + v^2$, $F = uv$, $G = u^2 + 8v^2$. **2.212.** $E = \cosh 2u$, $F = 0$, $G = \sinh^2 u$. **2.213.** $E = 1 + f'^2$, $F = af'$, $G = a^2 + u^2$. **2.214.** $E = 1 + (z'_x)^2$, $F = z'_x z'_y$, $G = 1 + (z'_y)^2$. **2.215.** The first quadratic form is positive definite, but forms a), b), d) are not. **2.216.** $E' = \frac{1}{J^2}\left[E\left(\frac{\partial v'}{\partial v}\right)^2 - 2F\frac{\partial v'}{\partial u}\frac{\partial v'}{\partial v} + G\left(\frac{\partial v'}{\partial v}\right)^2\right]\frac{\partial v'}{\partial u}\frac{\partial v'}{\partial v}$,
$G' = \frac{1}{J^2}\left[E\left(\frac{\partial u'}{\partial u}\right)^2 - 2F\frac{\partial u'}{\partial u}\frac{\partial u'}{\partial v} + G\left(\frac{\partial u'}{\partial u}\right)^2\right]$,
$F' = \frac{1}{J^2}\left[-E\frac{\partial u'}{\partial v}\frac{\partial v'}{\partial v} + F\left(\frac{\partial u'}{\partial u}\frac{\partial v'}{\partial v} + \frac{\partial v'}{\partial u}\frac{\partial u'}{\partial v}\right) + G\frac{\partial u'}{\partial u}\frac{\partial v'}{\partial u}\right]$, $H' = \frac{H}{|J|}$,
$J = \frac{D(u', v')}{D(u, v)} \neq 0$. **2.218.** $\tilde{u}$ is the natural parameter of the meridian; sphere: $dl^2 = d\tilde{u}^2 + R^2\cos^2(\tilde{u}/R)d\tilde{v}^2$; torus: $dl^2 = d\tilde{u}^2 + \big(a + b\cos(\tilde{u}/b)\big)^2 d\tilde{v}^2$; catenoid: $dl^2 = d\tilde{u}^2 + (a^2 + \tilde{u}^2)d\tilde{v}^2$; pseudosphere: $dl^2 = d\tilde{u}^2 + e^{-2\tilde{u}/a}d\tilde{v}^2$. **2.219.** $dl^2 = d\tilde{u}^2 + e^{-2\tilde{u}/a}d\tilde{v}^2$. Assuming $u^* = \tilde{v}$, $v^* = e^{\tilde{u}/a}$, we obtain $dl^2 = \frac{a^2}{\tilde{v}^2}(d\tilde{u}^2 + d\tilde{v}^2)$. **2.220.** $\sqrt{2}a\,|v_2 - v_1|$. **2.221.** $\sqrt{2}|u_2 - u_1|$. **2.222.** $\frac{1}{2}\left(\sqrt{5} + 4\,\mathrm{arcsh}\,\frac{1}{2}\right)$. **2.223.** $\sqrt{2}\,|\sinh u_2 - \sinh u_1|$. **2.224.** $\frac{1}{3}(5\sqrt{5} - 8)$. **2.225.** $|\sinh u_2 - \sinh u_1|$. **2.228.** $\pi - \arccos\frac{1}{2\sqrt{2}}$. **2.229.** $\frac{\pi}{2}$. **2.230.** $\arccos\frac{2}{3}$. **2.231.** $\pi - \arccos\frac{3}{5}$. **2.232.** $\cos\varphi = \frac{a^2xy}{\sqrt{1 + a^2x^2}\sqrt{1 + a^2y^2}}$. **2.233.** Let $A(0,0)$, $B(4,2)$ and $C(-4,2)$ be vertexes of the triangle. Then $p = \frac{64}{3}$, $\cos\angle A = 1$, $\cos\angle B = \cos\angle C = \frac{2}{3}$. **2.234.** $p = 2 + \sqrt{3} + 2\ln\frac{1 + \sqrt{3}}{\sqrt{2}}$; if $A(0,0)$, $B(1,1)$, $C(-1,1)$, then $\cos\angle A = 0$, $\cos\angle B = \cos\angle C = \frac{1}{\sqrt{3}}$. **2.235.** $4\pi^2 ab$. **2.236**. $2a^2(\pi - 2)$, a is the sphere radius. **2.239.** $\ln(u + \sqrt{u^2 + a^2}) \pm v = \mathrm{const}$. **2.240.** The first quadratic form of a surface of revolution $d\vec{r}^{\,2} = (\varphi'(u)^2 + \psi'(u)^2)du^2 + \varphi(u)^2dv^2$ can be rewritten as $d\vec{r}^{\,2} = dU^2 + G(U)dv^2$, where $U = \int\sqrt{\varphi'(u)^2 + \psi'(u)^2}du$. Then the equation of a

loxodrome is $v\cot\alpha = \pm\int\limits_{u_1}^{u_2}\frac{\mathrm{d}U}{\sqrt{G(U)}}$. **2.241.** $v\cot\alpha = \pm R\ln\tan\left(\frac{\pi}{4}+\frac{u}{2}\right)$. **2.242.** $\left(E\frac{\partial\varphi}{\partial v} - F\frac{\partial\varphi}{\partial u}\right)\mathrm{d}u + \left(F\frac{\partial\varphi}{\partial v} - G\frac{\partial\varphi}{\partial u}\right)\mathrm{d}v = 0$. **2.243.** $v - \tan u = C$. **2.244.** $u^2+u+1 = Ce^{-v}$. **2.245.** $X = \frac{U-V}{2}\cos V$, $Y = \frac{U-V}{2}\sin V$, $Z = \frac{U+V}{2}$, $U = 2u+v$, $V = v$. **2.246.** $(1+a^2x^2)y^2 = C_1$, $(1+a^2y^2)x^2 = C_2$. **2.248.** L/E and N/G. **2.249.** $k_n|_{u=\mathrm{const}} = \frac{1}{\sqrt{(u^2+a^2)(u^2+1)}}$, $k_n|_{v=\mathrm{const}} = -\frac{\sqrt{u^2+a^2}}{(u^2+1)^{3/2}}$. **2.250.** $\frac{2}{49\sqrt{5}}$. **2.251.** $k_1 = \frac{a}{b^2}$, $k_2 = \frac{a}{c^2}$. **2.252.** $k_1 = \frac{\sqrt{3}}{9}$, $k_2 = -\frac{\sqrt{3}}{3}$. **2.253.** $k_1 = \frac{1}{p}$, $k_2 = -\frac{1}{q}$. **2.254.** $k_1 = \frac{1}{2\sqrt{5}}$, $k_2 = 0$. **2.255.** $k_1 = \frac{1}{u\sqrt{2}}$, $k_2 = 0$. **2.257.** $K = \frac{c^2}{(a^2\cos^2 u + c^2\sin^2 u)^2}$, $H = -\frac{c}{2a}\frac{a^2(\cos^2 u+1)+c^2\sin^2 u}{a^2\cos^2 u + c^2\sin^2 u^2}$. **2.258.** $K = \frac{\cos u}{b(a+b\cos u)}$, $H = \frac{a+2b\cos u}{2b(a+b\cos u)}$. **2.259.** $K = -\frac{1}{\cosh^2 2u}$, $H = \frac{\sinh^2 u}{(\cosh 2u)^{3/2}}$. **2.260.** $K = \frac{1}{\cosh^2 2u}$, $H = \frac{\cosh^2 u}{(\cosh 2u)^{3/2}}$. **2.261.** $K = \frac{4}{(1+4u^2)^2}$, $H = \frac{2+4u^2}{(1+4u^2)^{3/2}}$. **2.262.** $K = -\frac{1}{\cosh^4 u}$, $H = 0$. **2.263.** $K = -\frac{1}{(u^2+1)^2}$, $H = 0$. **2.264.** $K = \frac{f'f''}{\rho(1+f'^2)^2}$, $H = \frac{f'}{2\rho\sqrt{1+f'^2}} + \frac{f''}{2(1+f'^2)^{3/2}}$, where $\rho = \sqrt{x^2+y^2}$ and f' and f'' are derivatives with respect to ρ.

2.265. a) $K = \frac{-1}{(\partial_x F)^2+(\partial_y F)^2+(\partial_z F)^2}\begin{vmatrix} \partial_{xx}F & \partial_{xy}F & \partial_{xz}F & \partial_x F \\ \partial_{yx}F & \partial_{yy}F & \partial_{yz}F & \partial_y F \\ \partial_{zx}F & \partial_{zy}F & \partial_{zz}F & \partial_z F \\ \partial_x F & \partial_y F & \partial_z F & 0 \end{vmatrix}$;

b) $K = \frac{z''_{xx}z''_{yy} - z''_{xy}z''_{yx}}{1+(z'_x)^2+(z'_y)^2}$. **2.266.** $1 - f'^2 - ff'' = 0$, hence $f(u) = a\cosh(u/a)$, $a = \mathrm{const}$. **2.267.** $K = -1$. **2.269.** $K = -\frac{1}{A^2}(\partial_{uu}\ln A + \partial_{vv}\ln A)$, see exercise 2.219.

Chapter 3

3.1. $2x - y = C$ — the family of parallel straight lines. **3.2.** $x^2 - y^2 = C$ — the family of hyperbols. **3.3.** $y = Cx$ — the pencil of straight lines. **3.4.** $(x-C)^2 + y^2 = C^2$ — the family of circles, passing through the origin of coordinates. **3.5.** $y^2 = Cx$ — the family of parabolas. **3.6.** $y = Cx^2 + 2x + 1$ — the family of parabolas. **3.7.** $x^2 = Cy$ — the family of parabolas. **3.8.** $x^2 + y^2 = C$ — the family of concentric circles. **3.9.** $x^2 + y^2 - z = C$ — the family of of paraboloids. **3.10.** $x^2 + y^2 + z^2 = C$ — the family of concentric spheres. **3.11.** $x - y^2 + z^2 = C$ — the family of hyperbolic paraboloids. **3.12.** $x^2 + y^2 = Cz$ — the family of paraboloids. **3.13.** $x + 2y - z = C$ — the family of parallel planes. **3.14.** $\vec{a} \cdot \vec{r} = C$ — the family of parallel planes with the normal vector $\vec{a}$. **3.15.** $(\vec{a} - C\vec{b}) \cdot \vec{r} = 0$ — the sheaf of planes. **3.16.** For $C > 16$ the level surfaces are the ellipsoids of revolution around Oz axis with foci at the point $(0, 0, 8)$ and $(0, 0, -8)$. **3.17.** $y = -x \ln C - C, (C > 0)$ — the family of straight lines. **3.25.** $\frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z^2}\vec{k}$. **3.26.** $3(x^2 - yz)\vec{i} + 3(y^2 - xz)\vec{j} + 3(z^2 - xy)\vec{k}$. **3.27.** $e^{x^2+y^2+z^2}(2xz\vec{i} + 2yz\vec{j} + (1 + 2z^2)\vec{k})$. **3.28.** $(y-z)(z-2x+y)\vec{i} + (z-x)(x-2y+z)\vec{j} + (x-y)(y-2z+x)\vec{k}$. **3.29.** $e^{x+y+z}(yz(x+1)\vec{i} + xz(y+1)\vec{j} + xy(z+1)\vec{k})$. **3.30.** $(y-2)(z-3)\vec{i} + (x-1)(z-3)\vec{j} + (x-1)(y-2)\vec{k}$. **3.31.** $(1+x^2)^{-1}\vec{i} + (1+y^2)^{-1}\vec{j} + (1+z^2)^{-1}\vec{k}$. **3.32.** $\frac{u}{u^2 - xy}(y\vec{i} + x\vec{j})$. **3.33.** $\frac{\vec{i} + \vec{j}}{e^u - 1}$. **3.34.** $-\vec{i} - \vec{j}$. **3.35.** $\frac{3}{\sqrt{17}}$. **3.36.** $-\frac{7}{3}$. **3.37.** $\frac{\sqrt{15}}{5}$. **3.38.** $-\frac{2}{5}$. **3.39.** 0. **3.40.** $\frac{2}{\sqrt{3}}$. **3.41.** 0. **3.42.** $\frac{3}{5}\sqrt{2}$. **3.43.** $\frac{1}{4}$. **3.44.** -2. **3.45.** $M_1(0,0),\ M_2(1,1)$. **3.46.** $M(-2,1,1)$. **3.47.** $M(7,2,1)$. **3.48.** $M(0,0,0)$. **3.49.** 0. **3.50.** $\pi - \arccos\frac{8}{9}$. **3.51.** 0. **3.52.** $\frac{2u}{|\vec{r}|}$; if $a = b = c$. **3.53.** $e^x\sqrt{y^2 + z^2 + y^2z^2}$. **3.54.** $\frac{(\operatorname{grad} u) \cdot (\operatorname{grad} v)}{|(\operatorname{grad} v)|}$. **3.55.** $x^2 = C_1 y$, $z = C_2$. **3.56.** $\frac{1}{x} - \frac{1}{y} = C_1,\ z = C_2$. **3.57.** $y^2 + z^2 = C_1,\ x = C_2$. **3.58.** $z = C_1 x$, $y = C_2$. **3.59.** $x = C_1 y,\ x^2 = C_2 z$. **3.60.** $\frac{1}{x} - \frac{1}{y} = C_1,\ \frac{1}{x} - \frac{1}{z} = C_2$. **3.61.** $x^2 + y^2 + z^2 = C_1,\ x + y + z = C_2$. **3.62.** $\frac{1}{x} - \frac{1}{z} = C_1,\ \frac{1}{x} + \frac{1}{2y^2} = C_2$. **3.69.** 0. **3.70.** $2(x^2 + y^2 + z^2)$. **3.71.** 0. **3.72.** 0. **3.73.** 3. **3.74.** $f_1(y,z) +$

$+f_2(x,z)+f_3(x,y)$. **3.75.** $\vec{0}$. **3.76.** $\vec{0}$. **3.77.** $-\dfrac{1}{x^2}\vec{j}-\dfrac{2y}{x^3}\vec{k}$. **3.78.** $2xy(3z^2-$ $-y)\vec{i}+2yz(y-z^2)\vec{k}$. **3.79.** $(1+2yz)\vec{j}-z^2\vec{k}$. **3.80.** $-\dfrac{1}{y}\vec{i}-\dfrac{1}{z}\vec{j}-\dfrac{1}{x}\vec{k}$. **3.81.** $x(x-$ $-2z)\vec{i}+y(y-2x)\vec{j}+z(z-2y)\vec{k}$. **3.82.** $\vec{i}+x(y-2)\vec{j}+(2-xz)\vec{k}$. **3.83.** $\dfrac{\partial\vec{a}}{\partial l_1}=$ $=y\vec{i}+z\vec{k}$, $\dfrac{\partial\vec{a}}{\partial l_2}=\dfrac{(x+y)\vec{i}+z\vec{j}+z\vec{k}}{\sqrt{2}}$, $\dfrac{\partial\vec{a}}{\partial l_3}=\dfrac{x\vec{i}+(y+z)\vec{j}+x\vec{k}}{\sqrt{2}}$, $\dfrac{\partial\vec{a}}{\partial l_4}=$ $=\dfrac{(x+y)\vec{i}+(y+z)\vec{j}+(x+z)\vec{k}}{\sqrt{3}}$. **3.92.** $(\vec{b}\cdot\vec{\nabla})\vec{a}=yz(x+y)\vec{i}+xz(y+z)\vec{k}+$ $+xy(z+x)\vec{k}$, $(\vec{a}\cdot\vec{\nabla})\vec{b}=y(z^2+x^2)\vec{i}+z(x^2+y^2)\vec{j}+x(y^2+z^2)\vec{k}$. **3.94.** $\dfrac{\vec{r}}{r}$. **3.95.** $-\dfrac{\vec{r}}{r^3}$. **3.96.** nr^n. **3.97.** $\vec{a}$. **3.98.** $f(r)\vec{a}+f'(r)\dfrac{(\vec{a}\cdot\vec{r})\vec{r}}{r}$. **3.99.** $\dfrac{\vec{a}}{r^3}-$ $-3\dfrac{(\vec{a}\cdot\vec{r})\vec{r}}{r^3}$. **3.100.** $2(\vec{a}^2\vec{r}-(\vec{a}\cdot\vec{r})\vec{a})$. **3.101.** $\dfrac{\vec{r}\times(\vec{a}\times\vec{b})}{(\vec{b}\cdot\vec{r})^2}$. **3.102.** 3. **3.103.** $\dfrac{2}{r}$. **3.104.** $\vec{a}\cdot\vec{b}$. **3.105.** $4\vec{a}\cdot\vec{r}$. **3.106.** $f'(r)\dfrac{\vec{a}\cdot\vec{r}}{r}$. **3.107.** $\dfrac{\vec{a}\cdot\vec{r}}{r}$. **3.108.** $2\vec{a}\cdot\vec{r}$. **3.109.** 0. **3.110.** $2\vec{a}\cdot\vec{b}$. **3.111.** $2\vec{a}\cdot\vec{r}$. **3.112.** $\vec{0}$. **3.113.** $\vec{a}\times\vec{b}$. **3.114.** $\vec{a}\times\vec{r}$. **3.115.** $\dfrac{\vec{r}\times\vec{a}}{r}$. **3.116.** $f'(r)\dfrac{\vec{r}\times\vec{a}}{r}$. **3.117.** $-\dfrac{\vec{a}}{r^3}+\dfrac{3\vec{r}(\vec{a}\cdot\vec{r})}{r^5}$. **3.118.** $2f(r)\vec{a}+$ $+f'(r)\dfrac{\vec{r}\times(\vec{a}\times\vec{r})}{r}$. **3.119.** $2\vec{a}$. **3.120.** $\vec{a}\times\vec{b}$. **3.121.** $3\vec{a}\times\vec{r}$. **3.122.** $\big(\vec{a}'(r)\cdot$ $\cdot\vec{b}(r)+\vec{a}(r)\cdot\vec{b}'(r)\big)\dfrac{\vec{r}}{r}$. **3.123.** $\vec{a}(r)+\dfrac{(\vec{a}'(r)\cdot\vec{r})\vec{r}}{r}$. **3.124.** $\dfrac{\vec{b}\cdot\vec{r}}{r}\big(f'(r)\vec{a}(r)+$ $+f(r)\vec{a}'(r)\big)$. **3.125.** $\dfrac{\vec{r}}{r}\cdot\big(f'(r)\vec{a}(r)+f(r)\vec{a}'(r)\big)$. **3.126.** $\dfrac{\vec{r}}{r}\times\big(f'(r)\vec{a}(r)+$ $+f(r)\vec{a}'(r)\big)$. **3.127.** $\operatorname{div}\vec{p}=nr^{n-2}(\vec{a}\cdot\vec{r})$, $\operatorname{curl}\vec{p}=nr^{n-2}(\vec{r}\times\vec{a})$, $(\vec{c}\cdot\vec{\nabla})\vec{p}=$ $=\vec{a}(\vec{r}\cdot\vec{c})nr^{n-2}$. **3.128**. $\operatorname{div}\vec{p}=(n+3)r^n$, $\operatorname{curl}\vec{p}=\vec{0}$, $(\vec{c}\cdot\vec{\nabla})\vec{p}=\vec{c}r^n+$ $+\vec{r}(\vec{c}\cdot\vec{r})nr^{n-2}$. **3.129.** $\operatorname{div}\vec{p}=(\vec{a}\cdot\vec{b})r^n+(\vec{a}\cdot\vec{r})(b\cdot\vec{r})nr^{n-2}$, $\operatorname{curl}\vec{p}=$ $=(\vec{b}\times\vec{a})r^n+(\vec{r}\times\vec{a})(\vec{b}\cdot\vec{r})nr^{n-2}$, $(\vec{c}\cdot\vec{\nabla})\vec{p}=\vec{a}(\vec{c}\cdot\vec{b})r^n+\vec{a}(\vec{c}\cdot\vec{r})(\vec{b}\cdot\vec{r})nr^{n-2}$. **3.130.** $\operatorname{div}\vec{p}=(n+4)(\vec{a}\cdot\vec{r})r^n$, $\operatorname{curl}\vec{p}=(\vec{a}\times\vec{r})r^n$, $(\vec{c}\cdot\vec{\nabla})\vec{p}=\vec{c}(\vec{a}\cdot\vec{r})r^n+$ $+\vec{r}(\vec{a}\cdot\vec{c})r^n+\vec{r}(\vec{a}\cdot\vec{r})(\vec{c}\cdot\vec{r})nr^{n-2}$. **3.131.** $\operatorname{div}\vec{p}=(n+5)(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})r^n$, $\operatorname{curl}\vec{p}=$ $=(\vec{a}\times\vec{r})(\vec{b}\cdot\vec{r})r^n+(\vec{b}\times\vec{r})(\vec{a}\cdot\vec{r})r^n$, $(\vec{c}\cdot\vec{\nabla})\vec{p}=\vec{c}(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})r^n+\vec{r}\big((\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{r})+$ $+(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{r})\big)r^n+\vec{r}(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})(\vec{c}\cdot\vec{r})nr^{n-2}$. **3.132.** $\operatorname{div}\vec{p}=0$, $\operatorname{curl}\vec{p}=\vec{a}(n+$ $+2)r^n-\vec{r}(\vec{a}\cdot\vec{r})nr^{n-2}$, $(\vec{c}\cdot\vec{\nabla})\vec{p}=(\vec{a}\times\vec{c})r^n+(\vec{a}\times\vec{r})(\vec{c}\cdot\vec{r})nr^{n-2}$. **3.133.** $\operatorname{div}\vec{p}=$ $=(\vec{r}\,\vec{b}\,\vec{a})r^n$, $\operatorname{curl}\vec{p}=(n+2)r^n(\vec{b}\cdot\vec{r})\vec{a}-r^n(\vec{a}\cdot\vec{b})\vec{r}-nr^{n-2}(\vec{b}\cdot\vec{r})(\vec{a}\cdot\vec{r})\vec{r}$, $(\vec{c}\cdot\vec{\nabla})\vec{p}=$

$= (\vec{a}\times\vec{c})(\vec{b}\cdot\vec{r})r^n + (\vec{a}\times\vec{r})(\vec{b}\cdot\vec{r})(\vec{c}\cdot\vec{r})nr^{n-2} + (\vec{a}\times\vec{r})(\vec{b}\cdot\vec{c})r^n$. **3.134.** $\vec{\nabla}u\times\vec{\nabla}v$. **3.135.** $f''(r) + \dfrac{2}{r}f'(r)$. **3.136.** $(\operatorname{curl}\vec{b}\cdot\vec{\nabla})\vec{a} - \operatorname{div}\vec{a}\operatorname{curl}\vec{b} - (\vec{a}\cdot\vec{\nabla})\operatorname{curl}\vec{b}$. **3.137.** 0. **3.138.** $\operatorname{grad}\operatorname{div}\vec{p} = n(n-2)r^{n-4}(\vec{a}\cdot\vec{r})\vec{r} + nr^{n-2}\vec{a}$, $\operatorname{curl}\operatorname{curl}\vec{p} = n(n-2)r^{n-4}(\vec{a}\cdot\vec{r})\vec{r} - n^2r^{n-2}\vec{a}$, $\Delta\vec{p} = n(n+1)r^{n-2}\vec{a}$. **3.139.** $\operatorname{grad}\operatorname{div}\vec{p} = n(n+3)r^{n-2}\vec{r}$, $\operatorname{curl}\operatorname{curl}\vec{p} = \vec{0}$, $\Delta\vec{p} = n(n+3)r^{n-2}\vec{r}$. **3.140.** $\operatorname{grad}\operatorname{div}\vec{p} = \big((\vec{a}\cdot\vec{b})\vec{r} + (\vec{b}\cdot\vec{r})\vec{a} + (\vec{a}\cdot\vec{r})\vec{b}\big)nr^{n-2} + \vec{r}(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})n(n-2)r^{n-4}$, $\operatorname{curl}\operatorname{curl}\vec{p} = \big((\vec{a}\cdot\vec{b})\vec{r} + (\vec{a}\cdot\vec{r})\vec{b} - (n+2)(\vec{b}\cdot\vec{r})\vec{a}\big)nr^{n-2} + \vec{r}(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})n(n-2)r^{n-4}$, $\Delta\vec{p} = -n(n+1)r^{n-2}(\vec{b}\cdot\vec{r})\vec{a}$. **3.141.** $\operatorname{grad}\operatorname{div}\vec{p} = (n+4)\big(r^n\vec{a} + n(\vec{a}\cdot\vec{r})r^{n-2}\vec{r}\big)$, $\operatorname{curl}\operatorname{curl}\vec{p} = (n+2)r^n\vec{a} - n(\vec{a}\cdot\vec{r})r^{n-2}\vec{r}$, $\Delta\vec{p} = 2r^n\vec{a} + n(n+3)(\vec{a}\cdot\vec{r})r^{n-2}\vec{r}$. **3.142.** $\operatorname{grad}\operatorname{div}\vec{p} = (n+5)\big(r^n(\vec{b}\cdot\vec{r})\vec{a} + r^n(\vec{a}\cdot\vec{r})\vec{b} + n(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})r^{n-2}\vec{r}\big)$, $\operatorname{curl}\operatorname{curl}\vec{p} = (n+4)r^n\big((\vec{b}\cdot\vec{r})\vec{a} + (\vec{a}\cdot\vec{r})\vec{b}\big)$, $\Delta\vec{p} = r^n\big((\vec{b}\cdot\vec{r})\vec{a} + (\vec{a}\cdot\vec{r})\vec{b}\big) + n(n+5)(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})r^{n-2}\vec{r}$. **3.143.** $\operatorname{grad}\operatorname{div}\vec{p} = \vec{0}$, $\operatorname{curl}\operatorname{curl}\vec{p} = n(n+3)r^{n-2}(\vec{r}\times\vec{a})$, $\Delta\vec{p} = n(n+3)r^{n-2}(\vec{a}\times\vec{r})$. **3.144.** $\operatorname{grad}\operatorname{div}\vec{p} = r^n(\vec{b}\times\vec{a}) + nr^{n-2}(\vec{b}\vec{a}\vec{r})\vec{r}$, $\operatorname{curl}\operatorname{curl}\vec{p} = (n+2)r^n(\vec{b}\times\vec{a}) + nr^{n-2}\big((n+3)(\vec{b}\cdot\vec{r})(\vec{r}\times\vec{a}) + n(\vec{a}\cdot\vec{r})(\vec{r}\times\vec{b})\big)$, $\Delta\vec{p} = (n+1)r^n(\vec{a}\times\vec{b}) + nr^{n-2}\big((\vec{b}\vec{a}\vec{r})\vec{r} - (n+3)(\vec{b}\cdot\vec{r})(\vec{r}\times\vec{a}) - n(\vec{a}\cdot\vec{r})(\vec{r}\times\vec{b})\big)$.

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4.1. 24. **4.2.** $\ln(\sqrt{5}+\sqrt{6})$. **4.3.** $4\sqrt{2}(3\cdot 4^{1/3}-4)$. **4.4.** $\sqrt{5}\ln 2$. **4.5.** $\dfrac{p^2}{3}(5\sqrt{5}-1)$. **4.6.** $\dfrac{5\sqrt{5}-3\sqrt{3}}{6}$. **4.7.** $\dfrac{17\sqrt{17}-2\sqrt{2}}{3}$. **4.8.** $\dfrac{\pi}{a}$. **4.9.** $\dfrac{1}{\sqrt{2}} + \dfrac{1}{256}\times\left(3326\sqrt{2} - 45\sqrt{10} + 5\ln\dfrac{(1+\sqrt{2})(3+\sqrt{10})}{(5+\sqrt{26})(7+\sqrt{50})}\right)$. **4.10.** $3\sqrt{2} + \dfrac{28\sqrt{5}}{3} - \ln(1+\sqrt{2})$. **4.11.** $\dfrac{a^3}{30}(25\sqrt{5} + 60\sqrt{2} + 1)$. **4.12.** $\dfrac{\pi a^3}{2}$. **4.13.** $2a^2$. **4.14.** $2\pi a^{2n+1}$. **4.15.** $\dfrac{\pi a^2}{2}$. **4.16.** $\dfrac{ab(a^2+ab+b^2)}{3(a+b)}$. **4.17.** $\dfrac{a^3}{6}(\cosh^{3/2}(2t_0)-1)$. **4.18.** $\dfrac{8}{3}(1+\sqrt{5}+3\ln(2+\sqrt{5}))$. **4.19.** $\dfrac{256}{15}a^3$. **4.20.** $4\pi a^{3/2}$. **4.21.** $\dfrac{3}{5}$. **4.22.** $\dfrac{9a^3}{2}$. **4.23.** $\dfrac{a^2}{3}\big((1+4\pi^2)^{3/2}-1\big)$. **4.24.** $\dfrac{1}{2a}\ln(1+4\pi^2)$. **4.25.** $\dfrac{\pi^2+12\pi-8\ln 2}{16}$. **4.26.** $\dfrac{(a^2+4)^{3/2}-8}{12}$. **4.27.** $\dfrac{a^2\sqrt{k^2+1}}{4k^2+1}\times$

$\times\left(\sin\frac{\ln a}{k}+2k\cos\frac{\ln a}{k}\right)$. **4.28.** $\frac{16}{5}$. **4.29.** $a\left(1+\frac{1}{12\sqrt{2}}\ln\frac{3+2\sqrt{2}}{3-2\sqrt{2}}\right)$.
4.30. $\sqrt{2}$. **4.31.** 0. **4.32.** $\frac{93}{5}a^2$. **4.33.** $\frac{4a^4}{5}$ **4.34.** πa^2. **4.35.** $\frac{2\sqrt{2}a^3}{3}$.
4.36. $\frac{56\sqrt{7}-1}{54}$. **4.37.** $\frac{1}{16}\left(6\sqrt{3}-2+\ln\left(5+\frac{26}{3\sqrt{3}}\right)\right)$. **4.38.** $\frac{7}{12}-$
$+\frac{80\sqrt{2}}{297}$. **4.39.** $\frac{\sqrt{a^2+b^2}}{ab}\arctan\frac{2\pi b}{a}$. **4.40.** $\frac{8\sqrt{2}}{3}\pi^3 a$. **4.41.** $\frac{2\pi}{3}\sqrt{a^2+b^2}\times$
$\times\left(3a^2+4\pi^2b^2\right)$. **4.42.** $\frac{2\sqrt{2}}{3}\left((1+2\pi^2)^{3/2}-1\right)$. **4.43.** $\frac{1}{4}\big(2\pi\sqrt{1+4\pi^2}(3+$
$+8\pi^2)+\ln(2\pi+\sqrt{1+4\pi^2})\big)$. **4.44.** $\frac{16\pi}{3}(\pi^2+9)a^3$. **4.45.** $16\pi a^3$.
4.46. $\frac{(5\pi^2+1)^{3/2}-1}{15}$. **4.47.** $\frac{2\sqrt{2}}{3}\left((1+2\pi^2)^{3/2}-1\right)$. **4.48.** $\frac{2\pi a^3}{3}$.
4.49. $\frac{a^2}{256\sqrt{2}}\left(100\sqrt{38}-72-17\ln\frac{25+4\sqrt{38}}{17}\right)$. **4.50.** $\frac{\sqrt{3}a^4}{32}$. **4.51.** $\frac{2}{3}(-1+$
$+2\sqrt{2})$. **4.52.** $\frac{1}{3}\left((1+x_2^2)^{3/2}-(1+x_1^2)^{3/2}\right)$. **4.53.** $\frac{17}{2}$. **4.54.** $2b$. **4.55.** $\frac{4}{3}ba^4$.
4.56. $4+\frac{18}{\sqrt{5}}\arcsin\frac{\sqrt{5}}{3}$. **4.57.** $2b\left(b+\frac{a}{\varepsilon}\arcsin\varepsilon\right)$, $\varepsilon=\frac{\sqrt{a^2-b^2}}{a}$.
4.58. $\frac{2}{3}a^2(2\sqrt{2}-1)$. **4.59.** $\frac{8\pi^3}{3}b^2\sqrt{a^2+b^2}$. **4.60.** $x_c=y_c=\frac{2a}{\pi}$.
4.61. $x_c=0$, $y_c=\frac{a}{4}\frac{2+\sinh 2}{\sinh 1}$. **4.62.** $x_c=\frac{2a}{e+1}$, $y_c=\frac{a}{4}\frac{2+\sinh 2}{\sinh 1}$.
4.63. $x_c=0$, $y_c=0$. **4.64.** $x_c=\frac{27-16\ln 2-4\ln^2 2}{8(3+2\ln 2)}$, $y_c=\frac{20}{3(3+2\ln 2)}$.
4.65. $x_c=\pi a$, $y_c=\frac{4a}{3}$. **4.66.** $x_c=y_c=\frac{4a}{3}$. **4.67.** $x_c=-2a$, $y_c=$
$=-\frac{3a}{\pi}$. **4.68.** $x_c=y_c=\frac{28+3\sqrt{2}\ln\left(2\sqrt{2}+3\right)}{64+16\sqrt{2}\ln\left(2\sqrt{2}+3\right)}$. **4.69.** $x_c=0$, $y_c=$
$=\frac{2a}{5}$. **4.70.** $x_c=\frac{a\sin\beta}{\beta}$, $y_c=\frac{a(1-\cos\beta)}{\beta}$. **4.71.** $x_c=0$, $y_c=0$,
$z_c=\pi b$. **4.72.** $x_c=\frac{2}{5}$, $y_c=-\frac{1}{5}$, $z_c=\frac{1}{2}$. **4.73.** $x_c=\frac{4a}{5}$, $y_c=\frac{4a}{5}$.
4.74. $x_c=y_c=z_c=\frac{4a}{3\pi}$. **4.75.** $x_c=\frac{\sqrt{2}a}{\pi}$, $y_c=0$, $z_c=\frac{2a}{\pi}$.
4.76. $x_c=y_c=\frac{1}{2}$, $z_c=1$. **4.77.** $x_c=\frac{3a}{2\pi^2}$, $y_c=-\frac{3a}{2\pi}$, $z_c=\frac{3}{2}\pi b$.

4.78. $x_c = a\pi$, $y_c = \dfrac{3a}{2}$, $z_c = \dfrac{32a}{3\pi}$. **4.79.** $I_x = \dfrac{7\sqrt{5}}{24}$. **4.80.** $I_x = \dfrac{1}{64}\left(102\sqrt{2} - 18\sqrt{5} + \ln(2+\sqrt{5}) - \ln(2\sqrt{2}+3)\right)$. **4.81.** $I_x = \dfrac{5}{1536} \times \left(87\ln(2+\sqrt{5}) + 34\sqrt{5}\right)$, $I_y = \dfrac{1}{64}\left(18\sqrt{5} - \ln(2+\sqrt{5})\right)$. **4.82.** $I_x = 9\sqrt{5}$, $I_y = 21\sqrt{5}$. **4.83.** $I_0 = \dfrac{3\sqrt{3}}{2}a^3$. **4.84.** $I_x = \dfrac{a^3}{4}(2\beta - \sin 2\beta)$; $I_y = \dfrac{a^3}{4}(2\beta + \sin 2\beta)$. **4.85.** $I_x = \dfrac{a^3}{768}\left(278 + \dfrac{9}{\sqrt{2}}\ln\dfrac{\sqrt{2}+1}{\sqrt{2}-1}\right)$, $I_0 = 2I_x$.
4.86. $I_y = I_x = \dfrac{3a^3}{4}$. **4.87.** $I_y = I_x = \dfrac{3a^3}{8}$. **4.88.** $I_x = I_y = 2\pi\sqrt{a^2+b^2} \times \left(\dfrac{a^2}{2} + \dfrac{4\pi^2 b^2}{3}\right)$, $I_z = 2\pi a^2\sqrt{a^2+b^2}$. **4.89.** $\dfrac{2GmM}{\pi R^2}\vec{j}$. **4.90.** $-\dfrac{2Gm\rho}{h}\vec{e}_\rho$.
4.91. $\dfrac{3a^2 g}{5}\vec{i} + \dfrac{3a^2 g}{5}\vec{j}$. **4.92.** $\dfrac{\lambda_0 R z_0}{2\varepsilon_0(R^2+z_0^2)^{3/2}}\vec{k}$. **4.93.** $-\dfrac{\lambda_0}{4R\varepsilon_0}\vec{i}$. **4.94.** $Q = \oint\limits_{\mathcal{L}}(\vec{v}\cdot\vec{n})\mathrm{d}l$, where $\vec{n}$ in the outward unit normal vector to the curve $\mathcal{L}$.
4.95. 6. **4.96.** $\dfrac{4}{3}$. **4.97.** $-\dfrac{14}{15}$. **4.98.** $-\dfrac{5}{12}$. **4.99.** 0. **4.100.** 0. **4.101.** $-2\pi a^2$.
4.102. $-\dfrac{\pi}{4\sqrt{2}}$. **4.103.** $\pi a(a+2b)$. **4.104.** $\dfrac{1}{35}$. **4.105.** $3\sqrt{3}$. **4.106.** $a^2(20 - 6\pi)/24$. **4.107.** $-\dfrac{7}{3}$. **4.108.** 18. **4.109.** $\dfrac{ab}{2}$. **4.110.** $\dfrac{\pi}{4}$. **4.111.** 0.
4.112. $-\dfrac{18}{5}$ **4.113.** -4. **4.114.** -2. **4.115.** $-2\pi a^2$. **4.116.** $\dfrac{\pi a^4}{2}$.
4.117. $-\dfrac{6+\pi}{8}$. **4.118.** $\dfrac{3\pi a^2}{4}$. **4.119.** $\dfrac{7a^3}{10}$. **4.120.** $\dfrac{8}{35}$. **4.121.** $\dfrac{33}{512} - \dfrac{3\ln 2}{32}$.
4.122. $\dfrac{22}{3}$ **4.123.** πe. **4.124.** $2\sqrt{2}\pi a^2$. **4.125.** 0. **4.126.** $-\dfrac{\pi a^2}{6}$. **4.127.** $\dfrac{33}{2}$.
4.128. 1. **4.129.** 4π. **4.130.** -4π. **4.131.** $-\dfrac{4}{3}$. **4.132.** -4. **4.133.** 2π.
4.134. 729π. **4.135.** 0. **4.136.** $\pi\sqrt{2}$. **4.137.** $a^3(4\ln 2 - 3)$. **4.138.** $-\dfrac{7}{30}a^3$.
4.139. $-\dfrac{\pi a^6}{16}$. **4.140.** $2\sqrt{2}\pi a^2 \sin\left(\dfrac{\pi}{4} - \alpha\right)$. **4.141.** $-\dfrac{\pi a^3}{4}$. **4.142.** $-FR$.
4.143. a) $\dfrac{4}{3}$; b) $\dfrac{17}{12}$; c) $\dfrac{3}{2}$ and 1. **4.144.** a) $k\dfrac{a^2-b^2}{2}$; b) 0. **4.145.** $-\dfrac{k\ln 2}{c} \times$

$\times\sqrt{a^2+b^2+c^2}$. **4.146.** $\frac{k}{2}\ln 2$. **4.147.** $\oint\limits_{\mathcal{L}} v_x \mathrm{d}y - v_y \mathrm{d}x$. **4.148.** 0. **4.149.** $-\frac{2\gamma I}{9}\int\limits_0^{2\pi}\sqrt{1+\frac{5}{4}\sin^2 t\mathrm{d}t} \approx -0,44\gamma I$. **4.150.** $-\frac{2\gamma I}{h^2}\vec{e}_\varphi$, where $\vec{e}_\varphi$ is the basis vector of the cylindrical coordinate system (see section 5.4). **4.151.** $\frac{\sqrt{3}}{120}$. **4.152.** $8\sqrt{3}$. **4.153.** $26\sqrt{14}$. **4.154.** $4\sqrt{61}$. **4.155.** $\frac{3-\sqrt{3}}{2}+$ $+(\sqrt{3}-1)\ln 2$. **4.156.** $40a^4$. **4.157.** $\frac{\pi a^3}{4}$. **4.158.** πa^3. **4.159.** $\frac{\pi a^4}{3}$ **4.160.** $\frac{2\pi a^6}{15}$. **4.161.** $4a^2h+\pi ah^2$. **4.162.** $2\pi a\left(\frac{a^3}{2}+a^2h+\frac{ah^2}{2}+\frac{h^3}{3}\right)$. **4.163.** $a^8\left(\frac{5\pi}{16}+\frac{20}{21}\right)$. **4.164.** $\frac{1}{420}\left(271\sqrt{2}+16-105\ln(1+\sqrt{2})\right)$. **4.165.** $\frac{\pi a^4}{2}$. **4.166.** $3\sqrt{2}\pi a^{5/2}$. **4.167.** $-\frac{8}{3}\sqrt{2}a^3$. **4.168.** $\frac{1}{512}\Bigg(-396\sqrt{6}+$ $+392\sqrt{34}+225\ln\frac{12\sqrt{6}+33}{8\sqrt{34}+49}\Bigg)$. **4.169.** $\frac{16}{3}a^4$. **4.170.** $\frac{\pi}{2}(1+\sqrt{2})$. **4.171.** $\frac{2\sqrt{2}\pi h^3}{3}$. **4.172.** $2\sqrt{2}\pi b^2(2b^2-1)$. **4.173.** $\frac{64\sqrt{2}a^4}{15}$. **4.174.** $\frac{19\pi}{6}a^3$. **4.175.** $\frac{4}{15}a^5$. **4.176.** $\frac{5\pi a^3}{6}$. **4.177.** $\frac{\pi R^4}{2}\sin\alpha\cos^2\alpha$. **4.178.** $\frac{125\sqrt{5}-1}{420}$. **4.179.** $\frac{4\pi}{15}(1+6\sqrt{3})$. **4.180.** $\frac{apq}{2}+\frac{\pi pqa^4}{16}$. **4.181.** $\frac{1}{8a}(8a^2b^2+\pi b^4)$. **4.182.** $\frac{\pi}{5}(9\sqrt{3}-1)$. **4.183.** 0. **4.184.** $\pi^2\left(a\sqrt{1+a^2}+\ln(a+\sqrt{1+a^2})\right)$. **4.185.** $\pi a(a^2+2b^2+2ab)$. **4.186.** $\pi a^2\left(1+\frac{1}{\sqrt{2}}\ln(1+\sqrt{2})\right)$. **4.187.** $\frac{128\pi a^4}{9}$. **4.188.** $\frac{4}{15}(64-43\sqrt{2})a^3$ **4.189.** 4π. **4.190.** $\frac{\sqrt{2}}{16}+\frac{5}{16}\ln(1+\sqrt{2})$. **4.191.** $(4\pi-2\sqrt{3})a^4$. **4.192.** $\frac{4\pi}{3}abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$. **4.194.** $F(t)=\frac{\pi}{18}\times$ $\times(3-t^2)^2$, if $|t|\leqslant\sqrt{3}$; $F(t)=0$, if $|t|>\sqrt{3}$. **4.195.** $F(t)=\frac{\pi(8-5\sqrt{2})}{6}t^4$. **4.196.** $F(t)=\frac{\pi t}{r}\left(a^2-(r-t)^2\right)$, if $|t-r|<a$; $F(t)=0$, if $|t-r|\geqslant a$. **4.197.** $\frac{2\pi}{3}(3\sqrt{3}-1)$. **4.198.** $\frac{2\pi}{15}(1+6\sqrt{3})$. **4.199.** $a^3(\pi+4)$. **4.200.** $\frac{19}{6}\pi a^3$.

4.201. $256\sqrt{2}\pi$. **4.202.** $\frac{8}{3}\pi a^4$. **4.203.** πa^2. **4.204.** $7\sqrt{2}\pi a^4$. **4.205.** $\frac{2^{11}\pi a^4}{35}$. **4.206.** $\frac{8}{3}\pi a^4$. **4.207.** $\frac{\pi a}{12}(3a^2+2b^2)\sqrt{a^2+b^2}$. **4.208.** $\left(\frac{\pi}{3}-\frac{26}{45}\right)a^4$. **4.209.** $\pi a^4\frac{\cos\alpha}{\sin^3\alpha}$. **4.210.** $40a^4$. **4.211.** $I_{xy}=I_{xz}=I_{yz}=\frac{\sqrt{3}}{12}$. **4.212.** $I_x=$ $=I_y=\pi^2ab(5a^2+2b^2)$, $I_z=2\pi^2ab(3a^2+2b^2)$. **4.213.** $\frac{4\pi}{15}(1+6\sqrt{3})a^4$. **4.214.** $\frac{2\pi a}{3}(2a^3-3a^2h+h^3)$. **4.215.** $x_c=y_c=0$, $z_c=\frac{a}{2}$. **4.216.** $x_c=y_c=$ $=z_c=\frac{a}{2}$. **4.217.** $x_c=y_c=0$, $z_c=\frac{3a}{8}$. **4.218.** $x_c=\frac{1}{5}\cdot\frac{55+\sqrt{3}}{26}$, $y_c=z_c=0$. **4.219.** $x_c=y_c=0$, $z_c=\frac{55+9\sqrt{3}}{130}a$. **4.220.** $x_c=y_c=0$, $z_c=\frac{2}{3}h$. **4.221.** $x_c=y_c=0$, $z_c=a\pi$. **4.222.** $\vec{F}=2\pi\rho_0 Gma\times$ $\times\left(\frac{1}{a}-\frac{1}{\sqrt{a^2+h^2}}\right)\vec{k}$. **4.223.** $\vec{F}=-\frac{qQ}{4\sqrt{2}\pi\varepsilon_0}\frac{\ln a-\ln b}{a^2-b^2}\vec{k}$. **4.224.** $\vec{E}=$ $=-\frac{R\vec{a}}{3\varepsilon_0}$. **4.225.** 3. **4.226.** $abc\left(\frac{f(a)-f(0)}{a}+\frac{g(b)-g(0)}{b}+\frac{h(c)-f(0)}{c}\right)$. **4.227.** $\frac{1}{4}$. **4.228.** $\frac{43}{24}$. **4.229.** $\frac{8}{3}$. **4.230.** $\frac{1}{3}\pi h^3$. **4.231.** 0. **4.232.** -90π. **4.233.** -12π. **4.234.** 2π. **4.235.** 0. **4.236.** 4π. **4.237.** 0. **4.238.** 8π. **4.239.** $-\frac{b^4}{48}$. **4.240.** $\frac{\pi h^4}{4}$. **4.241.** 0. **4.242.** $-\pi$. **4.243.** $-\frac{2^8\pi}{3}$. **4.244.** $-\pi b^4$. **4.245.** $-\frac{243\pi}{2}$. **4.246.** 0. **4.247.** -2π. **4.248.** 0. **4.249.** $\frac{2\pi a^3}{3}$. **4.250.** $\frac{23\pi}{60}$. **4.251.** $\frac{2}{3}ab(b^2+2a^2)$. **4.252.** 88. **4.253.** 0. **4.254.** $2\pi a^2h$. **4.255.** 2π. **4.256.** 0. **4.257.** $6\pi a^2$. **4.258.** $2\pi a^2h$. **4.259.** $-2\pi a^2h$. **4.260.** 0. **4.261.** π. **4.262.** 0. **4.263.** 8. **4.264.** 324. **4.265.** $-\frac{4}{15}\left(12ap^7q+20bp^3q+5cpq^3\right)$. **4.266.** $\frac{16}{3}$. **4.267.** $\frac{17\pi a^4}{32\sqrt{3}}$. **4.268.** $\frac{\pi}{2}$. **4.269.** $\frac{2\pi}{3}\left(1-\frac{\sqrt{2}}{4}\right)$. **4.270.** $\frac{3}{8}a^4$. **4.271.** $-\pi\sqrt{2}$. **4.272.** 2π. **4.273.** $\frac{8\pi}{3}R^3(a+b+c)$. **4.274.** $\frac{3\pi}{8}a^4$. **4.275.** $\pi/5$ **4.276.** πa^2R. **4.277.** $-4\pi abc$. **4.278.** $4\pi\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ac}{b}\right)$. **4.279.** $\frac{4\pi\left(a^2+2c^2\right)}{c\sqrt{a^2-c^2}}\times$

$\times \arccos \frac{c}{a}$. **4.280.** 24π. **4.281.** $\frac{432}{5}\pi$. **4.282.** $\frac{\pi}{4}$. **4.283.** πa^3. **4.284.** $\frac{5}{3}+$ $+\frac{\pi}{15}(10\sqrt{2}-11)$ **4.285.** $\frac{16a^4}{15}$ **4.286.** $\pi a(\pi+1)$. **4.287.** $\frac{869}{90}$. **4.288.** $-\frac{56}{3}$. **4.289.** $\frac{40}{3}$. **4.290.** $-\frac{1}{6}$ **4.291.** $\ln 2$. **4.292.** 0. **4.293.** $-\frac{\pi a^4}{4}$. **4.294.** $\frac{\pi a^2}{8}$. **4.295.** $-\frac{\pi a^3}{8}$. **4.296.** πab. **4.297.** 0. **4.298.** $-\frac{15\pi}{32}a^{8/3}$. **4.299.** $-\frac{5\pi}{2}a^3$. **4.300.** 3. **4.301.** $-\frac{40}{3}$. **4.302.** 0. **4.303.** $\frac{16}{\pi^3}$. **4.304.** $\frac{\pi a^3}{16} - \frac{a^3}{12} +$ $+\frac{a^2}{2}$. **4.305.** $\frac{\pi a^2}{2} - 2a$. **4.306.** $\frac{a^3}{3}(3\pi + 10)$. **4.307.** $\frac{\pi a^4}{32}$. **4.308.** $\frac{3}{8}\pi ab$. **4.309.** $\frac{4a^2}{3}$. **4.310.** $6\pi a^2$. **4.311.** $\frac{4}{3}$. **4.312.** $\frac{8\pi}{3}$. **4.313.** a^2. **4.314.** $\frac{5\pi a^2}{8}$. **4.315.** $\frac{2a^2}{3}$. **4.316.** $\frac{3}{2}$. **4.317.** $\frac{1}{3} + \frac{4\pi}{9\sqrt{3}}$. **4.318.** $\frac{1}{30}$. **4.319.** $\frac{a^2}{2}B(2m+1,$ $2n+1)$. **4.320.** $\frac{abc^2}{2(2n+1)}$. **4.321.** 0. **4.322.** $3^6\pi$. **4.323.** 0. **4.324.** -54π. **4.325.** -2π. **4.326.** $2\pi a^2$. **4.327.** -8π. **4.328.** $-2\sqrt{2}\pi a^3$. **4.329.** $\frac{3\pi a^4}{2}$. **4.330.** $2\pi ab^2$. **4.331.** $2\sqrt{2}\pi a^2 \sin\left(\frac{\pi}{4} - \alpha\right)$. **4.332.** $-2a^3$. **4.333.** a) 0; b) -2. **4.334.** $\frac{\pi a^2}{\sqrt{3}}$. **4.335.** -16π. **4.336.** -2π **4.337.** $\pi\sqrt{3}\left(a^2 - \frac{t^2}{3}\right)$. **4.338.** $-\frac{32}{3}$. **4.339.** $-\frac{4}{3}a^2$. **4.340.** $-2\pi a^3$. **4.341.** $-\frac{3\pi a^2}{4\sqrt{2}}$. **4.342.** $\frac{\pi a^4}{4}$. **4.343.** $-\frac{32}{15}a^3$. **4.344.** $-\frac{1}{12}$. **4.345.** 36. **4.346.** 0. **4.347.** $\frac{5}{6}$. **4.348.** 0. **4.349.** $\frac{2}{3}$. **4.350.** $\frac{12}{5}\pi a^5$. **4.351.** $8\pi^2 ab^2$. **4.352.** 0. **4.353.** $24\pi a^3$. **4.354.** 8π. **4.355.** $\frac{\pi}{3}$. **4.356.** 32π. **4.357.** $\frac{2\pi h^3}{3}$ **4.358.** $\frac{1}{7}$. **4.359.** $-\frac{19\pi}{2}$. **4.360.** $\frac{\pi a^5}{60}$. **4.361.** $-\frac{3\pi a^4}{8}$. **4.362.** $-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+5)}$. **4.363.** π. **4.364.** 0. **4.365.** 0. **4.366.** $\frac{2\pi a^5}{5} + \frac{\pi a^4}{4}$. **4.367.** $-\frac{\pi h^4}{4}$. **4.368.** $-\frac{\pi h^4}{2}$.

Chapter 5

5.13. $\oiint\limits_{\Omega} u \operatorname{curl} \vec{a} \cdot \mathrm{d}\vec{S}$. **5.18.** $-12\pi a^2$. **5.19.** 80π. **5.20.** $4\pi a^2$. **5.21.** 0. **5.23.** $I = 2S$, where S is the area of the domain D. **5.24.** $u = 2\pi$, if the point (x_0, y_0) is inside the closed curve $\mathcal{L}$; $u = \pi$, if the point (x_0, y_0) belongs to the curve $\mathcal{L}$; $u = 0$, if the point (x_0, y_0) is outside $\mathcal{L}$. **5.27.** a) $u = 0$; b) $u = 4\pi$. **5.30.** $u = xy + yz + zx + C$. **5.31.** $u = x + xyz + C$. **5.32.** $u = x^2y - y^2 + xz + C$. **5.33.** $u = \arctan(xyz) + C$. **5.34.** $u = \ln(x + y + z) + C$. **5.35.** $u = xy + e^z + C$. **5.36.** $u = x^2yz + C$. **5.37.** $u = x^2yz + xy^2z + xyz^2 + C$. **5.38.** $u = z + e^x \sin y + C$. **5.39.** $u = z\sin(xy) + C$. **5.40.** a) $u = r + C$; b) $u = \ln r + C$, c) $u = \frac{1}{3}r^3 + C$. **5.41.** 4. **5.42.** 1. **5.43.** 37. **5.44.** 4. **5.45.** 4. **5.46.** -2. **5.47.** $\int\limits_0^{a+b} f(u)\mathrm{d}u$. **5.48.** $\pi + 1$. **5.49.** $\arctan 3 - \arctan 5$. **5.50.** $-\arctan\frac{3}{2}$. **5.51.** 9. **5.52.** 1. **5.53.** 123. **5.54.** 4095. **5.55.** $-\frac{1}{3}$. **5.56.** $-\frac{9}{2}$. **5.57.** $b - a$. **5.58.** $\int\limits_0^{\sqrt{a^2+b^2+c^2}} uf(u)\mathrm{d}u$. **5.59.** $u = x^2\cos y + y^2\cos x + C$. **5.60.** $u = x + ye^{x/y} + C$. **5.61.** $u = \sqrt{x^2 + y^2} + \frac{y}{x}$. **5.62.** $u = x\sqrt{1 - y^2} + y\sqrt{1 - x^2} + \arctan\frac{y}{x} + \ln y + C$. **5.63.** $u = x^2y + y^2z + z^2x + xyz + C$. **5.64.** $u = x^2yz + \frac{x}{z} - \frac{y}{z^2} + C$. **5.65.** $\vec{A} = x\vec{j} + (y - x)\vec{k}$. **5.66.** $\vec{A} = (y^2 - 2xz)\vec{k}$. **5.67.** $\vec{A} = (e^x - xe^y)\vec{j}$. **5.68.** $\vec{A} = -\frac{2}{z}\sin(zx)\,\vec{k}$. **5.69.** $\vec{A} = -5x^2yz\,\vec{j}$. **5.70.** $\vec{A} = 9zx\,\vec{j} + 15yx\,\vec{k}$. **5.71.** $\vec{A} = -\ln\sqrt{x^2 + y^2}\,\vec{k}$. **5.72.** $\vec{A} = \left(\frac{x^2}{2} + xy\right)\vec{j} + \left(-\frac{x^2}{2} - zx + yz + \frac{y^2}{2}\right)\vec{k}$. **5.73.** $\vec{A} = x^2\vec{j} + (xz + y^2)\vec{k}$. **5.74.** $\vec{A} = \left(zy^2x - \frac{zx^3}{3}\right)\vec{j} + \left(z^2yx - \frac{yx^3}{3}\right)\vec{k}$. **5.75.** $\vec{A} = 3xy^2z\,\vec{j} + \left(\frac{x^3 + y^3}{3} + y^3x\right)\vec{k}$. **5.76.** $\vec{A} = (z^2xy - 2zxy + x)\vec{j} + (y^2zx + y)\vec{k}$. **5.77.** $\vec{A} = 3x^2\vec{j} + (2y^3 - 6xz)\vec{k}$. **5.78.** $\vec{A} = -(xz^2 + yze^{x^2})\vec{j} - 2xyz\vec{k}$. **5.79.** b) 1) $\frac{\pi}{2}$, 2) 0, 3) π, 4) 0; c) 1) at the point A: $\vec{i} = \vec{e}_\rho$, $\vec{j} = \vec{e}_\varphi$, $\vec{k} = \vec{e}_z$; at the point B: $\vec{i} = -\vec{e}_\varphi$, $\vec{j} = \vec{e}_\rho$, $\vec{k} = \vec{e}_z$; 2) at the point A: $\vec{e}_\rho = \vec{i}$, $\vec{e}_\varphi = \vec{j}$, $\vec{e}_z = \vec{k}$;

at the point B: $\vec{e}_\rho = \vec{j}$, $\vec{e}_\varphi = -\vec{i}$, $\vec{e}_z = \vec{k}$. **5.80.** b) 1) $\dfrac{\pi}{2}$, 2) 0, 3) π, 4) 0; c) 1) at the point A: $\vec{i} = \vec{e}_r, \vec{j} = \vec{e}_\varphi$, $\vec{k} = -\vec{e}_\theta$; at the point B: $\vec{i} = -\vec{e}_\varphi$, $\vec{j} = \vec{e}_r$, $\vec{k} = -\vec{e}_\theta$; 2) at the point A: $\vec{e}_r = \vec{i}$, $\vec{e}_\theta = -\vec{k}$, $\vec{e}_\varphi = \vec{j}$; at the point B: $\vec{e}_r = \vec{j}$, $\vec{e}_\theta = -\vec{k}$, $\vec{e}_\varphi = -\vec{i}$.

5.81. Elliptic cylindrical coordinates. Coordinate surfaces:
a) elliptic cylinders:

$$\frac{x^2}{a^2 \cosh^2 u} + \frac{y^2}{a^2 \sinh^2 u} = 1, \quad u = \text{const};$$

b) hyperbolic cylinders:

$$\frac{x^2}{a^2 \cos^2 v} - \frac{y^2}{a^2 \sin^2 v} = 1, \quad v = \text{const};$$

c) planes parallel to the plane xOy: $z = \text{const}$.
Lamé coefficients:

$$H_u = a\sqrt{\sinh^2 u + \sin^2 v}, \quad H_v = a\sqrt{\sinh^2 u + \sin^2 v}, \quad H_z = 1.$$

5.82. Parabolic cylindrical coordinates. Coordinate surfaces:
a) confocal straight parabolic cylinders:

$$\frac{x^2}{\xi^2} = 2y + \xi^2, \quad \xi = \text{const};$$

b) confocal straight parabolic cylinders:

$$\frac{x^2}{\eta^2} = -2y + \eta^2, \quad \eta = \text{const};$$

c) planes parallel to the plane xOy: $z = \text{const}$.
Lamé coefficients:

$$H_\xi = \sqrt{\xi^2 + \eta^2}, \quad H_\eta = \sqrt{\xi^2 + \eta^2}, \quad H_z = 1.$$

5.83. Bipolar coordinates. Coordinate surfaces:
a) circular cylinders:

$$x^2 + (y - a\cot\xi)^2 = \frac{a^2}{\sin^2 \xi}, \quad \xi = \text{const};$$

b) circular cylinders:

$$(x - a\cot\eta)^2 + y^2 = \frac{a^2}{\sinh^2\eta}, \quad \mu = \text{const};$$

c) planes parallel to the plane xOy: $z = \text{const}$.
Lamé coefficients:

$$H_\xi = \frac{a}{\cosh\eta - \cos\xi}, \quad H_\eta = \frac{a}{\cosh\eta - \cos\xi}, \quad H_z = 1.$$

5.84. Ellipsoidal coordinates. Relation to cartesian coordinates:

$$x^2 = \frac{(a^2+\xi)(a^2+\eta)(a^2+\zeta)}{(a^2-b^2)(a^2-c^2)},$$

$$y^2 = \frac{(b^2+\xi)(b^2+\eta)(b^2+\zeta)}{(b^2-c^2)(b^2-a^2)},$$

$$z^2 = \frac{(c^2+\xi)(c^2+\eta)(c^2+\zeta)}{(c^2-a^2)(c^2-b^2)}.$$

Lamé coefficients:

$$H_\xi = \frac{1}{2}\sqrt{\frac{(\xi-\eta)(\xi-\zeta)}{(a^2+\xi)(b^2+\xi)(c^2+\xi)}},$$

$$H_\eta = \frac{1}{2}\sqrt{\frac{(\eta-\zeta)(\xi-\eta)}{(a^2+\eta)(b^2+\eta)(c^2+\eta)}},$$

$$H_\zeta = \frac{1}{2}\sqrt{\frac{(\xi-\zeta)(\eta-\zeta)}{(a^2+\zeta)(b^2+\zeta)(c^2+\zeta)}}.$$

5.85. Prolate ellipsoidal system. Coordinate surfaces:
a) prolate ellipsoids of revolution:

$$\frac{x^2+y^2}{a^2\sinh^2 u} + \frac{z^2}{a^2\cosh^2 u} = 1, \quad u = \text{const};$$

b) two-sheeted hyperboloids of revolution:

$$-\frac{x^2+y^2}{a^2\sin^2 v} + \frac{z^2}{a^2\cos^2 v} = 1, \quad v = \text{const};$$

c) half-planes adjoining the axis Oz: $\varphi = \text{const}$.
Lamé coefficients:

$$H_u = a\sqrt{\sinh^2 u + \sin^2 v}, \quad H_v = a\sqrt{\sinh^2 u + \sin^2 v},$$

$$H_\varphi = a \sinh u \sin v.$$

5.86. Oblate ellipsoidal system. Coordinate surfaces:
a) oblate ellipsoids of revolution:

$$\frac{x^2 + y^2}{a^2 \cosh^2 u} + \frac{z^2}{a^2 \sinh^2 u} = 1, \quad u = \text{const};$$

b) one-sheeted hyperboloids of revolution:

$$\frac{x^2 + y^2}{a^2 \sin^2 v} - \frac{z^2}{a^2 \cos^2 v} = 1, \quad v = \text{const};$$

c) half-planes adjoining the axis Oz: $\varphi = \text{const}$.
Lamé coefficients:

$$H_u = a\sqrt{\sinh^2 u + \cos^2 v}, \quad H_v = a\sqrt{\sinh^2 u + \cos^2 v},$$

$$H_\varphi = a \cosh u \sin v.$$

5.87. Parabolic coordinates. Coordinate surfaces:
a) confocal paraboloids:

$$\frac{x^2 + y^2}{\xi^2} = 2z + \xi^2, \quad \xi = \text{const};$$

b) confocal paraboloids:

$$\frac{x^2 + y^2}{\eta^2} = -2z + \eta^2, \quad \eta = \text{const};$$

c) half-planes adjoining the axis Oz: $\varphi = \text{const}$.
Lamé coefficients:

$$H_\xi = \sqrt{\xi^2 + \eta^2}, \quad H_\eta = \sqrt{\xi^2 + \eta^2}, \quad H_z = \xi\eta.$$

5.88. Toroidal coordinates. Coordinate surfaces:

a) spheres:

$$x^2 + y^2 + (z - a\cot\xi)^2 = \frac{a^2}{\sin^2\xi}, \quad \xi = \text{const};$$

b) tori:

$$(\sqrt{x^2+y^2} - a\,\coth\eta)^2 + z^2 = \frac{a^2}{\sinh^2\eta}, \quad \eta = \text{const};$$

c) half-planes adjoining the axis Oz: $\varphi = \text{const}$.
Lamé coefficients:

$$H_\xi = \frac{a}{\cosh\eta - \cos\xi}, \quad H_\eta = \frac{a}{\cosh\eta - \cos\xi}, \quad H_\varphi = \frac{a\sinh\eta}{\cosh\eta - \cos\xi}.$$

5.89. $\rho = C_1\varphi,\ z = C_2$. **5.90.** $r = C_1\sin^2\theta,\ \varphi = C_2$. **5.91.** $\cos\varphi\vec{e}_z + \vec{e}_\rho - \frac{z}{\rho}\sin\varphi\,\vec{e}_\varphi$. **5.92.** $2(\rho+\cos\varphi)\vec{e}_\rho - \left(\frac{e^z}{\rho}\cos\varphi + 2\sin\varphi\right)\vec{e}_\varphi - e^z\sin\varphi\vec{e}_z$. **5.93.** $\cos\varphi\vec{e}_\rho + \left(\frac{z}{\rho}\sin 2\varphi - \sin\varphi\right)\vec{e}_\varphi + \sin^2\varphi\vec{e}_z$. **5.94.** $(2\rho\sin\varphi - 3z)\vec{e}_\rho + \left(\rho\cos\varphi - \frac{z}{\rho}\sin 2\varphi\right)\vec{e}_\varphi + (\cos^2\varphi - 3\rho)\vec{e}_z$. **5.95.** $2r\cos\theta\vec{e}_r - r\sin\theta\vec{e}_\theta$. **5.96.** $(-2+3\cos\theta+e^r\sin\varphi)\vec{e}_r - 3\sin\theta\vec{e}_\theta + \frac{e^r\cos\varphi}{r\sin\theta}\vec{e}_\varphi$. **5.97.** $\left(2r\sin\theta - \frac{1}{r^2}\cos\theta\right)\vec{e}_r + \left(r\cos\theta - \frac{1}{r^2}\sin\theta\right)\vec{e}_\theta$. **5.98.** $-\frac{2\cos\theta}{r^3}\vec{e}_r - \frac{\sin\theta}{r^3}\vec{e}_\theta$. **5.99.** $\operatorname{div}\vec{a} = 3\rho + e^\varphi\cos z - \frac{z}{\rho}\sin\varphi$, $\operatorname{curl}\vec{a} = \left(-\cos\varphi + \frac{e^\varphi}{\rho}\sin z\right)\vec{e}_\rho + \frac{z}{\rho}\cos\varphi\,\vec{e}_z$. **5.100.** $\operatorname{div}\vec{a} = -2ze^z - e^z z^2 + \frac{\varphi}{1+\rho^2} + \frac{\varphi}{\rho}\arctan\rho$, $\operatorname{curl}\vec{a} = \frac{2-\arctan\rho}{\rho}\vec{e}_z$. **5.101.** $\operatorname{div}\vec{a} = -\rho + \frac{1}{\rho}\sin\varphi - \frac{1}{\rho^2}\sin\varphi$, $\operatorname{curl}\vec{a} = z\vec{e}_\varphi - \frac{1}{\rho}\cos\varphi\,\vec{e}_z$. **5.102.** $\operatorname{div}\vec{a} = 1$, $\operatorname{curl}\vec{a} = 2\varphi\vec{e}_z$. **5.103.** $\operatorname{div}\vec{a} = 4r + \frac{1}{r(r^2+1)\sin\theta} - \frac{2\cos\theta}{r\sin\theta}\cos^2\varphi$, $\operatorname{curl}\vec{a} = \frac{1}{r\sin\theta}\left(\frac{\varphi\cos\theta}{1+r^2} - 2\sin 2\varphi\right)\vec{e}_r + \frac{\varphi}{r}\frac{r^2-1}{(1+r^2)^2}\vec{e}_\theta - \frac{2\cos^2\varphi}{r}\vec{e}_\varphi$. **5.104.** $\operatorname{div}\vec{a} = \frac{2}{r}(3r - \cos\theta + \cos\varphi)$, $\operatorname{curl}\vec{a} = \frac{\cos 2\theta}{\sin\theta}\vec{e}_r - \left(2\cos\theta + \frac{\sin\varphi}{r\sin\theta}\right)\vec{e}_\theta - \frac{\sin\theta}{r}\vec{e}_\varphi$. **5.105.** $\operatorname{div}\vec{a} = 4r +$

$+\dfrac{\cos^2\theta}{r\sin\theta}-\dfrac{3}{r}\sin\theta$, $\operatorname{curl}\vec{a} = -\dfrac{\varphi}{r}\cot\theta\vec{e}_r+\dfrac{\varphi}{r}\vec{e}_\theta+\dfrac{2\cos\theta}{r}\vec{e}_\varphi$. **5.106.** $\operatorname{div}\vec{a} = 0$, $\operatorname{curl}\vec{a} = \vec{0}$. **5.108.** 0. **5.109.** $-\dfrac{z\varphi}{\rho}+\dfrac{6z^2\varphi}{\rho^2}+4\varphi+2\varphi^3$. **5.110.** $-\dfrac{2}{r}(1+$ $+2\cos 2\theta)$. **5.111.** $\left(\dfrac{2\theta}{r^2}+\varphi\right)\cot\theta + 2\left(\dfrac{1}{r^2}+3\theta\varphi+6r\varphi^2+\dfrac{r}{\sin^2\theta}\right)$. **5.112.** 16π. **5.113.** $\dfrac{3\pi}{2}$. **5.114.** $2\pi R^4$. **5.115.** 0. **5.116.** $\dfrac{2}{3}\pi R^3$. **5.117.** $\dfrac{\pi R^2}{2}\left(R^2-\dfrac{\pi}{4}\right)$. **5.118.** $2\pi R^4$. **5.119.** $2\sqrt{2}$. **5.120.** $-2\pi a^2$. **5.121.** $2\pi R^2$. **5.122.** π^3. **5.123.** 1. **5.124.** $\dfrac{\pi}{4}+\dfrac{1}{\sqrt{2}}-1$. **5.125.** $\dfrac{\pi^2}{4}-\dfrac{1}{2}$. **5.126.** $-\pi R^2$. **5.127.** $\dfrac{\pi}{2}$. **5.128.** π. **5.129.** $4\pi R^2$. **5.130.** 2. **5.131.** $u = \rho+\varphi+z+C$. **5.132.** $u = \dfrac{\rho^2}{2}+z\varphi+C$. **5.133.** $u = \rho\varphi z+C$. **5.134.** $u = e^\rho\sin\varphi+z^2+C$. **5.135.** $u = \rho\varphi\sin z+C$. **5.136.** $u = \ln\rho\cdot\arctan z - \rho\sin\varphi+C$. **5.137.** $u = \ln\rho\cdot\sin\varphi+z^2+C$. **5.138.** $u = r\theta+C$. **5.139.** $u = e^r\theta+C$. **5.140.** $u = r\varphi\cos\theta+C$. **5.141.** $u = r^2+\theta+\varphi+C$. **5.142.** $u = \dfrac{r}{2}\varphi^2+\dfrac{\theta^2}{2}+C$. **5.143.** $u = r\sin\theta\cos\varphi+C$. **5.144.** $u = e^r\sin\theta+\ln(1+\varphi^2)+C$. **5.145.** $u = e^{\theta\varphi}\ln r+C$. **5.146.** $\operatorname{div}\vec{e}_{q^1} = \dfrac{1}{H_1H_2H_3}\dfrac{\partial(H_2H_3)}{\partial q^1}$, $\operatorname{div}\vec{e}_{q^2} = \dfrac{1}{H_1H_2H_3}\dfrac{\partial(H_1H_3)}{\partial q^2}$, $\operatorname{div}\vec{e}_{q^3} = \dfrac{1}{H_1H_2H_3}\times$ $\times\dfrac{\partial(H_1H_2)}{\partial q^3}$; $\operatorname{div}\vec{e}_\rho = \dfrac{1}{\rho}$, $\operatorname{div}\vec{e}_\varphi = 0$, $\operatorname{div}\vec{e}_z = 0$; $\operatorname{div}\vec{e}_r = \dfrac{2}{r}$, $\operatorname{div}\vec{e}_\theta = \dfrac{1}{r}\cot\theta$, $\operatorname{div}\vec{e}_\varphi = 0$. **5.147.** $\operatorname{curl}\vec{e}_\rho = \vec{0}$, $\operatorname{curl}\vec{e}_\varphi = \dfrac{1}{\rho}\vec{e}_z$, $\operatorname{curl}\vec{e}_z = \vec{0}$; $\operatorname{curl}\vec{e}_r = \vec{0}$, $\operatorname{curl}\vec{e}_\theta = \dfrac{1}{r}\vec{e}_\varphi$, $\operatorname{curl}\vec{e}_\varphi = \dfrac{1}{r}\cot\theta\vec{e}_r-\dfrac{1}{r}\vec{e}_\theta$. **5.148.** a) $u = C_1\ln\rho+C_2$, b) $u = C_1\varphi+C_2$, c) $u = C_1z+C_2$. **5.149.** a) $u = \dfrac{C_1}{r}+C_2$, b) $u = C_1\ln\tan\dfrac{\theta}{2}+C_2$, c) $u = C_1\varphi+C_2$. **5.150.** $u = \dfrac{r^{n+1}}{(n+1)(n+2)}+$ $+\dfrac{C_1}{r}+C_2$.

5.151. Elliptic cylindric coordinates:

$$\Delta\Phi = \frac{1}{a^2(\sinh^2 u+\sin^2 v)}\left(\frac{\partial^2\Phi}{\partial u^2}+\frac{\partial^2\Phi}{\partial v^2}\right)+\frac{\partial^2\Phi}{\partial z^2}.$$

Parabolic cylindrical coordinates:

$$\Delta\Phi = \frac{1}{\xi^2+\eta^2}\left(\frac{\partial^2\Phi}{\partial\xi^2}+\frac{\partial^2\Phi}{\partial\eta^2}\right)+\frac{\partial^2\Phi}{\partial z^2}.$$

Bipolar coordinates:

$$\Delta\Phi = \frac{1}{a^2}(\cosh^2\eta-\cos\xi)\left(\frac{\partial^2\Phi}{\partial\xi^2}+\frac{\partial^2\Phi}{\partial\eta^2}\right)+\frac{\partial^2\Phi}{\partial z^2}.$$

Ellipsoidal coordinates:

$$\Delta\Phi = \frac{4f(\xi)}{(\xi-\eta)(\xi-\zeta)}\frac{\partial}{\partial\xi}\left(\sqrt{f(\xi)}\frac{\partial\Phi}{\partial\xi}\right)+$$
$$+\frac{4f(\eta)}{(\eta-\xi)(\eta-\zeta)}\frac{\partial}{\partial\eta}\left(\sqrt{f(\eta)}\frac{\partial\Phi}{\partial\eta}\right)+$$
$$+\frac{4f(\zeta)}{(\zeta-\eta)(\zeta-\xi)}\frac{\partial}{\partial\zeta}\left(\sqrt{f(\zeta)}\frac{\partial\Phi}{\partial\zeta}\right),$$
$$f(t)=(a^2+t)(b^2+t)(c^2+t).$$

Prolate ellipsoidal coordinates:

$$\Delta\Phi = \frac{1}{a^2(\sinh^2 u+\sin^2 v)}\left[\frac{1}{\sinh u}\frac{\partial}{\partial u}\left(\sinh u\frac{\partial\Phi}{\partial u}\right)+\right.$$
$$\left.+\frac{1}{\sin v}\frac{\partial}{\partial v}\left(\sin v\frac{\partial\Phi}{\partial v}\right)\right]+\frac{1}{a^2\sinh^2 u\ \sin^2 v}\frac{\partial^2\Phi}{\partial\varphi^2}.$$

Oblate ellipsoidal coordinates:

$$\Delta\Phi = \frac{1}{a^2(\sinh^2 u+\cos^2 v)}\left[\frac{1}{\cosh u}\frac{\partial}{\partial u}\left(\cosh u\frac{\partial\Phi}{\partial u}\right)+\right.$$
$$\left.+\frac{1}{\cos v}\frac{\partial}{\partial v}\left(\cos v\frac{\partial\Phi}{\partial v}\right)\right]+\frac{1}{a^2\cosh^2 u\ \sin^2 v}\frac{\partial^2\Phi}{\partial\varphi^2}.$$

Parabolic coordinates:

$$\Delta\Phi = \frac{1}{\xi^2+\eta^2}\left[\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Phi}{\partial\xi}\right)+\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\Phi}{\partial\eta}\right)\right]+\frac{1}{\xi^2\eta^2}\frac{\partial^2\Phi}{\partial z^2}.$$

Toroidal coordinates:

$$\Delta\Phi = \frac{(\cosh\eta - \cos\xi)^3}{a^2 \sinh\eta}\left[\frac{\partial}{\partial\xi}\left(\frac{\sin\eta}{\cosh\eta - \cos\xi}\frac{\partial\Phi}{\partial\xi}\right) + \right.$$

$$\left. + \frac{\partial}{\partial\eta}\left(\frac{\sinh\eta}{\cosh\eta - \cos\xi}\frac{\partial\Phi}{\partial\eta}\right) + \frac{1}{\sinh\eta(\cosh\eta - \cos\xi)}\frac{\partial^2\Phi}{\partial\varphi^2}\right].$$

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Contents

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