Lower bounds for the number of integer polynomials with given order of discriminants and Diophantine Approximations

by

Vasili Bernik (Minsk), Friedrich Götze (Bielefeld)
and Olga Kukso (Minsk)

Abstract

Let $Q \in \mathbb{R}$ be a sufficiently large number, $P(x) \in \mathbb{Z}[x]$ denotes a polynomial with degree $\deg P \leqslant n$ and height $H(P) \leqslant Q$. The discriminant of the polynomial $P(x)$ we denote as $D(P)$. In the paper we obtain the lower bound for the number of the polynomials with the absolute value of their discriminants less than arbitrarily chosen value in assumption that $n \geqslant 3$.

1 Introduction

The discriminant of a polynomial is one of its main characteristics both in algebra and in number theory. For example, if one consider the polynomial of second degree $P(x) = ax^2+bx+c$, the value of the discriminant $D = b^2-4ac$ is necessary for calculating roots, determining whether they are real or not.

There are two ways to define the discriminant $D(P)$ of the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with height $H = H(P) = \max_{0 \leqslant j \leqslant n} |a_j|$ and roots α_1, α_2, ...,α_n.
$D(P)$ can be defined as a determinant

$$D(P) = (-1)^{C_n^2} \begin{vmatrix} 1 & a_{n-1} & a_{n-2} & \dots & a_0 & 0 & \dots \\ 0 & a_n & a_{n-1} & \dots & a_1 & a_0 & \dots \\ & & & \dots & & & \\ 0 & \dots & 0 & a_n & \dots & a_1 & a_0 \\ n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & a_1 & 0 & \dots \\ 0 & na_n & (n-1)a_{n-1} & \dots & 2a_2 & a_1 & \dots \\ & & & \dots & & & \\ 0 & \dots & \dots & 0 & na_n & \dots & a_1 \end{vmatrix} \qquad (1)$$

or as the transformed product of roots differences

$$D(P) = a_n^{2n-2} \prod_{1 \leqslant i < j \leqslant n} (\alpha_i - \alpha_j)^2. \qquad (2)$$

In the following we consider polynomials with integer rational coefficients only. Hence from (1) we obtain, that if $D(P) \neq 0$ then

$$|D(P)| \geqslant 1. \tag{3}$$

From (2) it obviously follows that $D(P) \neq 0$ if and only if $P(x)$ has no multiple roots.

Now fix $n \in \mathbb{N}$. Let Q be a sufficiently large number, $Q > Q_0(n)$. By $\mathcal{P}_n(Q)$ we denote the class of polynomials $P(x)$ with $\deg P \leqslant n$ and $H(P) \leqslant Q$. By $c(n)$, c_j, $j = 0, 1, \ldots,$ we denote constants depending on n only. We shall use Vinogradov's symbols: $A \ll B$ denotes that $A \leqslant c_0 B$, and $A \asymp B$ denotes $B \ll A \ll B$. From (1) we obtain that $|D(P)| < c(n)Q^{2n-2}$ and if $D(P) \neq 0$, by (3) we have

$$1 \leqslant |D(P)| < c(n)Q^{2n-2}. \tag{4}$$

From the restrictions on degree and height of polynomials we obtain that

$$\#\mathcal{P}_n(Q) < 2^{2n+2}Q^{n+1}.$$

The last estimate and (4) ensure the existence of intervals in $[1, c(n)Q^{2n-2}]$ of length $c(n)Q^{n-3}$, such that we can't find a polynomial from the class $\mathcal{P}_n(Q)$ with discriminant from these intervals. For $n \geqslant 4$ these intervals can be arbitrary large. In the paper we show how methods of Diophantine approximation theory allow to obtain lower boundary for the number of polynomials $P(x) \in \mathcal{P}_n(Q)$, whose values of discriminants are close to the maximum possible.

Note that the distribution of discriminants of integer polynomials is important for Diophantine equations [1, 2, 3] and Diophantine approximations [4, 5]. .

2 Main theorems

In the paper we prove two theorems.

Theorem 1 *There are at least $c(n)Q^{n+1-2v}$ polynomials $P(x)$ in $\mathcal{P}_n(Q)$ with discriminants*

$$|D(P)| < Q^{2n-2-2v}, \tag{5}$$

where $v \in [0, \frac{1}{2}]$.

Theorem 2 *Let Q denote a sufficiently large number, and let c_1 and c_2 denote constants depending on n only such that $c_1c_2 < n^{-1}2^{-n-11}$, $v \in [0, \frac{1}{2}]$. Define $\mathcal{L}_{n,Q}(c_1, c_2)$ as the set of $x \in I \subset [-\frac{1}{2}, \frac{1}{2}]$ such that the system of inequalities*

$$\begin{cases} |P(x)| < c_1 Q^{-n+v}, \\ |P'(x)| < c_2 Q^{1-v} \end{cases}$$

has solutions in polynomials $P(x) \in \mathcal{P}_n(Q)$. Then

$$\mu \mathcal{L}_{n,Q}(c_1, c_2) < \frac{|I|}{2}.$$

The first theorem deals with the topic described in the title. The second theorem shows that Minkowski's theorem about linear forms can't be considerably improved for most points of any interval $I \subset \mathbb{R}$. We shall show how Theorem 1 follows from Theorem 2. Let $P(x) \in \mathbb{Z}[x]$, $\deg P \leqslant n$, $|a_n| > cH$.

In Lemma 6 below we show that the last inequality does not impose a restriction and that polynomials with $|a_n| \leqslant cH$ can be transformed into polynomials with a large highest coefficient without changing the value of the discriminant.

For any point $x \in I \subset \mathbb{R}$ and $Q > 1$ we shall prove using Dirichlet's principle that there are two real positive numbers c_3 and c_4 wth $\max(c_3, c_4) \leqslant 1$ and $c_3c_4 > 8n$ such that the following system of inequalities holds

$$\begin{cases} |P(x)| < c_3 Q^{-n+v}, \\ |P'(x)| < c_4 Q^{1-v}, \ H(P) \leqslant Q, \end{cases} \tag{6}$$

for some polynomials $P \in \mathcal{P}_n(Q)$.

Let $c_3 = 1$, $c_4 = 8n$. Then the system (6) may be rewritten as

$$\begin{cases} |P(x)| < Q^{-n+v}, \\ |P'(x)| < 8nQ^{1-v}. \end{cases} \tag{7}$$

The existence of solutions to (7) and Theorem 2 imply that for $\gamma = n^{-1}2^{-n-12}$ the system of inequalities

$$\begin{cases} \gamma Q^{-n+v} < |P(x)| < Q^{-n+v}, \\ \gamma Q^{1-v} < |P'(x)| < 8nQ^{1-v} \end{cases} \tag{8}$$

has solutions in $P \in \mathcal{P}_n(Q)$ for all $x \in B_1$, $\mu B_1 \geqslant \frac{|I|}{2}$. Indeed if one of the inequalities in (8) doesn't hold then $|P(x)| \leqslant \gamma Q^{-n+v}$, $|P'(x)| < 8nQ^{-1-v}$ and $8n\gamma < 2^{-n-9}$. If $|P'(x)| < \gamma Q^{1-v}$, $|P(x)| \leqslant Q^{-n+v}$ then $c_1c_2 < n^{-1}2^{-n-12}$.

The claim reduces to the fact that the system of inequalities doesn't hold on the set B with measure $\mu B < \frac{|I|}{2}$ and holds for all $x \in B_1 = I \setminus B$, $\mu B_1 \geqslant \frac{|I|}{2}$.

Let us choose $x_1 \in B_1$. Then we can find a polynomial $P_1(x)$, for which the system (8) holds for $x = x_1$. For all x in the interval $|x - x_1| < Q^{-\frac{2}{3}}$, we obtain by Lagrange's theorem about finite increments gives

$$P_1'(x) = P_1'(x_1) + P_1''(\xi_1)(x - x_1), \text{ for some } \xi_1 \in [x, x_1]. \tag{9}$$

The evident estimate $|P''(\xi_2)| < n^3 Q$ implies $|P''(\xi_1)(x - x_1)| < n^3 Q^{\frac{1}{3}}$. But $|P_1'(x_1)| \gg Q^{\frac{1}{2}}$ for $v \leqslant \frac{1}{2}$ and therefore for sufficiently large Q from (9) and the second inequality in (8) we obtain

$$\frac{\gamma}{2} Q^{1-v} < \frac{1}{2}|P_1'(x_1)| < |P_1'(x)| < 2|P_1'(x_1)| < 16nQ^{1-v}.$$

In view of the values of $P(x_1)$ and $P'(x_1)$ in (8) we can distinguish four possible combinations of signs. We will consider one of them: $P_1(x_1) < 0$, $P_1'(x_1) > 0$. The remains ones can be dealt with in a similar manner. Again we use Lagrange's theorem.

$$P_1(x) = P_1(x_1) + P_1'(\xi_2)(x - x_1), \text{ for some } \xi_2 \in [x_1, x]. \tag{10}$$

Let $x = x_1 + \Delta$, $\Delta > 2\gamma^{-1} Q^{-n-1+2v}$. On the one hand side if $P_1(x_1) < P_1(x_1 + \Delta) < 0$, then the first inequality of (8) implies

$$0 < P_1(x_1 + \Delta) - P_1(x_1) < Q^{-n+v}.$$

On the other side we have

$$|P'(\xi_2)\Delta| > \frac{\gamma}{2} Q^{1-v} 2\gamma^{-1} Q^{-n-1+2v} = Q^{-n+v}.$$

We thus obtain a contradiction to equality (10). This means that $P_1(x_1 + \Delta) > 0$ and there is a real root α of the polynomial $P_1(x)$ between x_1 and $x_1 + \Delta$.

At the same time

$$|x_1 - \alpha| < 2\gamma^{-1} Q^{-n-1+2v} = n 2^{n+13} Q^{-n-1+2v}. \tag{11}$$

Let us obtain a lower bound for the absolute value of the difference $|x_1 - \alpha|$. Again we consider only one of four possibilities $P_1(x_1) > 0$, $P_1'(x_1) < 0$. At the point $x = x_1 + \Delta_1$ we have

$$P_1(x) = P_1(x_1) + P_1'(\xi_3)\Delta_1, \ \xi_3 \in [x_1, x]. \tag{12}$$

If $\Delta_1 < 2^{-4}n^{-1}\gamma Q^{-n-1+2v}$, then in (12) the following holds: $|P_1(x_1)| > \gamma Q^{-n+v}$ and $|P'(\xi_3)\Delta_1| < \gamma Q^{-n+v}$. It implies that the polynomial $P_1(x)$ can't have any root in the interval $[x_1, x_1 + \Delta_1]$ and therefore for any root α, we have

$$n^{-1}2^{-n-13}Q^{-n-1+2v} < |x - \alpha|.$$

Let α be the closest root to x_1 of the polynomial $P_1(x)$. Using the representation

$$P_1'(\alpha) = P_1'(x_1) + P_1''(\xi_4)(x_1 - \alpha),\ \xi_4 \in [x, \alpha],$$

we obtain the estimate $|P_1''(\xi)| < n^3 Q$ and (11) for sufficiently large Q we get

$$n^{-1}2^{-n-13}Q^{1-v} < |P_1'(\alpha)| < 16nQ^{1-v}.$$

The square of derivative is a factor in the discriminant of the polynomial. Taking into account that for $|a_n| \asymp H(P)$ all roots of the polynomial are bounded [see Lemma 3] we can estimate the differences $|\alpha_i - \alpha_j|$, $2 \leqslant i < j \leqslant n$, by a constant $c(n)$. Thus, in the point $x_1 \in B_1$ we can construct a polynomial $P_1(x)$ with discriminant:

$$|D(P_1)| \ll Q^{2n-2-2v}.$$

Let us introduce $x_{01} = inf\{x:\ x \in I \cap B_1\}$. Clearly that the point $x_1 \in B_1$ can be taken from the interval $J_1 = [x_{01},\ x_{01} + Q^{-n-1}]$. Let us $J_1' = [x_{01}, x_{01} + Q^{-n-1} + 4\gamma^{-1}Q^{-n-1+2v}]$ and $x_{02} = inf\{x:\ x \in (I \setminus J_1') \cap B_1\}$. Let choose $x_2 \in J_2 = [x_{02}, x_{02} + Q^{-n-1}]$ and at the same time $x_2 \in B_2$. By construction we have

$$|x_2 - x_1| > 4\gamma^{-1}Q^{-n-1+2v} \tag{13}$$

Let choose a point $x_2 \in B_1$ such, that

$$|x_2 - x_1| > 4\gamma^{-1}Q^{-n-1+2v}. \tag{14}$$

For this point we can construct a polynomial $P_2(x)$ again such that it will satisfy the system of inequalities (8) in the point x_2. Let show that $P_2(x) \neq P_1(x)$. Consider the value of the polynomial $P_1(x)$ in the point $x = x_2$. Then

$$P_1(x_2) = P_1(x_1) + P_1'(\xi_5)(x_2 - x_1),\ \xi_5 \in [x_1, x_2].$$

Using $|P_1(x_1)| < Q^{-n+v}$, $|P_1'(\xi_5)| > \frac{\gamma}{2}Q^{1-v}$ and (14) we obtain

$$|P_1(x_2)| > Q^{-n+v},$$

which contradicts the first inequality of (8). Thus, the polynomial, constructed at the point x_2, is different from $P_1(x)$. Its discriminant $D(P_2)$ also satisfies (5). Futhermore, the point $x_3 \in B_1$ $x_3 - x_2 > 4\gamma^{-1}Q^{-n-1+2v}$ we construct a polynomial $P_3(x)$ different from $P_1(x)$ and $P_2(x)$, that satisfies conditions (5) and (8). It's clear that repeating the described procedure we can construct $c(n)Q^{n+1-2v}$ polynomials $P(x)$, with discriminants satisfying (5).

Let introduce several lemmas, that will be useful for proof of the Theorem 2.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ are roots of the polynomial $P(x)$.

For each polynomial $P(x) \in \mathbb{Z}_n[x]$,

$$P(x) = a_n(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n),$$

we may choose one of its roots (say α_1) and consider only those $x \in I$ such that $\min_{1\leqslant i\leqslant n} |x - \alpha_i| = |x - \alpha_1|$. Furthermore, assume that the roots are ordered such that

$$|\alpha_1 - \alpha_2| \leqslant |\alpha_1 - \alpha_3| \leqslant \ldots \leqslant |\alpha_1 - \alpha_n|.$$

Denote

$$|\alpha_1 - \alpha_j| = H^{-\mu_j},\ l_j - 1 = [\mu_j T],\ j = \overline{2, n},$$

where $T = [\frac{n}{\varepsilon}] + 1$ and ε is a small positive value. Therefore $(l_j - 1)T^{-1} \leqslant \mu_j < l_j T^{-1}$

We introduce $p_j = \frac{l_{j+1} + \cdots + l_n}{T}$, $j = \overline{1, n-1}$.

Thus all polynomials $P(x)$ can be divided into subclasses according the vector $\bar{s} = (l_2, \ldots, l_n)$. We denote them by $\mathcal{P}_n(H, \bar{s})$.

We denote

$$S(\alpha_i) = \{x \in \mathbb{R} :\ |x - \alpha_i| = \min_{1\leqslant j\leqslant n} |x - \alpha_j|\}.$$

Lemma 1 (See [6]). *If $P \in \mathcal{P}_n(H)$ and $x \in S(\alpha_1)$, then*

$$|x - \alpha_1| \leqslant 2^n |P(x)||P'(\alpha_1)|^{-1},$$

$$|x - \alpha_1| \leqslant \min_{2\leqslant j\leqslant n} (2^{n-j}|P(x)||P'(\alpha_1)|^{-1} \prod_{k=2}^{j} |\alpha_1 - \alpha_k|)^{\frac{1}{j}}.$$

Lemma 2 *If $x \in S(\alpha_1)$, then*

$$|x - \alpha_1| < n\frac{|P(x)|}{|P'(x)|}.$$

Proof. From the representation $P(x) = a_n(x-\alpha_1)\dots(x-\alpha_n)$ we obtain

$$\frac{|P'(x)|}{|P(x)|} = \sum_{j=1}^{n} \frac{1}{|x-\alpha_j|} \leqslant \frac{n}{|x-\alpha_1|}.$$

Lemma 3 (See [5]). *If $|a_n| \gg H$, then for any root of the polynomial the following is true*

$$|\alpha_j| \ll 1.$$

Lemma 4 (See [5]). *Let $k, m \in \mathbb{Z}$, $P \in \mathcal{P}_n(H)$. Then*

$$\max_{k \leqslant m \leqslant k+n} |P(m)| > c(n)H.$$

Lemma 5 (See [7]). *For any $n \in \mathbb{N}$, $n > 1$ and real $\delta > 0$ there is an effectively calculable bound $H_0(\delta, n)$ such that for any $H > H_0$ and positive real μ, τ, η the following holds. If $P_1(x), P_2(x) \in \mathbb{Z}[x]$ are coprime,*

$$\max(\deg(P_1), \deg(P_2)) \leqslant n,$$

$$\max(H(P_1), H(P_2)) \leqslant H^{\mu}$$

and there is an interval $I \subset \mathbb{R}$ with

$$|I| = H^{-\eta},$$

such that for all $x \in I$

$$\max(|P_1(x)|, |P_2(x)|) < H^{-\tau},$$

then

$$\tau + \mu + 2\max\{\tau + \mu - \eta, 0\} < 2n\mu + \delta.$$

Lemma 6 *Let $P(x) \in \mathcal{P}_n(H)$ and $|D(P)| < Q^{2n-2-2v}$. Then there is a polynomial $T(x) = b_n x^n + b_{n-1}x^{n-1} + \dots + b_0$ that additionally satisfies*

$$|D(T)| = |D(P)|, \ H(T) \ll H, \ |b_n| \gg H.$$

Proof. Assume that $P(x) = a_n x^n + \dots + a_1 x + a_0$ has roots $\alpha_1, \alpha_2, \dots, \alpha_n$. By Lemma 4 there is a number m_0, $1 \leqslant m_0 \leqslant n+1$, that

$$|P(m_0)| > c(n)H.$$

Let us consider the polynomial $P_1(x) = P(x+m_0) = a_n x^n + a'_{n-1}x^{n-1} + \cdots + a'_1 x + P(m_0)$. Its roots are $\beta_j = \alpha_j - m_0$, $1 \leqslant j \leqslant n$, and the absolute value of the discriminant equals

$$|D(P_1)| = a_n^{2n-2}|\prod_{1\leqslant i<j\leqslant n}(\beta_i - \beta_j)^2| = a_n^{2n-2}|\prod_{1\leqslant i<j\leqslant n}(\alpha_i - \alpha_j)^2| = |D(P)|.$$

The polynomial $T(x) = x^n P_1(\frac{1}{x}) = P(m_0)x^n + a''_{n-1}x^{n-1} + \cdots + a''_1 x + a_n))$ has roots $\gamma_j = \frac{1}{\beta_j} = \frac{1}{\alpha_j - m_0}$, $1 \leqslant j \leqslant n$. The absolute value of its discriminant equals

$$|D(T)| = P(m_0)^{2n-2}|\prod_{1\leqslant i<j\leqslant n}(\beta_i - \beta_j)^2 \beta_i^{-2}\beta_j^{-2}|.$$

But $|\prod_{1\leqslant i<j\leqslant n}\beta_i^{-2}\beta_j^{-2}| = (P(m_0)a_n^{-1})^{2n-2}$, therefore $|D(T)| = |D(P)|$. The condition $H(T) < c(n)H$ is obviously satisfied because $H(T) = H(P_1)$ and $H(P_1) \asymp H(P)$.

Lemma 7 (See [6]). *Let $P(x) \in \mathcal{P}_n(H)$. Then*

$$|P^{(l)}(\alpha_1)| \ll H^{1-p_l},\ 1 \leqslant l \leqslant n-1.$$

Lemma 8 (See [7]). *The measure of those x such that the inequality*

$$|P_n(x)| < H^{-w},$$

holds for $w > n-1$ and $H > H_0$, has infinitely many solutions in reducible polynomials $P(x)$ tends to zero measure when $H_0 \to \infty$.

3 Proof of the Theorem 2

We will start with estimating the measure of those x such that the system

$$\begin{cases} |P(x)| < c_1 Q^{-n+v}, \\ Q^{1-v_1} < |P'(x)| < c_2 Q^{1-v} \end{cases} \tag{15}$$

is solvable, where v_1, $v < v_1 < 1$, will be specified later.

In the second inequality of (15) we shall replace $P'(x)$ by $P'(\alpha)$, where α denotes the closest root to x. This is done by Lagrange formula for $P'(x)$

$$P'(x) = P'(\alpha) + P''(\xi_1)(x-\alpha),\ \xi_1 \in (\alpha, x)$$

and the estimate of $|x-\alpha|$ by Lemma 2

$$|x-\alpha| < n\frac{|P(x)|}{|P'(x)|}.$$

Then
$$|P'(\alpha)| = |P'(x) - P''(\xi_1)(x-\alpha)|.$$
As
$$|P''(\xi_1)(x-\alpha)| \leqslant n^3 Q c_1 n Q^{-n-1+v+v_1} = c_1 n^4 Q^{-n+v+v_1}$$
for sufficiently large Q we obtain
$$\frac{3}{4}Q^{1-v_1} \leqslant \frac{3}{4}|P'(x)| \leqslant |P'(\alpha)| \leqslant \frac{4}{3}|P'(x)| \leqslant \frac{4}{3}c_2 Q^{1-v}$$
and
$$\frac{3}{4}|P'(\alpha)| \leqslant |P'(x)| \leqslant \frac{4}{3}|P'(\alpha)|.$$
Therefore for sufficiently large Q we can consider the following system
$$\begin{cases} |P(x)| < c_1 Q^{-n+v} \\ \frac{3}{4}Q^{1-v_1} < |P'(\alpha)| < \frac{4}{3}c_2 Q^{1-v} \\ |a_j| \leqslant Q. \end{cases} \tag{16}$$
Let $\mathcal{L}'_n(v)$ denotes the set of x, for which the system (16) is solvable. Now we are able to prove that $\mu\mathcal{L}'_n(v) < \frac{3}{8}|I|$.

Consider the intervals:
$$\sigma_1(P): \ |x-\alpha| < \frac{4}{3}c_1 n Q^{-n+v}|P'(\alpha)|^{-1}$$
and
$$\sigma_2(P): \ |x-\alpha| < c_5 Q^{-1+v}|P'(\alpha)|^{-1}.$$
The value of c_5 we will specify below. Obviously
$$|\sigma_1(P)| \leqslant \frac{4}{3}c_1 c_5^{-1} n Q^{-n+1}|\sigma_2(P)|. \tag{17}$$
Fix the vector $\bar{b} = (a_n, \ldots, a_2)$ of part of the coefficients of $P(x)$. All polynomials with the same vector $\bar{b}$ form the class $\mathcal{P}(\bar{b})$. Thus we divide $\mathcal{P}_n(Q)$ in classes according the vector $\bar{b}$.

The interval $\sigma_2(P_1)$, $P_1 \in \mathcal{P}(\bar{b})$ is called *inessential* if there is another interval $\sigma_2(P_2)$, $P_2 \in \mathcal{P}(\bar{b})$ such that
$$|\sigma_2(P_1) \cap \sigma_2(P_2)| \geqslant 0,5|\sigma_2(P_1)|.$$
Otherwise the interval $\sigma_2(P_2)$ is called *essential* and for any $P_2 \in \mathcal{P}(\bar{b})$ holds
$$|\sigma_2(P_1) \cap \sigma_2(P_2)| < 0,5|\sigma_2(P_1)|.$$

The case of essential intervals. In this case every point $x \in I$ belongs to not more than two essential intervals $\sigma_2(P)$. Therefore for any vector $\overline{b}$

$$\sum_{P \in \mathcal{P}_1(\overline{b})} |\sigma_2(P)| \leqslant 2|I|. \tag{18}$$

We have to sum over the lengths of essential intervals $\sigma_1(P)$ inside the class $\mathcal{P}(\overline{b})$ with fixed vector $\overline{b}$, and then over all classes. We can estimate the number of classes as the number of all possible vectors $\overline{b}$

$$(2Q+1)^{n-1} = (2Q)^{n-1}(1+\frac{1}{2Q})^{n-1} \leqslant 2^{n-1}Q^{n-1}e^{\frac{n-1}{2Q}} < 2^nQ^{n-1}$$

From (17) and (18) obtain

$$\sum_{\overline{b}, |a_j| \leqslant Q} \sum_{P \in \mathcal{P}(\overline{b})} |\sigma_1(P)| < \frac{4}{3}c_1c_5^{-1}nQ^{-n+1}2|I|2^nQ^{n-1} = n2^{n+2}c_1c_5^{-1}.$$

Thus for $c_5 = n2^{n+5}c_1$ the measure μ_1 will be not larger than $\frac{|I|}{8}$.

The case of inessential intervals. Let us estimate the values of $|P_j(x)|$, $j = 1, 2$, on the intersection $\sigma_2(P_1, P_2)$ of the intervals $\sigma_2(P_1)$ and $\sigma_2(P_2)$. By Lagrange's formula

$$P_j(x) = P_j'(\alpha)(x-\alpha) + \frac{1}{2}P_j''(\xi_2)(x-\alpha)^2, \text{ for some } \xi_2 \in (\alpha, x).$$

and

$$P_j'(x) = P_j'(\alpha) + P_j''(\xi_3)(x-\alpha), \text{ for some } \xi_3 \in (\alpha, x).$$

The second summand is estimated by

$$|P''(\xi_2)(x-\alpha)^2| \leqslant 2n^3c_5^2Q^{-3+2v+2v_1},$$

while

$$|P'(\alpha)(x-\alpha)| < c_5Q^{-1+v}.$$

As $2v_1 < 2 - v$ for an appropriate choice of $v_1 < \frac{3}{4}$ we obtain

$$|P_j(x)| \leqslant \frac{4}{3}c_5Q^{-1+v}, \ j = 1, 2. \tag{19}$$

Similarly we obtain the following estimate for $P_j'(x)$

$$|P_j'(\alpha)| < \frac{4}{3}c_2Q^{1-v},$$

$$|P_j''(\xi_3)(x-\alpha)| < 2n^3 c_5 Q^{-1+v+v_1}$$

and for $v_1 \leqslant 1$

$$|P_j'(x)| \leqslant \frac{4}{3} c_2 Q^{1-v}, \ j = 1, 2. \tag{20}$$

Denote the difference of the chosen polynomials $P_1(x)$ and $P_2(x)$ by $K(x) = P_2(x) - P_1(x)$, $K(x)$ is not identically zero. Obviously it can be represented as $K(x) = b_1 x + b_0$. Besides, (19) and (20) imply

$$|b_1 x + b_0| < \frac{8}{3} c_5 Q^{-1+v} \tag{21}$$

and

$$|b_1| = |K'(x)| < \frac{8}{3} c_2 Q^{1-v}.$$

For fixed b_0 and b_1 the measure of those $x \in I$ that satisfy (21) doesn't exceed $\frac{16}{3} c_5 Q^{-\lambda} b_1^{-1}$. Thus, provided that $x \in I$ and taking into account (21) it follows that b_0 can have not more than $|I||b_1| + 2$ values. Summing over all b_0 we obtain an estimate for the measure when b_1 is fixed

$$\frac{16}{3} c_5 Q^{-1+v} b_1^{-1}(|I||b| + 2) < 6 c_5 Q^{-1+v}. \tag{22}$$

After summing (22) over all $|b_1|$ we have

$$2^5 c_2 c_5 Q^{1-v-\lambda} |I| = n 2^{n+8} c_1 c_2 |I| = \frac{1}{8} |I|.$$

From $c_1 c_2 < n^{-1} 2^{-n-11}$ we can estimate the total measure both for essential and nonessential intervals by $\frac{|I|}{4}$.

Now we are to consider the remaining cases. Our task is to estimate the measure of $\mathcal{L}_n''(v)$ of the set of all x such that the system

$$\begin{cases} |P(x)| < Q^{-n+v} \\ |P'(x)| < Q^{1-v_1} \\ |a_j| \leqslant Q \end{cases} \tag{23}$$

is solvable in $P \in \mathcal{P}_n(Q)$.

To prove the Theorem 2 it remains to show that

$$\mu \mathcal{L}_n''(v) \ll \frac{1}{4} |I|.$$

The proof splits into the following cases:

1. $l_2T^{-1}+p_1 \geqslant n+1-v$,

2. $n+0,1 \leqslant l_2T^{-1}+p_1 < n+1-v$,

3. $\frac{7}{4} \leqslant l_2T^{-1}+p_1 < n+0,1$,

4. $l_2T^{-1}+p_1 < \frac{7}{4}$.

Case 1.

$$l_2T^{-1}+p_1 \geqslant n+1-v. \tag{24}$$

Denote the class $\mathcal{P}_t(\bar{s}) = \bigcup_{2^t \leqslant H < 2^{t+1}} \mathcal{P}_n(H, \bar{s})$. Since Q is a sufficiantly large number and $H \leqslant Q$, we have $t_0 < t \ll \log Q$. Let consider two estimates of $|x-\alpha_1|$, obtained from (23) and Lemma 1 for $x \in S(\alpha_1)$

$$|x-\alpha_1| \leqslant 2^n \frac{|P(x)|}{|P'(\alpha_1)|} \ll 2^{t(-n+v-1+p_1+(n-1)\varepsilon)}, \tag{25}$$

and

$$|x-\alpha_1| \leqslant \left(2^{n-1}\frac{|P(x)||\alpha_1-\alpha_2|}{|P'(\alpha_1)|}\right)^{\frac{1}{2}} \ll 2^{\frac{t}{2}(-n+v-1+p_2+(n-2)\varepsilon)}. \tag{26}$$

In the case (24) we use the estimate (26). Let us divide the interval I into smaller parts I_j, $\mu I_j = 2^{-t(\frac{n+1-v-p_2}{2}-\gamma)}$, where γ is a positive constant.

For an integer polynomial $P(x)$ and an interval I_j we shall write "$P(x)$ belongs to I_j" or "I_j contains $P(x)$", if there is a point $x \in I_j$, that satisfies the system (23). Let $\sigma(P)$ denote the measure of those $x \in S(\alpha_1)$, that satisfy (23).

a) Assume that not more than one polynomial $P(x) \in \mathcal{P}_t(\bar{s})$ belongs to every I_j. Then for every polynomial the measure of those x, that satisfy (23), doesn't exceed $c(n)2^{-t(\frac{n+1-v-p_2-(n-2)\varepsilon}{2})}$ and the number of I_j is less than $2^{t(\frac{n+1-v-p_2}{2}-\gamma)}|I|$. Therefore

$$\sum_{P \in \mathcal{P}_t(\bar{s})} \sigma(P) \ll \sum_{P(x) \in \mathcal{P}_t(\bar{s})} 2^{t(\frac{n+1-v-p_2}{2}-\gamma)}|I| \cdot c(n)2^{-t(\frac{n+1-v-p_2-(n-2)\varepsilon}{2})} \ll 2^{-t\gamma_1}, \tag{27}$$

where $\gamma_1 = \gamma - \frac{n-2}{2}\varepsilon$.

The sum (27) extends over all $t \geqslant t_0$. Since $\sum_{t>t_0} 2^{-t\gamma_1} \ll 2^{-t_0\gamma_1}$ then for sufficiently large t_0 the measure of those x such that the system (23) holds and polynomials $P(x)$ satisfy the case 1 a), doesn't exceed $\frac{|I|}{32}$

b) Suppose to thecontrary, that there are intervals I_j, that contains at least two polynomials, i.e. we can find polynomials P_1 and P_2 from the class

$\mathcal{P}_t(\bar{s})$, points x_1 and x_2 from I_j, that satisfy the system of inequalities

$$\begin{cases} |P_1(x_1)| \ll 2^{t(-n+v)}, \\ |P_1'(x_1)| \ll 2^{t(1-v_1)} \end{cases}$$

$$\begin{cases} |P_2(x_2)| \ll 2^{t(-n+v)}, \\ |P_2'(x_2)| \ll 2^{t(1-v_1)}. \end{cases}$$

Let us estimate the value of $P_1(x)$ and $P_2(x)$ for points of the interval I_j. Using Taylor's expansion for $P_i(x)$ at the point say α_1

$$P_i(x) = \sum_{j=1}^{n} \frac{P_i^{(j)}(\alpha_1) \cdot (x - \alpha_1)^j}{j!}.$$

and estimates $|P^j(\alpha_1)|$ from the Lemmas 1 and 7 we get

$$|P_j(x)| \ll 2^{t(1-p_j+j(\frac{-n+v-1+p_j+(n-j)\varepsilon}{j}+\gamma))} \ll 2^{t(-n+v+n\gamma_1)}.$$

Now for polynomials P_1 and P_2 without common roots we can apply Lemma 5.

Since we have $\tau = n - v - n\gamma_1$, $\mu = 1$, $\eta = \frac{n+1-v-p_2}{2} - \gamma$ then

$$n - v - n\gamma_1 + 1 + 2(n - v - n\gamma_1 + 1 - \frac{n+1-v-p_2}{2} + \gamma) < 2n + \delta.$$

Hence it follows

$$2 - 2v < \delta + (3n-2)\gamma_1,$$

which leads to a contradiction for $v \leqslant \frac{1}{2}$ and sufficiently small γ, ε and δ.

Case 2.

$$n + 0.1 \leqslant l_2T^{-1} + p_1 < n + 1 - v. \tag{28}$$

Let us divide the interval I into intervals I_j, where $|I_j| = 2^{t(-\frac{l_2}{T}+\gamma)}$.

a) Assume that not more than one polynomial $P(x) \in \mathcal{P}_t(\bar{s})$ belongs to every I_j. We use the inequality (25). Then for every polynomial the measure of those x, that satisfy (23), doesn't exceed $c(n)2^{-t(n+1-v-p_1-(n-1)\varepsilon)}$ and the number of I_j is less than $2^{t(n+1-v-p_1-\gamma)}|I|$. Therefore

$$\sum_{P \in \mathcal{P}_t(\bar{s})} \sigma(P) \ll \sum_{P(x) \in \mathcal{P}_t(\bar{s})} 2^{t(\frac{l_2}{T}-\gamma)} \cdot 2^{-t(n+1-v-p_1-(n-1)\varepsilon)} \ll 2^{-t\gamma_2}, \tag{29}$$

where $\gamma_2 = \gamma - (n-1)\varepsilon$. Again we sum the estimate (29) over all $t > t_0$ as in formula (27). It is clear that the total sum is less than $\frac{|I|}{32}$

b) Assuming like above in Case 1 above the existence of an interval I_j that contains at least two different polynomials $P_1(x)$ and $P_2(x)$, for any $x \in I_j$, by Taylor expansion we get

$$|P_i(x)| \ll 2^{-t(l_2 T^{-1}+p_1-1-2\gamma)},\ i = 1, 2.$$

For $P_1(x)$ and $P_2(x)$, which have no common roots, on I_j we apply may Lemma 5 with $\mu = 1$, $\eta = l_2 T^{-1} - \gamma$, $\tau + 1 = l_2 T^{-1} + p_1 - 2\gamma$. Note that $l_2 T^{-1} \leqslant p_1$. Then

$$l_2 T^{-1} + 3p_1 - 4\gamma < 2n + \delta.$$

This together with (28) implies the inequality

$$2n + \frac{1}{5} - 4\gamma \leqslant l_2 T^{-1} + 3p_1 - 4\gamma < 2n - \delta,$$

$$\frac{1}{5} < \delta + 4\gamma.$$

which is a contradiction for small δ and γ.

Case 3.

$$\frac{7}{4} \leqslant l_2 T^{-1} + p_1 < n + \frac{1}{10}. \tag{30}$$

This case represents the largest interval for $l_2 T^{-1} + p_1$ and is the most difficult case. We divide I into intervals I_j of length $2^{-tl_2 T^{-1}}$. At first let us estimate the value of a polynomial $P \in \mathcal{P}_n$ and its derivative on the interval I_j. For this purpose expand by Taylor's formula in neighborhood of the point α_1:

$$P(x) = \sum_{j=1}^{n} \frac{P^{(j)}(\alpha_1) \cdot (x - \alpha_1)^j}{j!}.$$

$$|P'(\alpha_1) \cdot (x - \alpha_1)| \ll 2^{t(1-p_1-l_2 T^{-1})},$$

$$|P''(\alpha_1) \cdot (x - \alpha_1)^2| \ll 2^{t(1-p_2-2l_2 T^{-1})} \ll 2^{t(1-p_1-l_2 T^{-1})},$$

$$|P^{(i)}(\alpha_1) \cdot (x - \alpha_1)^i| \ll 2^{t(1-p_i-il_2 T^{-1})} \ll 2^{t(1-p_1-l_2 T^{-1})},\ 3 \leqslant i \leqslant n.$$

For the derivative we have use

$$P'(x) = \sum_{j=0}^{n-1} \frac{P^{(j+1)}(\alpha_1) \cdot (x - \alpha_1)^j}{j!}.$$

$$|P'(\alpha_1)| \asymp 2^{t(1-p_1)},$$

$$|P^{(i)}(\alpha_1)\cdot(x-\alpha_1)^{i-1}| \ll 2^{t(1-p_i-(i-1)l_2T^{-1})} \ll 2^{t(1-p_1)},\ 2\leqslant i\leqslant n.$$

Thus, if the polynomial $P(x)$ belongs to the interval I_j it should satisfy the system

$$\begin{cases} |P(x)| \ll 2^{t(1-p_1-l_2t^{-1})}, \\ |P'(x)| \asymp 2^{t(1-p_1)}. \end{cases} \tag{31}$$

Consider those intervals that contain $c(n)2^{t\rho}$ polynomials. Then the measure of $x\in I$, that satisfy (23) is

$$2^{t(-n+v-1+p_1+(n-1)\varepsilon)}c(n)2^{t\rho}2^{tl_2T^{-1}}.$$

If $\rho < n+1-v-(p_1+l_2T^{-1})$ and $t>t_0$ the measure can be estimated by $\frac{|I|}{32}$.

To simplify further calculations we introduce

$$u := n+1-v-p_1-l_2T^{-1},$$

From (30) and $v\leqslant\frac{1}{2}$ it follows that for $u\geqslant\frac{2}{5}$. Let introduce $u_1=u-\frac{1}{5}\geqslant\frac{1}{5}$ and represent u_1 as a sum $u_1=[u_1]+\{u_1\}$.

Let $n+1-v-p_1-\rho-l_2T^{-1}\leqslant 0$, i.e. $\rho\geqslant u$. By Dirichlet's principle there exist at least $c(n)\cdot 2^{t(\{u_1\}+0.2)}$ polynomials $P_1(x)$, $P_2(x)$, $\dots$, $P_k(x)$, where $k\gg 2^{t(\{u_1\}+0.2)}$, with $[u_1]$ first identical coefficients.

Consider the polynomials $R_j(x)=P_{j+1}(x)-P_1(x)$, which obviously satisfy:

$$\deg R_j(x)\leqslant n-[u_1],$$
$$H(R_j)\ll 2^t.$$

From (31) we get

$$\begin{cases} |R_j(x)| \ll 2^{t(1-p_1-l_2T^{-1})},\ i=1,\dots,k, \\ |R_j'(x)| \ll 2^{t(1-p_1)}. \end{cases} \tag{32}$$

Every coefficient of the polynomial R_j ranges in the segment $[-2^{t+1};2^{t+1}]$. We divide all segments into equal parts of length $2^{t\mu}$, where $\mu=1-\frac{\{u_1\}}{n-[u_1]}$.

Then at least $c(n)2^{\frac{1}{5}t}$ polynomials will fall into the same segments. Hence their differences $R_j(x)$ will have a height less than $c(n)2^{t\mu}=c(n)2^{t(1-\frac{\{u_1\}}{n-[u_1]})}$.

Introduce $S_j(x) = R_{j+1}(x) - R_1(x)$. Thus the system of inequalities (32) may be written as

$$\begin{cases} |S_j(x)| \ll 2^{t(1-p_1-l_2\cdot T^{-1})},\ j=1,2\dots,k-1,\\ |S'_j(x)| \ll 2^{t(1-p_1)},\ j=1,2\dots,k-1,\\ \deg S_j(x) \leqslant n-[u_1],\\ H(S_i) < 2^{t(1-\frac{\{u_1\}}{n-[u_1]-1})}. \end{cases} \tag{33}$$

a) Suppose there are coprime polynomials of type $S_i(x)$. Then applying Lemma 5 to the interval I_j with

$$\tau = p_1 + l_2\cdot T^{-1} - 1,$$

$$\mu = 1 - \frac{\{u_1\}}{n-[u_1]},$$

$$\max\{\deg S_1, \deg S_2\} \leqslant \deg S = n-[u_1],$$

$$\eta = l_2 T^{-1}$$

we get

$$p_1 + l_2T^{-1} - \frac{\{u_1\}}{n-[u_1]} + 2(p_1 + l_2T^{-1} - \frac{\{u_1\}}{n-[u_1]} - l_2T^{-1}) \leqslant$$

$$\leqslant 2(n-[u_1])(1-\frac{\{u_1\}}{n-[u_1]}) + \delta.$$

This implies

$$3p_1 - l_2T^{-1} - \frac{3\{u_1\}}{n-[u_1]} \leqslant 2p_1 + 2l_2T^{-1} + 2v + 0.4 - 2 + \delta.$$

Replacing p_1 by l_2T^{-1} and representing $n-[u_1]-1$ as $p_1+l_2\cdot T^{-1}+0,2+\{u_1\}-1$ we obtain

$$\frac{8}{5} - 2v < \frac{3\{u_1\}}{p_1 + l_2T^{-1} + v + 0,2 + \{u_1\} - 1} + \delta. \tag{34}$$

Having written the right side of the inequality as the function of $\{u_1\}$ and v in $[0;1)\times[0;1/4)$, we show that it doesn't exceed $0.4+\frac{3}{p_1+l_2\cdot T^{-1}+0.2}+\delta$, but our assumption $p_1 + l_2\cdot T^{-1} > 1$ leads to a contradictin to inequality (34) for small enough δ.

b) If all polynomials $S_j(x)$ are of the type $lS_0(x)$, it implies $|2^{0.4t}S_0(x)| \ll 2^{t(1-p_1-l_2\cdot T^{-1})}$

$$|S_0(x)| \ll H(S_0)^{\frac{1-p_1-l_2T^{-1}-0.2}{1-\frac{\{u_1\}}{n-[u_1]}-0.2}}.$$

If the inequality

$$p_1 + l_2 \cdot T^{-1} + 0.2 - 1 > (n - [u_1])(1 - \frac{\{u_1\}}{n - [u_1]} - 0.2) \tag{35}$$

fails to hold Sprindzuk's theorem [5] implies that we can estimate in this case 2 b) by $\frac{|I|}{32}$.

Since the product on the right side of (35) is $p_1 + l_2 \cdot T^{-1} - 0.8$ the inequality (35) leads again to a contradiction.

c) If there are reducible polynomials among the already constructed $S_i(x)$ then the system (33) for one of the multipliers T_1 or T_2 (say T_1), such that $S_i(x) = T_1(x)T_2(x)$, implies the bounds

$$\begin{cases} |T_1(x)| \ll H(T_1)^{(1-p_1-l_2T^{-1})(1-\frac{\{u_1\}}{n-[u_1]})^{-1}}, \\ \deg T(x) \le n - [u_1] - 1. \end{cases}$$

To apply the Sprindzuk's theorem we again need to check the inequality

$$p_1 + l_2 \cdot T^{-1} - 1 > (n - [u_1] - 1)(1 - \frac{\{u_1\}}{n - [u_1]}). \tag{36}$$

We rewrite the right hand side of (36) as

$$p_1 + l_2 \cdot T^{-1} + 0.2 - 2 + \frac{1}{p_1 + l_2 \cdot T^{-1} + 0.2}.$$

Determing the maximum of this expression as the function of $\{u_1\}$ we obtain that the right side does not exceed $p_1 + l_2 \cdot T^{-1} + d_2 - 2 + \frac{1}{p_1 + l_2 \cdot T^{-1} + 0.2}$. Thus we again obtain that the measure in case c) doesn't exceed $\frac{|I|}{32}$.

Case 4.

$$l_2T^{-1} + p_1 < \frac{7}{4}.$$

Let us estimate the expression $l_2T^{-1} + p_1$ below. To do this we have to prove that $|P'(x)| \asymp 2^{t(1-p_1)}$. By the Taylor's formula for $P'(x)$ at α (the closest root to x) we get

$$P'(x) = \sum_{j=0}^{n-1} \frac{P^{(j+1)}(\alpha) \cdot (x - \alpha)^j}{j!}.$$

We have $|P'(\alpha)| \asymp 2^{t(1-p_1)}$, and for the remaining summands of the expansion we note

$$|P^{(i)}(\alpha)(x-\alpha)^{i-1}| \ll 2^{t(1-p_1)},\ 2 \leqslant i \leqslant n.$$

Since $|P'(x)| \ll 2^{t/3}$, then $1-p_1 \leqslant \frac{1}{3}$ or $\frac{2}{3} \leqslant p_1$.

Thus we need to consider the system

$$\begin{cases} |P(x)| < 2^{t(-n+v)}, \\ |P'(x)| < 2^{t/3}, \\ \frac{2}{3} < l_2 T^{-1} + p_1 < \frac{7}{4}. \end{cases} \tag{37}$$

All solutions of the system of the inequalities (37), with α_1 being the closest root to x, are contained in the interval

$$\sigma(P) = \{x \in I:\ |x-\alpha_1| < 2^{t(-n+v)}|P'(\alpha_1)|^{-1}\}. \tag{38}$$

Besides $\sigma(P)$ we also introduce the following interval $\sigma_1(P)$, that involves $\sigma(P)$.

$$\sigma_1(P) = \{x \in I:\ |x-\alpha_1| < 2^{t(v-0.9)}|P'(\alpha_1)|^{-1}\}. \tag{39}$$

From (38) and (39) we get

$$\mu\sigma(P) \ll 2^{t(-n+v+1-v)}\mu\sigma_1(P) = 2^{t(-n+0.9)}\mu\sigma_1(P).$$

Divide all polynomials in $\mathcal{P}_n$ into classes $\mathcal{P}_{\bar{b}}$ according the $n-1$ first coefficients $\bar{b} = (a_n, a_{n-1}, \dots, a_2)$. Obviously $\#\bar{b} \asymp 2^{t(n-1)}$.

a) If $\mu\sigma_1(P_1) \cap \mu\sigma_1(P_2) < \frac{1}{2}\mu\sigma_1(P_1)$ then $\sum_{P\in\mathcal{P}_{\bar{b}}} \mu\sigma_1(P) \ll |I|$. Summing over all classes we obtain

$$\sum_{\bar{b}} \sum_{P\in\mathcal{P}_{\bar{b}}} \mu\sigma(P) \leqslant 2^n 2^{t(n-1)} n 2^{t(-n+0.9)} 2|I| \leqslant n2^{n+1}2^{-0.1t}|I|.$$

b) If $\mu\sigma_1(P_1) \cap \mu\sigma_1(P_2) \geqslant \frac{1}{2}\mu\sigma_1(P_1)$ we denote $R(x) = P_1(x) - P_2(x)$. Since P_1 and P_2 belong to the same class $\mathcal{P}_{\bar{b}}$, then $R(x)$ is of the type $ax+b$. Moreover taking into account the estimates of the polynomials and their derivatives we obtain

$$\begin{cases} |ax+b| \ll 2^{t(-0.9+v)}, \\ |a| \ll 2^{t(1-p_1)}. \end{cases}$$

$$|x + \frac{b}{a}| \ll 2^{t(-0.9+v)}|a|^{-1}. \tag{40}$$

It is clear that the inequality (40) holds in the entire essential interval. Summing the estimates (40) first over all b, which do not exceed $c(n)|a||I|$, and then over all a we obtain $c(n)2^{t(-0.9+v+1-p_1)}|I| = c(n)2^{t(v-p_1+0.1)}|I| \ll 2^{-0.1t}|I|$. Let us sum the estimates of the cases a) and b) over all $t > t_0$. We obtain that in the Case 4 the measure of those x that satisfy (23) doesn't exceed $\frac{|I|}{32}$, and for the cases 1 – 4 together the measure of the set $\mathcal{L}''_n(v)$ doesn't exceed $\frac{|I|}{4}$, thus proving the theorem.

References

[1] *V.I. Bernik, M.M. Dodson*, Metric Diophantine approximation on manifolds // Cambridge Tracts in Math. Vol. **137**. Cambridge Univ. Press, 1999.

[2] *S.A. Katre, S.A. Khule*, A discriminant criterion for matrices over orders in algebraic numbers fields to be sums of squares // Proc. Symp. on Alg. and Number Th. (Cochin, 1990), **20**, Centre Math. Sci., Trivandrum, 1990, P. 31 – 38.

[3] *S.A. Katre, S.A. Khule*, Matrices over orders in algebraic number fields as sums of k–th powers // Proc. Amer. Math. Soc. **128**, 2000, No. 3, P. 671 – 675.

[4] *K. Győry*, Polynomials and binary forms with given discriminant // Publ. Math. Debrecen, **69/4**, 2006, P. 473 – 499.

[5] *V.G. Sprindzuk*, Mahler's problem in metric Number Theory // Nauka i Tehnika, Minsk 1967 [Transl. Math. Monogr. **25**, Amer. Math. Soc., Providenca, R.I., 1969].

[6] *V. Bernik*, The metric theorem on the simultaneous approximation of zero by values of integral polynomials // Izv. Akad. Nauk SSSR, Ser. Mat. **44**, 1980, P. 24 – 45.

[7] *V. Bernik*, On the exact order of approximation of zero by values of integral polynomials // Acta Arith. **53**, 1989, P. 17 – 28.

Institute of Mathematics
Surganova str. 11
220072, Minsk,
Belarus
E-mail: bernik@im.bas-net.by
olga_kukso@tut.by

Universität Bielefeld
Fakultät für Mathematik
Postfach 10 01 31
33501 Bielefeld
Germany
E-mail: goetze@math.uni-bielefeld.de