Classical metric Diophantine approximation revisited

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Dedicated to Klaus Roth on the occasion of his 80th birthday

Abstract

The idea of using measure theoretic concepts to investigate the size of number theoretic sets, originating with E. Borel, has been used for nearly a century. It has led to the development of the theory of metrical Diophantine approximation, a branch of Number Theory which draws on a rich and broad variety of mathematics. We discuss some recent progress and open problems concerning this classical theory. In particular, generalisations of the Duffin-Schaeffer and Catlin conjectures are formulated and explored.

1   Dirichlet, Roth and the metrical theory

Diophantine approximation is based on a quantitative analysis of the property that the rational numbers are dense in the real line. Dirichlet's theorem, a fundamental result of this theory, says that given any real number x and any natural number N, there are integers p and q such that

$$\left| x - \frac{p}{q} \right| \leqslant \frac{1}{qN}, \qquad 0 < q < N.$$

There is an extraordinarily rich variety of analogues and generalisations of this fact. Although simple, Dirichlet's result is best possible for all real numbers. Also it implies that for any irrational x there are infinitely many rational numbers p/q satisfying $|x - p/q| < 1/q^2$. The latter inequality can be sharpened a little by the multiplicative factor $1/\sqrt{5}$ but no further improvement is possible for the golden ratio $(\sqrt{5} - 1)/2$ and its equivalents [10].

This leads to the notion of badly approximable numbers: the irrational number x is *badly approximable* if there is a constant $c > 0$ such that

$$\left| x - \frac{p}{q} \right| < \frac{c}{q^2} \tag{1}$$

for no rationals p/q. Dirichlet's theorem can be improved for and only for badly approximable numbers [12]. In the other direction, if for any constant $c > 0$ there is a rational p/q satisfying (1) then β is called *well approximable*. The sets of badly and well approximable numbers will be denoted by $\mathcal{B}$ and $\mathcal{W}$ respectively. Further, *very well approximable* numbers x satisfy the stronger condition that there exists an $\varepsilon = \varepsilon(x) > 0$ such that the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}}$$

holds for infinitely many rationals p/q. The set of such numbers will be denoted by $\mathcal{V}$. The numbers which are not very well approximable will be called *relatively badly approximable* and will be denoted by $\mathcal{R}$, thus $\mathcal{R} := \mathbb{R} \setminus \mathcal{V}$. One readily sees that $\mathcal{B} \subset \mathcal{R}$. In 1912 Borel [7] proved that $\mathcal{V}$ is of Lebesgue measure zero (or null), so that its complement $\mathcal{R}$ is of full measure. The Lebesgue measure of set A will be written as $|A|$; there should be no confusion with the symbol for modulus. As usually, we say that 'almost no' number lies in $\mathcal{V}$ and 'almost all' numbers lie in its complement $\mathcal{R}$ or that numbers in $\mathcal{R}$ are 'typical'. If numbers in a unit interval are considered, there is a natural interpretation in terms of probability: a number lies in $\mathcal{V}$ with probability zero and in $\mathcal{R}$ with probability one.

There are uncountably many badly approximable numbers, as they are characterized by having bounded partial quotients in their continued fraction expansion. This characterisation also implies that $\mathcal{B}$ has Lebesgue measure zero (this also follows from Khintchine's theorem discussed below) and full Hausdorff dimension: $\dim \mathcal{B} = 1$. The quadratic irrationals are known to be badly approximable and a natural but as yet unanswered question is whether other irrational algebraic numbers *are* or *are not* badly approximable. $\mathcal{R}$oth proved the following remarkable result.

Theorem 1 (Roth) *All real irrational algebraic numbers are $\mathcal{R}$-numbers.*

From the metrical point of view, Roth's theorem [28] shows that every real algebraic irrational number behaves typically.

2 Khintchine's theorem

Loosely speaking, in the above section we have been dealing with variations of Dirichlet's theorem in which the right hand side or error of approximation is either of the form cq^{-2} or $q^{-2-\varepsilon}$. It is natural to broaden the discussion to general error functions. More precisely,

given a function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$, a real number x is said to be ψ–*approximable* if there are infinitely many $q \in \mathbb{N}$ such that

$$\|qx\| < \psi(q) . \tag{2}$$

The function ψ governs the 'rate' at which the rationals approximate the reals and will be referred to as an *approximating function*. Here and throughout, $\|x\|$ denotes the distance of x from the nearest integer and $\mathbb{R}^+ = [0, \infty)$. One can readily verify that the set of ψ-approximable numbers is invariant under translations by integer vectors. Therefore without any loss of generality and to ease the 'metrical' discussion which follows, we shall restrict our attention to ψ–approximable numbers in the unit interval $\mathbb{I} := [0, 1)$. The set of such numbers is clearly a subset of $\mathbb{I}$ and will be denoted by $\mathcal{A}(\psi)$; i.e.

$$\mathcal{A}(\psi) := \{x \in \mathbb{I} \colon \|qx\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\} .$$

In 1924, Khintchine [20] established a beautiful and strikingly simple criterion for the 'size' of the set $\mathcal{A}(\psi)$ expressed in terms of Lebesgue measure. We will write the n-dimensional Lebesgue measure of a set X in $\mathbb{R}^n$ by $|X|_n$; when there is no risk of confusion the suffix will be omitted. There should be no confusion with the notation for a norm or modulus of a number or a vector. We give an improved modern version of this fundamental result – see [10, 17, 32, 33].

Theorem 2 (Khintchine)

$$|\mathcal{A}(\psi)| = \begin{cases} 0 & if \quad \displaystyle\sum_{r=1}^{\infty} \psi(r) < \infty , \\[2em] 1 & if \quad \displaystyle\sum_{r=1}^{\infty} \psi(r) = \infty \quad and \ \psi \ is \ monotonic. \end{cases}$$

Remark. Regarding the above theorem and indeed the theorems and conjectures below, it is straightforward to establish the complementary convergent statements; i.e. if the sum in question converges then the set in question is of zero measure.

In Khintchine's theorem, the divergence case constitutes the main substance and involves the extra monotonicity condition. This condition cannot in general be relaxed, as was shown by Duffin and Schaeffer [14] in 1941. They constructed a non-monotonic approximating function ϑ for which the sum $\sum_q \vartheta(q)$ diverges but $|\mathcal{A}(\vartheta)|_1 = 0$ (see [17, 33] for details). Nevertheless, in the case of arbitrary ψ, Duffin and Schaeffer produced a conjecture that we now discuss.

The integer p implicit in the inequality (2) satisfies

$$\left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} . \tag{3}$$

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To relate the rational p/q with the error of approximation $\psi(q)/q$ uniquely, we impose the coprimeness condition $(p, q) = 1$. In this case, let $\mathcal{A}'(\psi)$ denote the set of x in $\mathbb{I}$ for which the inequality (3) holds for infinitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with $(p, q) = 1$. Clearly,

$$\mathcal{A}'(\psi) \subset \mathcal{A}(\psi)$$

so that the convergence part of Khintchine's theorem remains valid if $\mathcal{A}(\psi)$ is replaced by $\mathcal{A}'(\psi)$. In fact, for any approximating function $\psi \colon \mathbb{N} \to \mathbb{R}^+$ one easily deduces that

$$|\mathcal{A}'(\psi)| = 0 \quad \text{if} \quad \sum_{r=1}^{\infty} \varphi(r) \, \frac{\psi(r)}{r} \;<\; \infty \;.$$

Here, and throughout, φ is the Euler function. A less obvious fact is that the divergence part of Khintchine's theorem (when ψ is required to be monotonic) holds for $\mathcal{A}'(\psi)$, *i.e.*, for ψ monotonic, the coprimeness condition $(p, q) = 1$ is irrelevant. As already mentioned above, this is not the case if we remove the monotonicity condition and the appropriate statement is given by a famous conjecture.

The Duffin-Schaeffer conjecture: *For any approximating function $\psi \colon \mathbb{N} \to \mathbb{R}^+$*

$$|\mathcal{A}'(\psi)| = 1 \quad if \quad \sum_{r=1}^{\infty} \varphi(r) \frac{\psi(r)}{r} = \infty \;.$$

Although various partial results have been established – see [17] for details and references, the full conjecture represents one of the most difficult and profound unsolved problems in metric number theory.

We now turn our attention to the 'raw' set $\mathcal{A}(\psi)$ on which no monotonicity or coprimeness conditions are imposed. It is known (see [17, 33]) that the one-dimensional Lebesgue measure of $\mathcal{A}(\psi)$ is either zero or one but this is far from providing a criterion for $|\mathcal{A}(\psi)|$ akin to the Duffin-Schaeffer conjecture for $\mathcal{A}'(\psi)$ or Khintchine's theorem for $\mathcal{A}(\psi)$ with ψ monotonic. The following conjecture [11] provides such a criterion.

Catlin's conjecture: *For any approximating function $\psi \colon \mathbb{N} \to \mathbb{R}^+$*

$$|\mathcal{A}(\psi)| = 1 \quad if \quad \sum_{r=1}^{\infty} \varphi(r) \, \max_{t \geqslant 1} \frac{\psi(rt)}{rt} = \infty \;.$$

Thinking geometrically, given a rational point s/r with $(s, r) = 1$ consider all its representations p/q with $p = ts$ and $q = tr$ for some $t \in \mathbb{N}$. The length of the largest interval given by (3) is precisely the maximum term appearing in the above conjecture.

To the best of our knowledge, it is not known whether the Duffin-Schaeffer conjecture is equivalent to Catlin's conjecture. For the current situation regarding the conjectures of Duffin & Schaeffer and Catlin see [17, pp. 27–29]. It is remarkable that in the simultaneous setup the analogues of the these conjectures have been completely settled – see §3.

2.1 Back to Roth

We end this section by returning to Roth. In connection with Roth's theorem, Waldschmidt [34, pp. 260] has made the following conjecture which currently seems well beyond reach.

Waldschmidt's conjecture: *Let ψ be a monotonic approximating function such that $\sum_{r=1}^{\infty} \psi(r) < \infty$ and let $\alpha \in \mathbb{I}$ be a real algebraic irrational number. Then, $\alpha \notin \mathcal{A}(\psi)$.*

The conjecture is a general version of Lang's conjecture [24]. The latter corresponds to the case that $\psi : r \to r^{-1}(\log q)^{-1-\varepsilon}$ with $\varepsilon > 0$ arbitrary; i.e. the inequality

$$\|q\alpha\| \; < \; q^{-1}(\log q)^{-1-\varepsilon}$$

has only finitely many solution for every positive ε. Note that in view of the imposed convergent sum condition, we have that $|\mathcal{A}(\psi)| = 0$. Thus, from the metrical point of view, Waldschmidt's conjecture simply states that α behaves typically in the sense that α belongs to the set $\mathbb{I} \setminus \mathcal{A}(\psi)$ of full measure. This clearly strengthens the notion of typical as implied by Roth's theorem.

Within the statement of Waldschmidt's conjecture, it is natural to question the relevance of the monotonicity assumption. In other words, does it make sense to consider the following stronger form of the conjecture? *Let ψ be an approximating function such that $\sum_{r=1}^{\infty} \varphi(r)\,\psi(r)/r < \infty$ and let $\alpha \in \mathbb{I}$ be a real algebraic irrational number. Then, $\alpha \notin \mathcal{A}(\psi)$.* This statement is easily seen to be false. For $q \in \mathbb{N}$, let $\psi^*(q) := \|q\alpha\|$. Obviously, $\liminf_{q\to\infty} \psi^*(q) = 0$. Therefore, there exists a sequence $\{q_n\}_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} \psi^*(q_n) < \infty$. Now, set $\psi(q)$ to be $2\psi^*(q)$ if $q = q_n$ for some n and 0 otherwise. Thus for the non-monotonic approximating function ψ, we have that

$$\sum_{r=1}^{\infty} \varphi(r)\,\psi(r)/r \; < \; \sum_{r=1}^{\infty} \psi(r) < \infty \quad \text{but} \quad \alpha \in \mathcal{A}(\psi) \, .$$

The upshot of this is that the monotonicity assumption in Waldschmidt's conjecture can not be removed. However, the example given above is somewhat 'artificial' and it makes perfect sense to study 'density' questions of the following type: *for a given approximating function ψ how large is the set of ψ-approximable algebraic numbers of degree n compared to the set of all algebraic numbers of degree n?* Even for monotonic approximating functions, considering this question would give a partial 'metrical' or 'density' answer to Waldschmidt's conjecture.

3 Simultaneous approximation by rationals

In simultaneous Diophantine approximation, one considers the set $\mathcal{A}_m(\psi)$ of points $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{I}^m := [0, 1)^m$ for which the inequality

$$\|q\mathbf{x}\| := \max\{\|qx_1\|, \ldots, \|qx_m\|\} < \psi(q)$$

holds for infinitely many positive integers q. Khintchine extended his one-dimensional result discussed in §2 to simultaneous approximation [21] (see also [10, Chapter VII]).

Theorem 3 (Khintchine)

$$|\mathcal{A}_m(\psi)|_m = \begin{cases} 0 & if \ \sum_{r=1}^{\infty} \psi(r)^m < \infty \ , \\ 1 & if \ \sum_{r=1}^{\infty} \psi(r)^m = \infty \ \ and \ \psi \ is \ monotonic. \end{cases}$$

As with the one-dimensional statement, Khintchine originally had a stronger monotonicity condition. It turns out that the monotonicity condition can be safely removed from Khintchine's theorem for $m \geqslant 2$. In fact, that this is the case, is a simple consequence of the next result that deals with the setup in which a natural coprimeness condition on the rational approximates is imposed.

Let $\mathcal{A}'_m(\psi)$ denote the set of points $\mathbf{x} := (x_1, \ldots, x_m) \in \mathbb{I}^m$ for which the inequality

$$\left| \mathbf{x} - \frac{\mathbf{p}}{q} \right| < \frac{\psi(q)}{q} \quad with \quad (p_1, \ldots, p_m, q) = 1 \tag{4}$$

is satisfied for infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{N}$. The coprimeness condition imposed in the definition of $\mathcal{A}'_m(\psi)$ ensures that the points in $\mathbb{I}^m$ are approximated by distinct rationals; i.e. the points $\mathbf{p}/q := (p_1/q, \ldots, p_m/q)$ are distinct. The following metric result concerning the set $\mathcal{A}'_m(\psi)$ is due to Gallagher [16] and is free from any monotonicity condition.

Theorem 4 (Gallagher) *Let $m \geqslant 2$. Then*

$$|\mathcal{A}'_m(\psi)|_m = \begin{cases} 0 & if \ \sum_{r=1}^{\infty} \psi(r)^m < \infty, \\ 1 & if \ \sum_{r=1}^{\infty} \psi(r)^m = \infty. \end{cases}$$

To see that Gallagher's theorem removes the monotonicity requirement from Khintchine's theorem for $m \geqslant 2$, simply note that divergent/convergent sum condition is the same in both statements and that $\mathcal{A}'_m(\psi) \subset \mathcal{A}_m(\psi)$. The latter implies that $|\mathcal{A}_m(\psi)|_m = 1$ whenever $|\mathcal{A}'_m(\psi)|_m = 1$. Thus, for $m \geqslant 2$ we are able to establish a criterion for the size of the 'raw' set $\mathcal{A}_m(\psi)$ on which no monotonicity or coprimeness conditions are imposed. In particular, we have the following divergent statement as a corollary to Gallagher's theorem.

Corollary 5 *Let $m \geqslant 2$. For any approximating function $\psi \colon \mathbb{N} \to \mathbb{R}^+$*

$$|\mathcal{A}_m(\psi)|_m = 1 \quad if \quad \sum_{r=1}^{\infty} \psi(r)^m = \infty \ .$$

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This naturally settles the (simultaneous) higher dimensional analogue of Catlin's conjecture which we now formulate.

The simultaneous Catlin conjecture: *For any approximating function $\psi \colon \mathbb{N} \to \mathbb{R}^+$*

$$|\mathcal{A}_m(\psi)|_m = 1 \quad if \quad \sum_{r=1}^{\infty} \mathcal{N}_m(r) \, \max_{t \geqslant 1} \left(\frac{\psi(rt)}{rt} \right)^m = \infty \;, \tag{5}$$

where $\mathcal{N}_m(r) := \#\{(p_1, \ldots, p_m) \in \mathbb{Z}^m : (p_1, \ldots, p_m, r) = 1, \; 0 \leqslant p_i < r \text{ for all } i\}$ is the number of distinct rational points with denominator r in the unit cube $\mathbb{I}^m$.

In the one dimensional case $\mathcal{N}_m(r)$ is simply $\varphi(r)$ and (5) reduces to Catlin's original conjecture. For $m \geqslant 2$, it is not difficult to verify that

$$\mathcal{N}_m(r) \;\asymp\; r^m \;,$$

where $\asymp$ means comparable and is defined to be the 'double' Vinogradov symbol, that is both $\ll$ and $\gg$. This together with a nifty geometric argument enables us to conclude that

$$\sum_{r=1}^{\infty} \mathcal{N}_m(r) \, \max_{t \geqslant 1} \left(\frac{\psi(rt)}{rt} \right)^m \;\asymp\; \sum_{r=1}^{\infty} \psi(r)^m \;.$$

The following equivalence is now obvious:

$$Corollary\ 5 \quad \Longleftrightarrow \quad The\ simultaneous\ Catlin\ conjecture\ for\ m \geqslant 2. \tag{6}$$

The (simultaneous) higher dimensional analogue of the Duffin-Schaeffer conjecture [33] requires the stronger coprimality condition that the coordinates $p_1, \ldots, p_m$ of the vector $\mathbf{p} \in \mathbb{Z}^m$ are pairwise coprime to q. The corresponding set of ψ-approximable points will be denoted by $\mathcal{A}''_m(\psi)$ and consists of points $\mathbf{x} \in \mathbb{I}^m$ for which the inequality in (4) is satisfied for infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{N}$ with $(p_j, q) = 1$ for $j = 1, \ldots, m$. In 1990, Pollington and Vaughan [26] established the **simultaneous Duffin-Schaeffer conjecture** for $m \geqslant 2$:

Theorem 6 (Pollington & Vaughan) *Let $m \geqslant 2$. For any approximating function $\psi \colon \mathbb{N} \to \mathbb{R}^+$*

$$|\mathcal{A}''_m(\psi)|_m = \begin{cases} 0 & if \; \sum_{r=1}^{\infty} \varphi(r)^m \left(\frac{\psi(r)}{r} \right)^m < \infty, \\[4mm] 1 & if \; \sum_{r=1}^{\infty} \varphi(r)^m \left(\frac{\psi(r)}{r} \right)^m = \infty. \end{cases}$$

Notice that this theorem does not imply Gallagher's theorem nor does it imply the simultaneous Catlin's conjecture for $m \geqslant 2$. The books of Sprindzuk [33] and Harman [17] contain a variety of generalisations of the above results including asymptotic formulae for the number of solutions, inhomogeneous version of Gallagher's theorem [17, Theorem 3.4] and approximation with different approximating functions in each coordinate.

4   Dual approximation and Groshev's theorem

Instead of approximation by rational points as considered in the previous section, one can consider the closeness of the point $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ to rational hyperplanes given by the equations $\mathbf{q} \cdot \mathbf{x} = p$ with $p \in \mathbb{Z}$ and $\mathbf{q} \in \mathbb{Z}^n$. The point $\mathbf{x} \in \mathbb{R}^n$ will be called *dually ψ-approximable* if the inequality

$$|\mathbf{q} \cdot \mathbf{x} - p| < \psi(|\mathbf{q}|)$$

holds for infinitely many $(p, \mathbf{q}) \in \mathbb{Z} \times \mathbb{Z}^n$, where $|\mathbf{q}| := |\mathbf{q}|_\infty = \max\{|q_1|, \ldots, |q_n|\}$. The set of dually ψ-approximable points in $\mathbb{I}^n$ will be denoted by $\mathcal{A}_n^*(\psi)$. The argument of the approximating function depends on the sup norm of $\mathbf{q}$ but it could also easily be chosen to depend on $\mathbf{q} \in \mathbb{Z}^n$. In what follows, we shall consider the sup norm $|\mathbf{q}|$ and will continue to use ψ to denote an approximating function on $\mathbb{N}$; i.e. we consider $\psi(|\mathbf{q}|)$ where $\psi \colon \mathbb{N} \to \mathbb{R}^+$. However in the next section we shall discuss the general case, when we will use Ψ to denote an approximating function with argument in $\mathbb{Z}^n$; i.e. we consider $\Psi(\mathbf{q})$ where $\Psi \colon \mathbb{Z}^n \to \mathbb{R}^+$.

The simultaneous and dual forms of approximation are special cases of a system of linear forms, covered by a general extension due to A. V. Groshev (see [33]). This treats real $n \times m$ matrices X, regarded as points in $\mathbb{R}^{nm}$, which are ψ-approximable. More precisely, $X = (x_{ij}) \in \mathbb{R}^{nm}$ is said to be ψ-approximable if the inequality

$$\|\mathbf{q}X\| < \psi(|\mathbf{q}|)$$

is satisfied for infinitely many $\mathbf{q} \in \mathbb{Z}^n$. Here $\mathbf{q}X$ is the system

$$q_1 x_{1j} + \cdots + q_n x_{n,j} \qquad (1 \leq j \leq m)$$

of m real linear forms in n variables and $\|\mathbf{q}X\| := \max_{1 \leq j \leq m} \|\mathbf{q} \cdot X^{(j)}\|$, where $X^{(j)}$ is the j'th column vector of X. As the set of ψ-approximable points is translation invariant under integer vectors, we can restrict attention to the nm-dimensional unit cube $\mathbb{I}^{nm}$. The set of ψ-approximable points in $\mathbb{I}^{nm}$ will be denoted by

$$\mathcal{A}_{n,m}(\psi) := \{X \in \mathbb{I}^{nm} : \|\mathbf{q}X\| < \psi(|\mathbf{q}|) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\}.$$

Thus, $\mathcal{A}_m(\psi) = \mathcal{A}_{1,m}(\psi)$ and $\mathcal{A}_n^*(\psi) = \mathcal{A}_{n,1}(\psi)$. The following result naturally extends Khintchine's simultaneous theorem to the linear forms setup.

Theorem 7 (Groshev) *Let $\psi : \mathbb{N} \to \mathbb{R}^+$. Then*

$$
|\mathcal{A}_{n,m}(\psi)|_{nm} = \begin{cases} 0 & if \ \ \displaystyle\sum_{r=1}^{\infty} r^{n-1} \psi(r)^m < \infty, \\[4mm] 1 & if \ \ \displaystyle\sum_{r=1}^{\infty} r^{n-1} \psi(r)^m = \infty \ \ and \ \psi \ is \ monotonic. \end{cases}
$$

The counterexample due to Duffin and Schaeffer mentioned in §2 means that the monotonicity condition cannot be dropped from Groshev's theorem when $m = n = 1$. To avoid this situation, let $m + n > 2$. Then for $n = 1$, we have already seen (Corollary 5) that as a consequence of Gallagher's theorem the monotonicity condition can be removed. Furthermore, the monotonicity condition can also be removed for $n > 2$ – this time due to a result of Sprindzuk which we discuss in §5. However, the $n = 2$ situation seems to be unresolved and we make the following conjecture.

Conjecture A *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ and suppose that $m + n > 2$. Then*

$$
|\mathcal{A}_{n,m}(\psi)|_{nm} = 1 \qquad if \ \ \sum_{r=1}^{\infty} r^{n-1} \psi(r)^m = \infty \ .
$$

To reiterate the discussion immediately before the statement of the conjecture, it is only the $n = 2$ case which is problematic. It is plausible that it can be resolved using existing techniques. Note that for $m + n > 2$, Conjecture A provides a criterion for the size of the 'raw' set $\mathcal{A}_{n,m}(\psi)$ on which no monotonicity or coprimeness conditions are imposed. In view of this, for approximating functions with sup norm argument, Conjecture A should naturally be equivalent to the linear forms analogue of Catlin's conjecture.

It is possible to formulate the linear forms analogue of both the Duffin-Schaeffer conjecture and Catlin's conjecture. However, this will be postponed till the next section in which we consider multi-variable approximating functions $\Psi \colon \mathbb{Z}^n \to \mathbb{R}^+$ and thereby formulate the conjectures in full generality. We shall indeed see that Conjecture A is equivalent to the linear forms analogue of Catlin's conjecture for approximating functions with sup norm argument.

5   More general approximating functions

Throughout this section we assume that $n \geqslant 2$ unless stated otherwise. In Theorem 13 of Chapter 1 of [33] Sprindzuk describes a very general setting for a Diophantine system of linear forms. Let $\{S_{\mathbf{q}}\}$ be a sequence of measurable sets in $\mathbb{I}^m$ indexed by integer points $\mathbf{q} \in \mathcal{Z}$, where $\mathcal{Z} \subset \mathbb{Z}^n \smallsetminus \{0\}$. Define $\mathcal{A}_{n,m}(\{S_{\mathbf{q}}\})$ to consist of points $X \in \mathbb{I}^{nm}$ such that there are infinitely many $\mathbf{q} \in \mathcal{Z}$ satisfying $\mathbf{q}X \in S_{\mathbf{q}} \pmod 1$. Then

$$|\mathcal{A}_{n,m}(\{S_\mathbf{q}\})|_{nm} = \begin{cases} 0 & \text{if } \sum_{\mathbf{q}\in\mathcal{Z}} |S_\mathbf{q}| < \infty, \\ \\ 1 & \text{if } \sum_{\mathbf{q}\in\mathcal{Z}} |S_\mathbf{q}| = \infty \text{ and any two vectors in } \mathcal{Z} \text{ are non-parallel.} \end{cases} \tag{7}$$

This extremely general result enables us to generalise the setup of §4 in two significant ways. Firstly, we are able to consider arbitrary approximating functions $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$ rather than restrict the argument of Ψ to the sup norm of $\mathbf{q}$. Secondly, we are naturally able to consider inhomogeneous problems. Define

$$\mathcal{A}_{n,m}^{\mathbf{b}}(\Psi) := \{X \in \mathbb{I}^{nm} : \|\mathbf{q}X + \mathbf{b}\| < \Psi(\mathbf{q}) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\} . \tag{8}$$

Here $\mathbf{b} \in [0,1)^m$ is a fixed vector that represents the 'inhomogeneous' or 'shifted' part of approximation. Now let $\mathbf{P}^n$ denote the set of *primitive* vectors in $\mathbb{Z}^n$ – non-zero integer vectors with coprime components. It is easy to see that the statement given by (7) with $\mathcal{Z} = \mathbf{P}^n$ specializes to give the following result – essentially Theorem 14 in [33].

Theorem 8 (Sprindzuk) *Let* $\Psi \colon \mathbb{Z}^n \to \mathbb{R}^+$ *and suppose that* $\Psi(\mathbf{q}) = 0$ *for* $\mathbf{q} \notin \mathbf{P}^n$. *Then for* $n \geqslant 2$

$$|\mathcal{A}_{n,m}^{\mathbf{b}}(\Psi)|_{nm} = \begin{cases} 0 & \text{if } \sum_{\mathbf{q}\in\mathbb{Z}^n} \Psi(\mathbf{q})^m < \infty , \\ \\ 1 & \text{if } \sum_{\mathbf{q}\in\mathbb{Z}^n} \Psi(\mathbf{q})^m = \infty . \end{cases} \tag{9}$$

Of course the primitivity condition in the theorem, namely that Ψ vanishes on non-primitive integer vectors, imposes a primitivity condition on the set $\mathcal{A}_{n,m}^{\mathbf{b}}(\Psi)$ – namely that the vectors $\mathbf{q} \in \mathbb{Z}^n$ associated with (8) are primitive.

We now consider a special case of Sprindzuk's theorem in which the argument of Ψ is restricted to the sup norm. In keeping with the notation used in §4, we write ψ for Ψ and so $\Psi(\mathbf{q}) = \psi(|\mathbf{q}|)$ for $\mathbf{q}$ in $\mathbb{Z}^n$. Let $n \geqslant 3$. Then the number of primitive vectors $\mathbf{q}$ in $\mathbb{Z}^n$ with $|\mathbf{q}| = r$ is comparable to r^{n-1}. Thus the number of primitive vectors is comparable to the number of vectors without any primitivity restriction. It follows that the divergence condition in (9) is equivalent to $\sum_{r=1}^{\infty} r^{n-1}\psi(r)^m = \infty$. The latter is precisely the divergent sum appearing in Groshev's theorem. The upshot of this is that Sprindzuk's theorem removes the monotonicity requirement from Groshev's theorem when $n \geqslant 3$. A similar argument in the case $n = 2$ does not yield an equivalent improvement in Groshev's theorem. The reason for this is simply the fact that the number of primitive vectors $\mathbf{q}$ in $\mathbb{Z}^2$ with $|\mathbf{q}| = r$ is comparable to $\varphi(r)$ whereas the number of vectors without any primitivity restriction is comparable to r. In short, when $n = 2$ the divergent sum appearing in Sprindzuk's theorem is not equivalent to that appearing in Groshev's theorem.

The primitivity condition cannot in general be omitted from Sprindzuk's theorem. To give a counterexample, we consider the case that $m = 1$ and $n \geqslant 2$. For $\mathbf{q} = (q_1, \ldots, q_n) \in$

$\mathbb{Z}^n \smallsetminus \{0\}$ let

$$\Psi_\vartheta(\mathbf{q}) := \begin{cases} \vartheta(|q_1|) & \text{if } \mathbf{q} = (q_1, 0, \ldots, 0), \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

where ϑ is the function constructed by Duffin and Schaeffer (see §2). Obviously

$$\sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} \Psi_\vartheta(\mathbf{q}) = \sum_{q \in \mathbb{Z} \smallsetminus \{0\}} \vartheta(|q|) = \infty.$$

On the other hand, $\mathcal{A}_{n,1}(\Psi) = \mathcal{A}(\vartheta) \times \mathbb{I}^{n-1}$ so that $|\mathcal{A}_{n,1}(\Psi_\vartheta)| = |\mathcal{A}(\vartheta)| \cdot |\mathbb{I}^{n-1}| = 0 \cdot 1 = 0$. Clearly Ψ_ϑ does not satisfy the primitivity condition in Sprindzuk's theorem.

The above counterexample implies that the primitivity condition cannot be omitted from Sprindzuk's theorem for the dual form of approximation; namely when considering the dual set $\mathcal{A}_n^*(\Psi) := \mathcal{A}_{n,1}(\Psi)$. However, if $m > 1$ so that we are dealing with a system of more than one linear form, no similar counterexample appears to be possible. Indeed we strongly believe in the truth of the following conjecture concerning the set

$$\mathcal{A}'_{n,m}(\Psi) := \{X \in \mathbb{I}^{nm} : \ |\mathbf{q}X + \mathbf{p}| < \Psi(\mathbf{q}) \text{ for infinitely many } (\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$$
$$\text{with } (p_1, \ldots, p_m, q_1, \ldots, q_n) = 1\} \tag{11}$$

Conjecture B *Let $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$ and suppose that $m > 1$. Then*

$$|\mathcal{A}'_{n,m}(\Psi)|_{nm} = 1 \qquad \text{if} \qquad \sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} \Psi(\mathbf{q})^m = \infty \ .$$

Obviously for every $\mathbf{q} \in \mathbf{P}^n$ the coprimality condition in (11) is satisfied. Therefore $\mathcal{A}_{n,m}(\Psi) = \mathcal{A}'_{n,m}(\Psi)$ for any Ψ vanishing outside of $\mathbf{P}^n$. Thus Conjecture B covers Sprindzuk's theorem in the case of $m > 1$. Furthermore, since $\mathcal{A}'_{n,m}(\Psi) \subset \mathcal{A}_{n,m}(\Psi)$, Conjecture B would imply the following statement for the 'raw' set $\mathcal{A}_{n,m}(\Psi)$ on which no monotonicity or coprimeness conditions are imposed.

Conjecture C *Let $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$ and suppose that $m > 1$. Then*

$$|\mathcal{A}_{n,m}(\Psi)|_{nm} = 1 \qquad \text{if} \qquad \sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} \Psi(\mathbf{q})^m = \infty \ .$$

As one should expect, we will see below that Conjecture C is naturally equivalent to the linear forms analogue of Catlin's conjecture for $m > 1$. Recall that the above conjectures, are straightforwardly established in the complementary convergent cases.

The counterexample given by (10) shows that Conjectures B and C are not valid when $m = 1$; i.e. the dual form of approximation. We now deal with the case $m = 1$. It is relatively easy to show that the set $\mathcal{A}'_{n,1}(\Psi)$ has measure zero if the sum

$$\sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} \frac{\varphi(\gcd(\mathbf{q}))}{\gcd(\mathbf{q})} \Psi(\mathbf{q}) \ = \ \sum_{d=1}^{\infty} \frac{\varphi(d)}{d} \sum_{\mathbf{q}' \in \mathbf{P}^n} \Psi(d\mathbf{q}') \tag{12}$$

converges. Here $\gcd(\mathbf{q})$ denotes the greatest common divisor of the components of $\mathbf{q} \in \mathbb{Z}^n$. The following can be regarded as a generalisation of the Duffin-Schaeffer conjecture to the case of dual approximation.

The dual Duffin-Schaeffer conjecture: *Let $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$ and suppose that $m = 1$. Then*

$$|\mathcal{A}'_{n,1}(\Psi)|_n = 1 \qquad if \quad (12) \quad diverges.$$

It is clear that this conjecture includes the original Duffin-Schaeffer conjecture. It would be desirable to find natural conditions on Ψ which make the conjecture genuinely multi-dimensional. For example, in the genuine multi-dimensional case it is natural to exclude approximating functions Ψ like Ψ_ϑ given by (10). The hope is that the genuine multi-dimensional case is 'easier' than the one dimensional case. Recall that in the case of simultaneous approximation, the multi-dimensional Duffin-Schaeffer conjecture has been proved. Also notice that if $n \geqslant 2$ and there exists $d \in \mathbb{N}$ such that

$$\sum_{\mathbf{q}' \in \mathbf{P}^n} \Psi(d\mathbf{q}') \, ,$$

the internal sum in (12), diverges, then the conjecture is reduced to Sprindzuk's theorem. Therefore, it is also reasonable to assume that the internal sum in (12) is finite irrespective of d.

Regarding the 'raw' set $\mathcal{A}_{n,1}(\Psi)$ on which no monotonicity or coprimeness conditions are imposed, it is natural to formulate the analogue of Catlin's conjecture. For $\mathbf{q} \in \mathbb{Z}^n$ let

$$\mathcal{N}_n^*(\mathbf{q}) \; := \; \#\{0 < p \leqslant |\mathbf{q}| : (p, \mathbf{q}) = 1\}.$$

It is easy to see that

$$\mathcal{N}_n^*(\mathbf{q}) \; \asymp \; \frac{\varphi\big(\gcd(\mathbf{q})\big)}{\gcd(\mathbf{q})} \, |\mathbf{q}|$$

and moreover that the set $\mathcal{A}_{n,1}(\Psi)$ has measure zero if the sum

$$\sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} \mathcal{N}_n^*(\mathbf{q}) \; \max_{t \geqslant 1} \frac{\Psi(t\mathbf{q})}{t|\mathbf{q}|} \tag{13}$$

converges. The following can be regarded as a generalisation of Catlin's conjecture to the case of dual approximation:

The dual Catlin conjecture : *Let $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$ and suppose that $m = 1$. Then*

$$|\mathcal{A}_{n,1}(\Psi)|_n = 1 \qquad if \quad (13) \quad diverges.$$

For $n \geqslant 2$ and $\Psi(\mathbf{q}) = \psi(|\mathbf{q}|)$ the above conjecture is equivalent to Conjecture A with $m = 1$. Recall, that for $n \geqslant 3$ the latter is known to be true.

Finally, for the sake of completeness, we extend the above dual conjectures to the general linear forms setup. For the analogue of the Duffin-Schaeffer conjecture it is natural to impose a coprimality condition on each linear form. Let

$$\mathcal{A}''_{n,m}(\Psi) := \{X \in \mathbb{I}^{nm} : \ |\mathbf{q}X + \mathbf{p}| < \Psi(\mathbf{q}) \text{ for infinitely many } (\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$$
$$\text{with } (p_j, q_1, \dots, q_n) = 1 \text{ for every } j = 1, \dots, m\}.$$

It is easy to show that the set $\mathcal{A}''_{n,m}(\Psi)$ has measure zero if the sum

$$\sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} \left(\frac{\varphi(\gcd(\mathbf{q}))}{\gcd(\mathbf{q})} \Psi(\mathbf{q}) \right)^m \tag{14}$$

converges. This leads to the following complementary problem.

The linear forms Duffin-Schaeffer conjecture: *Let $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$. Then*

$$|\mathcal{A}''_{n,m}(\Psi)|_{nm} = 1 \qquad \textit{if the sum (14) diverges.}$$

Regarding the 'raw' set $\mathcal{A}_{n,m}(\Psi)$ on which no monotonicity or coprimeness conditions are imposed, we formulate the analogue of Catlin's conjecture. For $\mathbf{q} \in \mathbb{Z}^n$ let

$$\mathcal{N}_{n,m}(\mathbf{q}) := \#\{\mathbf{p} \in \mathbb{Z}^m : 0 \leqslant |\mathbf{p}| \leqslant |\mathbf{q}|, \ (\mathbf{p}, \mathbf{q}) = 1\}. \tag{15}$$

It can be verified that the set $\mathcal{A}_{n,m}(\Psi)$ has measure zero if the sum

$$\sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} \mathcal{N}_{n,m}(\mathbf{q}) \left(\max_{t \geqslant 1} \frac{\Psi(t\mathbf{q})}{t|\mathbf{q}|} \right)^m \tag{16}$$

converges. This leads to the following complementary problem.

The linear forms Catlin conjecture: *Let $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$. Then*

$$|\mathcal{A}_{n,m}(\Psi)|_{nm} = 1 \qquad \textit{if} \quad (16) \textit{ diverges.}$$

On modifying the argument that enables us to establish the equivalence (6) within the simultaneous setup, we obtain the following statements in which the divergence of (16) is simplified.

$$\textit{Conjecture C} \qquad \Longleftrightarrow \qquad \textit{Linear forms Catlin's conjecture for } m \geqslant 2 \,.$$

$$\textit{Conjecture A} \qquad \Longleftrightarrow \qquad \textit{Linear forms Catlin's conjecture for } m + n > 2$$
$$\textit{and } \Psi(\mathbf{q}) = \psi(|\mathbf{q}|) \,.$$

6   The theorems of Jarník and Besicovitch

The results of §2–5 can be regarded as the *probabilistic theory of Diophantine approximation*. Indeed, these results indicate the probability of a certain Diophantine property and include both qualitative results like Khintchine's theorem and their quantitative versions (see [17, 33]). Furthermore, the results are rigid in the sense that the indicated probability is always either *zero* or *one*. Even in the case of the profound and as yet unsolved problem of Duffin-Schaeffer, it is known (see [17, 33]) that the measure of $\mathcal{A}'(\psi)$ and indeed $\mathcal{A}(\psi)$ must satisfy this rigid 'zero-one' law.

As the results considered obey zero-one laws, they always involve 'exceptional' sets of measure zero. The probabilistic theory of Diophantine approximation doesn't tell us anything more about the 'size' of these exceptional sets, although it is intuitively clear that it should depend on the choice of the approximating function. This leads us to a more delicate study which makes use of various concepts from geometric measure theory – in particular Hausdorff measure and dimension.

6.1   Hausdorff measures and dimension

In what follows, a *dimension function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a left continuous, monotonic function such that $f(0) = 0$. Suppose F is a subset of $\mathbb{R}^n$. Given a ball B in $\mathbb{R}^n$, let $r(B)$ denote the radius of B. For $\rho > 0$, a countable collection $\{B_i\}$ of balls in $\mathbb{R}^n$ with $r(B_i) \le \rho$ for each i such that $F \subset \bigcup_i B_i$ is called a *ρ-cover for F*. Define

$$\mathcal{H}_\rho^f(F) := \inf \sum_i f(r(B_i)),$$

where the infimum is taken over all ρ-covers of F. The *Hausdorff f–measure of F* denoted by $\mathcal{H}^f(F)$ is defined as

$$\mathcal{H}^f(F) := \lim_{\rho \to 0} \mathcal{H}_\rho^f(F) \;=\; \sup_{\rho > 0} \mathcal{H}_\rho^f(F) \,.$$

In the case that $f(r) = r^s$ ($s \geqslant 0$), the measure $\mathcal{H}^f$ is the more common *s-dimensional Hausdorff measure* $\mathcal{H}^s$, the measure $\mathcal{H}^0$ being the cardinality of F. Note that when s is a positive integer, $\mathcal{H}^s$ is a constant multiple of Lebesgue measure in $\mathbb{R}^s$ and that when $s = 1$, the measures coincide. Thus if the s-dimensional Hausdorff measure of a set is known for each $s > 0$, then so is its n-dimensional Lebesgue measure for each $n \geqslant 1$. The following easy property

$$\mathcal{H}^s(F) < \infty \quad \Longrightarrow \quad \mathcal{H}^{s'}(F) = 0 \qquad \text{if } s' > s$$

implies that there is a unique real point s at which the Hausdorff s-measure drops from infinity to zero (unless the set F is finite so that $\mathcal{H}^s(F)$ is never infinite). This point is called the Hausdorff dimension of F and is formally defined as

$$\dim F := \inf \left\{ s > 0 : \mathcal{H}^s(F) = 0 \right\}.$$

The Hausdorff dimension has been established for many number theoretic sets, e.g. $\mathcal{A}(\tau)$ (this is the Jarník-Besicovitch theorem discussed below), and is easier than determining the Hausdorff measure. Further details regarding Hausdorff measure and dimension can be found in [15, 25].

6.2   The theorems

The first step towards the study of Hausdorff measure of the set of ψ-approximable points was made by Jarník [18] in 1929 and independently by Besicovitch [6] in 1934. They determined the Hausdorff dimension of the set $\mathcal{A}(q \mapsto q^{-\tau})$, usually denoted by $\mathcal{A}(\tau)$, where $\tau > 0$.

Theorem 9 (Jarník-Besicovitch)

$$\dim \mathcal{A}(\tau) = \begin{cases} \frac{2}{\tau+1} & \text{if } \tau > 1 , \\ \\ 1 & \text{if } \tau \le 1 . \end{cases}$$

Note that for $\tau \le 1$ the result is trivial since $\mathcal{A}(\tau) = \mathbb{I}$ as a consequence of Dirichlet's theorem. Thus the main content is when $\tau > 1$. In this case, the dimension result implies that

$$\mathcal{H}^s (\mathcal{A}(\tau)) = \begin{cases} 0 & \text{if } s > \frac{2}{\tau+1} , \\ \\ \infty & \text{if } s < \frac{2}{\tau+1} , \end{cases}$$

but gives no information regarding the s–dimensional Hausdorff measure of $\mathcal{A}(\tau)$ at the critical value $s = \dim \mathcal{A}(\tau)$. In a deeper study, Jarník [19] essentially established the following general Hausdorff *measure* result for simultaneous Diophantine approximation.

Theorem 10 (Jarník) *Let f be a dimension function such that $r^{-m} f(r) \to \infty$ as $r \to 0$ and $r^{-m} f(r)$ is decreasing. Then*

$$\mathcal{H}^f (\mathcal{A}_m(\psi)) = \begin{cases} 0 & \text{if } \displaystyle\sum_{r=1}^{\infty} r^m f\left(\frac{\psi(r)}{r}\right)^m < \infty , \\ \\ \infty & \text{if } \displaystyle\sum_{r=1}^{\infty} r^m f\left(\frac{\psi(r)}{r}\right)^m = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

With $m = 1$ and $f(r) = r^s$, Jarník's theorem not only gives the above dimension result but implies that

$$\mathcal{H}^{\frac{2}{\tau+1}} (\mathcal{A}(\tau)) = \infty \quad \text{if } \tau > 1. \tag{17}$$

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For monotonic approximating functions $\psi : \mathbb{N} \to \mathbb{R}^+$, Jarník's theorem provides a beautiful and simple criteria for the 'size' of the set $\mathcal{A}_m(\psi)$ expressed in terms of Hausdorff measures. Naturally, it can be regarded as the Hausdorff measure version of Khintchine's simultaneous theorem. As with the latter, the divergence part constitutes the main substance. Notice, that the case when $\mathcal{H}^f$ is comparable to m–dimensional Lebesgue measure (i.e. $f(r) = r^m$) is excluded by the condition $r^{-m} f(r) \to \infty$ as $r \to 0$. Analogous to Khintchine's original statement, *in Jarník's original statement the additional hypotheses that $r\psi(r)^m$ is decreasing, $r\psi(r)^m \to 0$ as $r \to \infty$ and that $r^{m+1} f(\psi(r)/r)$ is decreasing were assumed.* Thus, even in the simple case when $m = 1$, $f(r) = r^s$ $(s \geqslant 0)$ and the approximating function is given by $\psi(r) = r^{-\tau} \log r$ $(\tau > 1)$, Jarník's original statement gives no information regarding the s–dimensional Hausdorff measure of $\mathcal{A}(\psi)$ at the critical exponent $s = 2/(\tau + 1) = \dim \mathcal{A}(\psi)$. That this is the case is due to the fact that $r^2 f(\psi(r)/r)$ is not decreasing. Recently, however, it has been shown in [2] that the monotonicity of ψ suffices in Jarník's theorem. In other words, the additional hypotheses imposed by Jarník are unnecessary. Furthermore, with the theorems of Khintchine and Jarník as stated above it is possible to combine them to obtain a single unifying statement.

Theorem 11 (Khintchine-Jarník) *Let f be a dimension function such that $r^{-m} f(r)$ is monotonic. Then*

$$
\mathcal{H}^f\left(\mathcal{A}_m(\psi)\right) = \begin{cases} 0 & if \quad \displaystyle\sum_{r=1}^{\infty} r^m f\left(\frac{\psi(r)}{r}\right)^m < \infty \,, \\[4mm] \mathcal{H}^f(\mathbb{I}^m) & if \quad \displaystyle\sum_{r=1}^{\infty} r^m f\left(\frac{\psi(r)}{r}\right)^m = \infty \quad and \ \psi \ is \ monotonic. \end{cases}
$$

For monotonic approximating functions, the Khintchine-Jarník theorem provides a complete measure theoretic description of $\mathcal{A}_m(\psi)$. Clearly, when $f(r) = r^m$ the theorem corresponds to Khintchine's theorem. It would be quite natural to suspect that such a unifying statement is established by combining two independent results: the Lebesgue measure statement (Khintchine's theorem) and the Hausdorff measure statement (Jarník's theorem). Indeed, the underlying method of proof of the individual statements are dramatically different. However, this is not the case. In view of the Mass Transference Principle recently established in [3] one actually has that

$$\text{Khintchine's Theorem} \quad \Longrightarrow \quad \text{Jarník's Theorem.}$$

Thus, the Lebesgue theory of $\mathcal{A}_m(\psi)$ underpins the general Hausdorff theory. At first glance this is rather surprising because the Hausdorff theory had previously been thought to be a subtle refinement of the Lebesgue theory. Nevertheless, the Mass Transference Principle allows us to transfer Lebesgue measure theoretic statements for limsup sets to Hausdorff statements and naturally obtain a complete metric theory. That this is the case is by no means a coincidence – see [3, 4, 5] for various points of view.

7   Mass Transference Principle

Given a dimension function f, define the following transformation on balls in $\mathbb{R}^m$:

$$B = B(x, r) \mapsto B^f := B(x, f(r)^{1/m}) \ .$$

When $f(x) = x^s$ for some $s > 0$ we also adopt the notation B^s for B^f. Clearly $B^m = B$. Recall that $\mathcal{H}^m$ is comparable to the m-dimensional Lebesgue measure. The limsup of a sequence of balls B_i, $i = 1, 2, 3, \ldots$ is

$$\limsup_{i \to \infty} B_i := \bigcap_{j=1}^{\infty} \bigcup_{i \geqslant j} B_i \ .$$

For such limsup sets, the following statement (the Mass Transference Principle) is the key to obtaining Hausdorff measure statements from Lebesgue statements.

Theorem 12 (Beresnevich & Velani) *Let $\{B_i\}_{i \in \mathbb{N}}$ be a sequence of balls in $\mathbb{R}^m$ with* $\mathrm{diam}(B_i) \to 0$ *as* $i \to \infty$. *Let f be a dimension function such that $x^{-m} f(x)$ is monotonic. For any finite ball B in $\mathbb{R}^m$, if*

$$\mathcal{H}^m\big(B \cap \limsup_{i \to \infty} B_i^f\big) = \mathcal{H}^m(B)$$

then

$$\mathcal{H}^f\big(B \cap \limsup_{i \to \infty} B_i^m\big) = \mathcal{H}^f(B) \ .$$

There is one point that is well worth making. The Mass Transference Principle is purely a statement concerning limsup sets arising from a sequence of balls. There is absolutely no monotonicity assumption on the radii of the balls. Even the imposed condition that $\mathrm{diam}(B_i) \to 0$ as $i \to \infty$ is redundant but is included to avoid unnecessary tedious discussion.

7.1   A Hausdorff measure Duffin-Schaeffer conjecture

As an application of the Mass Transference Principle, we shall see that the simultaneous Duffin-Schaeffer conjecture implies the corresponding conjecture for Hausdorff measures.

Let f be a dimension function. A straightforward covering argument making use of the lim sup nature of $\mathcal{A}''_m(\psi)$ implies that

$$\mathcal{H}^f(\mathcal{A}''_m(\psi)) = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} f\left(\frac{\psi(q)}{q}\right) \varphi(q)^m \ < \ \infty.$$

It is therefore natural to make the following conjecture (see [3]) which can be regarded as the simultaneous Duffin-Schaeffer conjecture for Hausdorff measures.

Conjecture D *Let f be a dimension function such that $r^{-m}f(r)$ is monotonic. Then*

$$\mathcal{H}^f(\mathcal{A}''_m(\psi)) = \mathcal{H}^f(\mathbb{I}^m) \qquad \text{if} \qquad \sum_{q=1}^{\infty} f\left(\frac{\psi(q)}{q}\right) \varphi(q)^m = \infty\,.$$

When ψ is monotonic, Conjecture D reduces to the Khintchine-Jarník theorem. It turns out that Conjecture D, a refinement of the Duffin-Schaeffer problem, is simply its consequence [3].

Theorem 13 (Beresnevich & Velani)

$$\text{The simultaneous Duffin-Schaeffer conjecture} \qquad \Longleftrightarrow \qquad \text{Conjecture D}$$

Conjecture D contains the simultaneous Duffin-Schaeffer conjecture. In order to prove the converse note that $\mathcal{A}''_m(\psi)$ is the limsup set of the sequence of balls given by

$$|qx - \mathbf{p}| < \psi(q) \text{ with } (q, \mathbf{p}) \in \mathbb{N} \times \mathbb{Z}^m \text{ and } 0 \leqslant p_j \leqslant q \text{ for all } j = \overline{1, m}.$$

First we can dispose of the case that $\psi(q)/q \nrightarrow 0$, as $q \to \infty$ as otherwise the result is trivial. We are given that $\sum f(\psi(q)/q)\,\varphi(q)^m = \infty$. Let $\theta(q) := q\,f(\psi(q)/q)^{1/m}$. Then θ is an approximating function and $\sum(\varphi(q)\,\theta(q)/q)^m = \infty$. Thus, on using the supremum norm, the Duffin-Schaeffer conjecture implies that $\mathcal{H}^m(B \cap \mathcal{A}''_m(\theta)) = \mathcal{H}^m(B)$ for any ball B in $\mathbb{I}^m$. It now follows via the Mass Transference Principle with $B = \mathbb{I}^m$ that $\mathcal{H}^f(\mathcal{A}''_m(\psi)) = \mathcal{H}^f(\mathbb{I}^m)$ and establishes Theorem 13. Since the simultaneous Duffin-Schaeffer conjecture is know to be true for $m \geqslant 2$, Theorem 13 implies the following result.

Corollary 14 *Conjecture D holds for $m \geqslant 2$.*

In a similar fashion, the Mass Transference Principle yields the following generalisation of Gallagher's theorem.

Theorem 15 *Let $m \geqslant 2$. Let f be a dimension function such that $r^{-m}f(r)$ is monotonic. Then*

$$\mathcal{H}^f(\mathcal{A}'_m(\psi)) = \begin{cases} 0 & \text{if} \quad \displaystyle\sum_{r=1}^{\infty} f\left(\frac{\psi(r)}{r}\right) r^m < \infty\,, \\[2ex] \mathcal{H}^f(\mathbb{I}^m) & \text{if} \quad \displaystyle\sum_{r=1}^{\infty} f\left(\frac{\psi(r)}{r}\right) r^m = \infty\,. \end{cases}$$

Note that since $\mathcal{A}'_m(\psi) \subseteq \mathcal{A}_m(\psi)$, Theorem 15 implies the divergent part of the Khintchine-Jarník theorem. Furthermore, we deduce that *the monotonicity condition in the Khintchine-Jarník theorem is redundant if $m \geqslant 2$.*

It is remarkable that by using the Mass Transference Principle one can deduce the Jarník-Besicovitch theorem from Dirichlet's theorem. Moreover, one obtains the stronger measure statement given by (17). Finally, we point out that all the Lebesgue measure statements on simultaneous Diophantine approximation can be generalised to the Hausdorff measure setting as above.

8 Mass Transference Principle for systems of linear forms

The Mass Transference Principle of §7 deals with $\limsup$ sets which are defined as a sequence of balls. However, the 'slicing' technique introduced in [4] extends the Mass Transference Principle to deal with $\limsup$ sets defined as a sequence of neighborhoods of 'approximating' planes. This naturally enables us to generalise the Lebesgue measure statements of §4-5 for systems of linear forms to Hausdorff measure statements. In particular, Groshev's theorem can be extended to obtain a linear forms analogue of the Khintchine–Jarník theorem.

Throughout $k, m \geqslant 1$ and $l \geqslant 0$ are integers such that $k = m + l$. Let $\mathcal{R} = (R_\alpha)_{\alpha \in J}$ be a family of planes in $\mathbb{R}^k$ of common dimension l indexed by an infinite countable set J. For every $\alpha \in J$ and $\delta \geqslant 0$ define

$$\Delta(R_\alpha, \delta) := \left\{ \mathbf{x} \in \mathbb{R}^k : \operatorname{dist}(\mathbf{x}, R_\alpha) < \delta \right\} .$$

Thus $\Delta(R_\alpha, \delta)$ is simply the δ–neighborhood of the l–dimensional plane R_α. Note that by definition, $\Delta(R_\alpha, \delta) = \emptyset$ if $\delta = 0$. Next, let

$$\Upsilon : J \to \mathbb{R}^+ : \alpha \mapsto \Upsilon(\alpha) := \Upsilon_\alpha$$

be a non-negative, real valued function on J. We assume that for every $\epsilon > 0$ the set $\{\alpha \in J : \Upsilon_\alpha > \epsilon\}$ is finite. This condition implies that $\Upsilon_\alpha \to 0$ as α runs through J. We now consider the following '$\limsup$' set,

$$\Lambda(\Upsilon) := \left\{ \mathbf{x} \in \mathbb{R}^k : \mathbf{x} \in \Delta(R_\alpha, \Upsilon_\alpha) \text{ for infinitely many } \alpha \in J \right\} .$$

Note that in view of the conditions imposed on k, l and m we have that $l < k$. Thus the dimension of the 'approximating' planes R_α is strictly less than that of the ambient space $\mathbb{R}^k$. The situation when $l = k$ is of little interest.

The following statement is a generalisation of the Mass Transference Principle to the case of systems of linear forms.

Theorem 16 (Beresnevich & Velani) *Let $\mathcal{R}$ and Υ as above be given. Let V be a linear subspace of $\mathbb{R}^k$ such that $\dim V = m = \operatorname{codim} \mathcal{R}$ and*

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$$(i) \quad V \cap R_\alpha \neq \emptyset \quad \text{for all } \alpha \in J,$$

$$(ii) \quad \sup_{\alpha \in J} \text{diam}(V \cap \Delta(R_\alpha, 1)) < \infty.$$

Let f and $g : r \to g(r) := r^{-l} f(r)$ be dimension functions such that $r^{-k} f(r)$ is monotonic. For any finite ball B in $\mathbb{R}^k$, if

$$\mathcal{H}^k\big(B \cap \Lambda\big(g(\Upsilon)^{\frac{1}{m}}\big)\big) = \mathcal{H}^k(B) \tag{18}$$

then

$$\mathcal{H}^f\big(B \cap \Lambda(\Upsilon)\big) = \mathcal{H}^f(B). \tag{19}$$

When $l = 0$, so that $\mathcal{R}$ is a collection of points in $\mathbb{R}^k$, conditions (i) and (ii) are trivially satisfied. When $l \geqslant 1$, so that $\mathcal{R}$ is a collection of l–dimensional planes in $\mathbb{R}^k$, condition (i) excludes planes R_α parallel to V and condition (ii) simply means that the angle at which R_α 'hits' V is bounded away from zero by a fixed constant independent of $\alpha \in J$. This in turn implies that each plane in $\mathcal{R}$ intersects V at exactly one point. The upshot is that the conditions (i) and (ii) are not particularly restrictive. However, we believe that they are actually redundant.

Conjecture E *Theorem 16 is valid without hypothesis (i) and (ii).*

As an application of the Mass Transference Principle for systems of linear forms, we shall obtain the following Hausdorff measure generalisation of Sprindzuk's theorem.

Theorem 17 *Let* $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$ *and suppose that* $\Psi(\mathbf{q}) = 0$ *for* $\mathbf{q} \notin \mathbf{P}^n$. *Let* f *and* $g : r \to g(r) := r^{-m(n-1)} f(r)$ *be dimension functions such that* $r^{-mn} f(r)$ *is monotonic. Then for* $n \geqslant 2$

$$\mathcal{H}^f(\mathcal{A}_{n,m}^{\mathbf{b}}(\Psi)) = \begin{cases} 0 & \text{if} \quad \displaystyle\sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} g\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right) |\mathbf{q}|^m < \infty, \\[3ex] \mathcal{H}^f(\mathbb{I}^{nm}) & \text{if} \quad \displaystyle\sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} g\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right) |\mathbf{q}|^m = \infty. \end{cases}$$

The convergence case is readily established using standard covering arguments. We will concentrate on the divergence case and assume that $r^{-k} f(r)$ is decreasing. The statement is almost obvious if the latter is not the case. When the sum given in the theorem diverges, there is a $j \in \{1, \ldots, n\}$ such that

$$\sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} g\left(\frac{\Psi_j(\mathbf{q})}{|\mathbf{q}|}\right) |\mathbf{q}|^m = \infty,$$

where $\Psi_j(\mathbf{q})$ vanishes on $\mathbf{q}$ with $|\mathbf{q}| \neq |q_j|$ and equals $\Psi(\mathbf{q})$ otherwise. Fix such a j. For each point $X \in \mathcal{A}^{\mathbf{b}}_{n,m}(\Psi_j)$ there are infinitely many $\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}$ such that

$$\|\mathbf{q}X + \mathbf{b}\| < \Psi_j(\mathbf{q}) . \tag{20}$$

In fact, we have $|\mathbf{q}| = |q_j|$ for every solution $\mathbf{q}$ of (20). Now, let

$$J := \{(\mathbf{q}, \mathbf{p}) \in \mathbb{Z}^n \setminus \{0\} \times \mathbb{Z}^m : |\mathbf{q}| = |q_j|\},$$

$\alpha := (\mathbf{q}, \mathbf{p}) \in J$, $R_\alpha := R_{\mathbf{q}, \mathbf{p}}$ where

$$R_{\mathbf{q}, \mathbf{p}} := \{X \in \mathbb{R}^{nm} : \mathbf{q}X + \mathbf{p} + \mathbf{b} = 0\} ,$$

and $\Upsilon_\alpha := \frac{\Psi_j(\mathbf{q})}{\sqrt{\mathbf{q} \cdot \mathbf{q}}}$. Then,

$$\mathcal{A}^{\mathbf{b}}_{n,m}(\Psi) \supset \mathcal{A}^{\mathbf{b}}_{n,m}(\Psi_j) = \Lambda(\Upsilon) \cap \mathbb{I}^{nm} . \tag{21}$$

Let

$$V := \{X = (\mathbf{x}_1, \ldots, \mathbf{x}_m) : x_{j,i} = 0 \ \forall \ j = 1, \ldots, m ; i = 2, \ldots, n, \} ,$$

where $\mathbf{x}_j = (x_{j,1}, \ldots, x_{j,n})$. Thus, V is an m–dimensional subspace of $\mathbb{R}^{nm}$ and we easily verify conditions (i) and (ii) of the Mass Transference Principle for linear forms, which is now applied with $k = mn$, $l = m(n-1)$ and $B = \mathbb{I}^{nm}$. Let

$$\tilde{\Psi}(\mathbf{q}) := g \left(\frac{\Psi_j(\mathbf{q})}{\sqrt{\mathbf{q} \cdot \mathbf{q}}} \right)^{1/m} \sqrt{\mathbf{q} \cdot \mathbf{q}} .$$

Then,

$$\mathcal{A}^{\mathbf{b}}_{n,m}(\tilde{\Psi}) = \Lambda(g(\Upsilon)^{1/m}) \cap \mathbb{I}^{nm} . \tag{22}$$

Since $r^{-m}g(r)$ is decreasing we have that

$$\tilde{\Psi}(\mathbf{q})^m \asymp g \left(\frac{\Psi_j(\mathbf{q})}{|\mathbf{q}|} \right) |\mathbf{q}|^m$$

so that $\sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} \tilde{\Psi}(\mathbf{q})^m = \infty$. Therefore, by Sprindzuk's theorem, we have that the set (22) has full measure in $\mathbb{I}^{nm}$ and (18) is fulfilled. Thus we have (19). The inclusion (21) completes the argument.

As a consequence of Theorem 17 we have the following statement for approximating functions with sup norm argument. The case $n = 1$, not covered by Theorem 17, corresponds to Theorem 15.

Theorem 18 *Let $n + m > 2$. Let f and $g : r \to g(r) := r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-mn} f(r)$ is monotonic. Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be an approximating function. If $n = 2$, assume that ψ is monotonic. Then*

$$\mathcal{H}^f(\mathcal{A}_{n,m}(\psi)) = \begin{cases} 0 & if \quad \displaystyle\sum_{r=1}^\infty g \left(\frac{\psi(r)}{r} \right) r^{n+m-1} < \infty , \\[3em] \mathcal{H}^f(\mathbb{I}^{nm}) & if \quad \displaystyle\sum_{r=1}^\infty g \left(\frac{\psi(r)}{r} \right) r^{n+m-1} = \infty . \end{cases}$$

We note that the validity of Conjecture A together with Mass Transference Principle for linear forms would remove the monotonicity condition on ψ in the above theorem. With ψ monotonic, the theorem corresponds to the linear forms analogue of the Khintchine-Jarník theorem as first established in [2]. Obviously, this can be deduced directly from Groshev's theorem.

Finally, it is easily verified that the Mass Transference Principle for linear forms yields the following generalisations of the Duffin-Schaeffer and Catlin conjectures stated in §5. In short, the Lebesgue conjectures imply the corresponding Hausdorff conjectures.

Conjecture F (General Duffin-Schaeffer) *Let f and $g : r \to g(r) := r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-mn} f(r)$ is monotonic. Let $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$ be an approximating function. Then*

$$\mathcal{H}^f(\mathcal{A}''_{n,m}(\Psi)) = \mathcal{H}^f(\mathbb{I}^{nm}) \qquad if \qquad \sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} g\left(\frac{\Psi(\mathbf{q})}{|\mathbf{q}|}\right) \times \left(\frac{\varphi(\gcd(\mathbf{q}))}{\gcd(\mathbf{q})}|\mathbf{q}|\right)^m = \infty \, .$$

Conjecture G (General Catlin) *Let f and $g : r \to g(r) := r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-mn} f(r)$ is monotonic. Let $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$ be an approximating function. Then*

$$\mathcal{H}^f(\mathcal{A}'_{n,m}(\Psi)) = \mathcal{H}^f(\mathbb{I}^{nm}) \qquad if \qquad \sum_{\mathbf{q} \in \mathbb{Z}^n \smallsetminus \{0\}} \max_{t \in \mathbb{N}} g\left(\frac{\Psi(t\mathbf{q})}{t|\mathbf{q}|}\right) \times \mathcal{N}_{n,m}(\mathbf{q}) = \infty \, ,$$

where $\mathcal{N}_{n,m}$ is defined in (15).

9 Twisted 'inhomogeneous' approximation

Throughout this section $\psi : \mathbb{N} \to \mathbb{R}^+$ will be a **monotonic** approximating function and we write $\mathcal{A}^{\mathbf{b}}_{n,m}(\psi)$ for the general 'inhomogeneous' set $\mathcal{A}^{\mathbf{b}}_{n,m}(\Psi)$; i.e. $\mathcal{A}^{\mathbf{b}}_{n,m}(\psi)$ is given by (8) with $\Psi(\mathbf{q}) = \psi(|\mathbf{q}|)$. The following clear cut statement, which is a direct consequence of the discussion above, provides a complete metric theory for $\mathcal{A}^{\mathbf{b}}_{n,m}(\psi)$.

Theorem 19 *Let f and $g : r \to g(r) := r^{-m(n-1)} f(r)$ be dimension functions such that $r^{-mn} f(r)$ is monotonic. Let ψ be a monotonic approximating function. Then*

$$\mathcal{H}^f(\mathcal{A}^{\mathbf{b}}_{n,m}(\psi)) = \begin{cases} 0 & if \quad \sum_{r=1}^{\infty} g\left(\frac{\psi(r)}{r}\right) \times r^{n+m-1} < \infty \, , \\[3ex] \mathcal{H}^f(\mathbb{I}^{nm}) & if \quad \sum_{r=1}^{\infty} g\left(\frac{\psi(r)}{r}\right) \times r^{n+m-1} = \infty \, . \end{cases}$$

In view of the discussion in §8, this general Hausdorff measure statement is easily seen to be a consequence of the corresponding Lebesgue statement (i.e. $f(r) = r^{mn}$ in Theorem 19) and the Mass Transference Principle for systems of linear forms. It is also worth mentioning, especially in the context of what is about to follow, that the behavior of $\mathcal{H}^f(\mathcal{A}_{n,m}^{\mathbf{b}}(\psi))$ is completely independent of the fixed inhomogeneous factor $\mathbf{b} \in \mathbb{I}^m$.

We now consider a somewhat 'twisted' version of the set $\mathcal{A}_{n,m}^{\mathbf{b}}(\psi)$ in which the inhomogeneous factor $\mathbf{b}$ becomes the object of approximation. More precisely, given $X \in \mathbb{I}^{mn}$ let

$$\mathcal{V}_{n,m}^X(\psi) := \{\mathbf{b} \in \mathbb{I}^m : \|\mathbf{q}X + \mathbf{b}\| < \psi(|\mathbf{q}|) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\} \ .$$

For the ease of motivation and indeed clarity of results we begin by describing the one–dimensional situation.

9.1   The one–dimensional theory

For any irrational x and any real number b, a theorem of Khintchine states that there are infinitely many integers q such that

$$\|qx - b\| < \frac{1 + \epsilon}{\sqrt{5}\, q} \ . \tag{23}$$

In this statement $\epsilon > 0$ is arbitrary and apart from this term it is equivalent to Hurwitz's homogeneous ($b = 0$) theorem. A weaker form, with $3/q$ appearing on the right hand side of (23), had been established earlier by Tchebychef. This enabled him to conclude that for any irrational x the sequence $\{qx\}_{q \in \mathbb{N}}$ modulo one is dense in the unit interval. Later this sequence has been shown to be uniformly distributed. In view of this density result, it is natural to consider the problem of approximating points in the unit interval with a pre-described rate of approximation by the sequence $qx \bmod 1$. That is to say, to investigate the set $\mathcal{V}^x(\psi) := \mathcal{V}_{1,1}^x(\psi)$.

Before describing a complete metric theory for $\mathcal{V}^x(\psi)$ we state two results which are simple consequences of (23) and the Mass Transference Principle (Theorem 7). Given $\tau \geqslant 0$, let $\psi : r \to r^{-\tau}$ and write $\mathcal{V}^x(\tau)$ for $\mathcal{V}^x(\psi)$.

Theorem 20 *Let x be irrational. For $\tau \geqslant 1$,*

$$\mathcal{H}^{\frac{1}{\tau}}(\mathcal{V}^x(\tau)) \ = \ \mathcal{H}^{\frac{1}{\tau}}(\mathbb{I}) \ .$$

It follows directly from the definition of Hausdorff dimension that $\dim \mathcal{V}^x(\tau) \geqslant 1/\tau$. The complementary upper bound result is easily establish and as a corollary we obtain the following statement.

Corollary 21 (Bugeaud [8] and Schmeling & Troubetzkoy [29]) *Let x be irrational. For $\tau > 1$,*

$$\dim \mathcal{V}^x(\tau) = \frac{1}{\tau} \ .$$

Thus the corollary is a simple consequences of (23) and the Mass Transference Principle. Moreover, we are able to deduce that the Hausdorff measure at the critical exponent is infinity; i.e. for $\tau > 1$,

$$\mathcal{H}^{\frac{1}{\tau}}(\mathcal{V}^x(\tau)) \ = \infty \ .$$

Next, for $\epsilon > 0$ let $\psi_\epsilon : r \to \epsilon/r$ and $f_\epsilon : r \to r/\epsilon$. Note that $\mathcal{H}^{f_\epsilon}$ is simply one dimensional Lebesgue measure scaled by a multiplying factor $1/\epsilon$. Now on combining (23) and the Mass Transference Principle in the obvious manner, we obtain that for any irrational x

$$\mathcal{H}^{f_\epsilon}(\mathcal{V}^x(\psi_\epsilon)) \ = \ \mathcal{H}^{f_\epsilon}(\mathbb{I}) \ = \ 1/\epsilon \ .$$

Alternatively,

$$\mathcal{H}^{f_\epsilon}(\mathbb{I} \setminus \mathcal{V}^x(\psi_\epsilon)) \ = \ 0$$

and via a simple covering argument we deduce that $\mathcal{H}^1(\mathbb{I} \setminus \mathcal{V}^x(\psi_\epsilon)) \ = \ 0$. Thus, for any $\epsilon > 0$ and any irrational x we have that

$$|\mathcal{V}^x(\psi_\epsilon)| \ = \ 1 \ , \tag{24}$$

and we have given a short and 'direct' proof of the following statement.

Theorem 22 (Kim [22]) *Let x be irrational. For almost every $b \in \mathbb{I}$*

$$\liminf_{q \to \infty} q||qx - b|| \ = \ 0 \ .$$

Note the above theorems and corollary are statements for any irrational x – the relevance of this will soon become apparent.

We now turn our attention towards developing a complete metric theory for $\mathcal{V}^x(\psi)$. To begin with, let us concentrate on the Lebesgue theory. On exploiting the limsup nature of the set $\mathcal{V}^x(\psi)$, it is easily verified that for any irrational x and any approximating function ψ

$$|\mathcal{V}^x(\psi)| \ = \ 0 \quad \text{if} \quad \sum_{r=1}^{\infty} \psi(r) < \infty \ .$$

It is worth stressing that the choice of the irrational x and the 'convergent' ψ are completely irrelevant. Naturally, one may suspect or even expect that $|\mathcal{V}^x(\psi)| = 1$ if the above sum diverges – irrespective of the irrational x and the 'divergent' ψ. This is certainly the situation in the 'standard' inhomogeneous setup; i.e. for the set $\mathcal{A}^b(\psi)$ the choice of the inhomogeneous factor b and the 'divergent' ψ are completely irrelevant. However, the

'twisted' setup throws up a few surprises which to some extent lead to a 'richer' theory – particularly in higher dimensions. Regarding the latter, it is slightly out of place to give a discussion here and we refer the reader to [9]. The following statement and indeed its higher dimension analogue (see §9.2) is due to Kurzweil [23].

Theorem 23 (Kurzweil) *Let ψ be given. Then for almost all irrational x*

$$|\mathcal{V}^x(\psi)| \; = \; 1 \quad if \quad \sum_{r=1}^{\infty} \psi(r) = \infty \; .$$

As already mentioned the complementary convergent part is valid for all irrational x. The above theorem indicates that the set of irrational x for which we obtain the full measure statement is dependent on the choice of the 'divergent' ψ. To clarify this and to take the discussion further, let

$$\mathcal{D} := \{\psi : \sum_{r=1}^{\infty} \psi(r) = \infty\} \; .$$

Thus the set $\mathcal{D}$ is the set of 'divergent' approximating functions ψ. Also, for $\psi \in \mathcal{D}$ let

$$\mathcal{V}(\psi) \; := \; \{x \in \mathbb{I} : |\mathcal{V}^x(\psi)| \; = \; 1\} \; .$$

With this notation in mind, Theorem 23 simply states that $|\mathcal{V}(\psi)| \; = \; 1$. Furthermore, Kurzweil in [23] solves a problem of H. Steinhaus by establishing the following elegant result which characterizes the set $\mathcal{B}$ of badly approximable numbers in terms of 'twisted' inhomogeneous approximation.

Theorem 24 (Kurzweil)

$$\bigcap_{\psi \in \mathcal{D}} \mathcal{V}(\psi) \; = \; \mathcal{B} \; .$$

Thus, for any given irrational x which is not badly approximable there is a 'divergent' approximating function ψ for which $|\mathcal{V}^x(\psi)| \; \neq \; 1$. In other words, Theorem 23 is in general false for all irrational x.

The truth of the statement of Theorem 24 can to some extent be explained by the fact that the distribution of $qx \bmod 1$ is best possible if $x \in \mathcal{B}$. More precisely, it is well known that the discrepancy $D(N)$ of $qx \bmod 1$ satisfies $D(N) \ll \log N$ if $x \in \mathcal{B}$ and that $D(N) \gg \log N$ infinitely often for any real number x. A theorem of Schmidt [31], building on the pioneering work of Roth [27], states that the latter is indeed the case for any sequence $x_n \bmod 1$.

Before moving onto the general Hausdorff theory, we point out that Theorem 23 implies (24) for almost almost all x. This leads to the weaker 'almost all irrational' (rather than all irrational) version of Theorem 22. The point is that for the particular function ψ_ϵ appearing (24), it is possible to replace 'almost all irrational' by 'all irrational' in the statement of Theorem 23. This then implies Theorem 22 as stated.

On applying the Mass Transference Principle in the obvious manner, we are able to deduce from Theorem 23 the following general Hausdorff measure statement – the convergent part is straightforward and is valid for all irrationals.

Theorem 25 *Let f be dimension function such that $r^{-1}f(r)$ is monotonic and let ψ be a monotonic approximating function. Then for almost all irrational x*

$$
\mathcal{H}^f(\mathcal{V}^x(\psi)) =
\begin{cases}
0 & \text{if} \quad \displaystyle\sum_{r=1}^{\infty} f(\psi(r)) < \infty \, , \\[2em]
\mathcal{H}^f(\mathbb{I}) & \text{if} \quad \displaystyle\sum_{r=1}^{\infty} f(\psi(r)) = \infty \, .
\end{cases}
$$

It is worth pointing out that in the case $\psi : r \to r^{-\tau}$ and $f : r \to r^s$, it is possible to strengthen the theorem to all irrational x – see Theorem 20. The key is that in this situation one can apply the Mass Transference Principle to (23), which is valid for any irrational x. This is the reason why we are able to prove a dimension result (Corollary 21) for any irrational rather than just almost all irrational. The latter is all that we can obtain from Theorem 25.

We end our discussion of the one-dimensional 'twisted' theory by attempting to generalize Theorem 24. Let f be a dimension function such that $r^{-1}f(r)$ is monotonic and let

$$
\mathcal{D}^f := \{\psi : \textstyle\sum_{r=1}^{\infty} f(\psi(r)) = \infty\} \, .
$$

The set $\mathcal{D}^f$ is the set of 'f-divergent' approximating functions ψ. Also, for $\psi \in \mathcal{D}^f$ let

$$
\mathcal{V}^f(\psi) := \{x \in \mathbb{I} : \mathcal{H}^f(\mathcal{V}^x(\psi)) = \mathcal{H}^f(\mathbb{I})\} \, .
$$

In this notation, the divergent part of Theorem 25 simply states that $|\mathcal{V}^f(\psi)| = 1$. Furthermore, on combining the Mass Transference Principle and Theorem 24 we obtain the following result.

Theorem 26 *Let f be dimension function such that $r^{-1}f(r)$ is monotonic. Then*

$$
\bigcap_{\psi \in \mathcal{D}^f} \mathcal{V}^f(\psi) \supseteq \mathcal{B} \, .
$$

The following conjecture is a natural refinement of Theorem 24.

Conjecture H *Let f be dimension function such that $r^{-1}f(r)$ is monotonic. Then*

$$
\bigcap_{\psi \in \mathcal{D}^f} \mathcal{V}^f(\psi) = \mathcal{B} \, .
$$

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Note that on combining Theorems 24 and 26 we obtain the following statement.

Corollary 27 *Let $\mathcal{F}$ be the set of dimension functions f such that $r^{-1}f(r)$ is monotonic. Then*

$$\bigcap_{f \in \mathcal{F}} \bigcap_{\psi \in \mathcal{D}^f} \mathcal{V}^f(\psi) = \mathcal{B} \, .$$

9.2 The higher dimensional theory

Starting with the Lebesgue theory, Kurzweil established the higher dimensional analogue of Theorem 23.

Theorem 28 (Kurzweil) *Let ψ be given. Then for almost all $X \in \mathbb{I}^{mn}$*

$$|\mathcal{V}_{n,m}^X(\psi)|_m \ = \ 1 \quad if \quad \sum_{r=1}^{\infty} r^{n-1} \psi(r)^m = \infty \, .$$

On applying the Mass Transference Principle (Theorem 7), we obtain the following general Hausdorff statement. The convergence part is again straightforward and is valid for all $X \in \mathbb{I}^{mn}$.

Theorem 29 *Let f be a dimension function such that $r^{-m}f(r)$ is monotonic and let ψ be a monotonic approximating function. Then for almost all $X \in \mathbb{I}^{nm}$*

$$\mathcal{H}^f(\mathcal{V}_{n,m}^X(\psi)) = \begin{cases} 0 & if \quad \displaystyle\sum_{r=1}^{\infty} f(\psi(r)) \, r^{n-1} < \infty \, , \\[2em] \mathcal{H}^f(\mathbb{I}^m) & if \quad \displaystyle\sum_{r=1}^{\infty} f(\psi(r)) \, r^{n-1} = \infty \, . \end{cases}$$

Let $\psi : r \to r^{-\tau}$ and write $\mathcal{V}_{n,m}^X(\tau)$ for $\mathcal{V}_{n,m}^X(\psi)$. As a consequence of the above theorem we have the following corollary.

Corollary 30 *Let $\tau > n/m$. Then for almost all $X \in \mathbb{I}^{nm}$*

$$\dim \mathcal{V}_{n,m}^X(\tau) = \frac{n}{\tau} \quad and \ moreover \quad \mathcal{H}^{\frac{n}{\tau}}(\mathcal{V}_{n,m}^X(\tau)) \ = \infty \ \, .$$

The above dimension statement is not new – see [9]. It is worth stressing that in higher dimensions it is not possible to obtain a dimension statement for all 'irrational' X as in the one dimensional theory – see [9]. The point is that the higher dimensional analogue of (23) is only valid for almost all $X \in \mathbb{I}^{mn}$ rather than all 'irrational' $X \in \mathbb{I}^{mn}$. Finally, we mention that Kurzweil also obtained the higher dimensional analogue of Theorem 24 and therefore the analogues of Theorem 26 and Corollary 27 for arbitrary n and m are also possible.

9.3 Back to algebraic irrationals and Roth again

We end up this paper with another discussion on the interactions of Roth's theorem and the metrical theory of Diophantine approximation. As mentioned in §1, quadratic real algebraic numbers are badly approximable and Roth's theorem states that algebraic numbers of degree $n \geqslant 3$ denoted by $\mathbb{A}_n$ are relatively badly approximable. It is also believed that $\mathbb{A}_n$ does not contain badly approximable numbers for $n \geqslant 3$. If the latter is the case then for any algebraic number α of degree $\geqslant 3$ we have $\alpha \notin \mathcal{V}(\psi)$ – see Theorem 24. In other words, one must be able to construct a monotonic approximating function ψ with $\sum_{r=1}^{\infty} \psi(r) = \infty$ such that

$$|\mathcal{V}^{\alpha}(\psi)| < 1 \,.$$

Restating this by making use of the definition of the set $\mathcal{V}^{\alpha}(\psi)$ we are naturally led to the following conjecture.

Conjecture I *For any $n \geqslant 3$ and any $\alpha \in \mathbb{A}_n$ there is a monotonic approximating function ψ and a subset B in $[0,1]$ of positive Lebesgue measure such that*

$$\sum_{r=1}^{\infty} \psi(r) = \infty$$

but for any $b \in B$ the inequality

$$\|q\alpha + b\| < \psi(q)$$

has only finitely many solutions $q \in \mathbb{N}$.

It is quite possible that one can prove the following inhomogeneous strengthening of Lang's conjecture that would imply Conjecture I.

Conjecture J *For any $n \geqslant 3$ and any $\alpha \in \mathbb{A}_n$. Then there is a subset B in $[0,1]$ of positive measure such that for any $b \in B$ the inequality*

$$\|q\alpha + b\| < \frac{1}{q \log q}$$

has only finitely many solutions $q \in \mathbb{N}$.

Possibly, the only condition that needs to be imposed on b to fulfil the above conjecture is that b and α are linearly (or algebraically) independent over $\mathbb{Q}$.

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around the theme that there are no integers between zero and one have been embedded in my grey matter forever. This latter fact leads me naturally onto Ayesha and Iona – so girls, the Scarecrow has finally found his brain and it's all thanks to Klaus Roth. Thus all is well in Oz and we can all go home – whoops this will inevitably lead to some other unknown journey!

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