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МОНОТОННАЯ РАЗНОСТНАЯ СХЕМА ПОВЫШЕННОГО ПОРЯДКА ТОЧНОСТИ ДЛЯ ДВУМЕРНЫХ УРАВНЕНИЙ КОНВЕКЦИИ – ДИФФУЗИИ

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Для двумерного стационарного уравнения конвекции – диффузии общего вида построена, теоретически обоснована и испытана на тестовой задаче устойчивая конечно-разностная схема, определенная на минимальном шаблоне равномерной сетки, удовлетворяющая принципу максимума и обладающая четвертым порядком аппроксимации. Монотонность схемы контролируется двумя параметрами регуляризации, введенными в разностный оператор. Схема ориентирована на решение прикладных задач конвекции – диффузии в условиях развитого пограничного слоя, включая гравитационную и термомагнитную конвекцию, диффузию частиц в магнитной жидкости. Схема апробирована на известной задаче высокоинтенсивной гравитационной конвекции в горизонтальном канале квадратного сечения при однородном нагреве сбоку. Проведено детальное сравнение с монотонной схемой Самарского второго порядка аппроксимации на последовательности квадратных сеток с числом разбиений от 10 до 1000 на каждой стороне квадрата во всем диапазоне чисел Рэлея, соответствующих режиму ламинарной конвекции. Показано значительное преимущество схемы четвертого порядка в скорости сходимости при уменьшении шага сетки.

Ключевые слова: гравитационная конвекция; термомагнитная конвекция; диффузия частиц; уравнение конвекции – диффузии; разностная схема повышенного порядка аппроксимации; принцип максимума; параметры регуляризации.

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A MONOTONE FINITE-DIFFERENCE HIGH ORDER ACCURACY SCHEME FOR THE 2D CONVECTION − DIFFUSION EQUATIONS

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A stable finite-difference scheme is constructed on a minimum stencil of a uniform mesh for a two-dimensional steady-state convection – diffusion equation of a general form; the scheme is theoretically studied and tested. It satisfies the maximum principle and has the fourth order of approximation. The scheme monotonicity is controlled by two regularization parameters introduced into the difference operator. The scheme is focused on solving applied convection – diffusion problems with a developed boundary layer, including gravitational convection, thermomagnetic convection, and diffusion of particles in a magnetic fluid. The scheme is tested on the well-known problem of a high-intensive gravitational convection in a horizontal channel of a square cross-section with a uniform heating from the side. A detailed comparison is performed with the monotone Samarskii scheme of the second order approximation on the sequences of square meshes with the number of partitions from 10 to 1000 on each side of the square domain and over the entire range of the Rayleigh numbers, corresponding to the laminar convection mode. A significant advantage of the fourth order scheme in the convergence rate is shown for the decreasing mesh step.

Keywords: gravitational convection; thermomagnetic convection; diffusion of particles; convection – diffusion equation; finite-difference high order approximation scheme; maximum principle; parameters of regularization.

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Introduction

A solution of the applied convective heat transfer problems requires a transition to the region of high values of the Rayleigh numbers, which is characterized by a formation of boundary layers with large velocity and temperature gradients and small-scale convective motions. Similarly, the concentration of solid suspended particles in colloidal systems is redistributed because of their diffusion under the action of mass forces. For example, the ferromagnetic particles in a magnetic fluid diffuse in the direction of the magnetic-field gradient, creating zones near the boundary with large gradients of the particle concentration [1; 2]. This imposes strong requirements on stabilization and approximation properties of a difference scheme. The problem is particularly crucial in a three-dimensional case. An increase of the approximation order of the difference scheme is one of the way to solve the problem, although it is very difficult to fulfill contradictory requirements of stability and accuracy.

A standard way to increase an approximation order of a difference scheme consists in a replacement of the high order derivatives in the main part of the approximation error by the lower order derivatives, which are suitable for a difference approximation on a minimum stencil, with the help of the original differential equation under assumption of sufficiently smooth functions of the equation. The stable schemes of fourth order approximation were constructed in this way in [3] for the two-dimensional Poisson equation with steps $\frac{1}{\sqrt{5}} \le \frac{h_1}{h_2} \le \sqrt{5}$ on a uniform mesh. In principle, it is not difficult to get the fourth order scheme for the convection – diffusion equation with variable coefficients, but a serious problem is ensuring the scheme monotonicity, i. e. fulfilling conditions of the maximum principle. The practice of numerical solution of convection and diffusion problems has shown that the property of monotonicity is an important factor of a scheme applicability in conditions of a developed boundary layer.

A lot of current publications in computational mathematics are devoted to the development of numerical methods for convection – diffusion problems including two-dimensional ones (see, e. g., [4–7]). To solve them, effective finite-difference and finite-element algorithms of the first or second order of accuracy are developed.

In this work, a monotone finite-difference scheme of the fourth order of approximation is constructed for the two-dimensional steady-state convection – diffusion equations in magnetic and non-magnetic fluids. The scheme is defined on a minimum nine-point stencil of a uniform mesh. Its monotonicity is provided by two regularization parameters introduced into the difference operator. The scheme is tested on the well-known problem of natural convection.

Equations of gravitational and thermomagnetic convection

One has to deal with the problem of controlling convective heat exchange in closed cavities in design of many technological devices (e. g., cooling systems for high-voltage electric cables, power transformers, electric generators and electric motors, nuclear reactors, etc.). There are two mechanisms for convection in a non-isothermal magnetic fluid located in gravitational and non-uniform magnetic fields: gravitational and magnetic one. The first mechanism is due to the dependence of density on temperature, the second one is due to the dependence of magnetization on temperature. The presence of the magnetic mechanism opens up real possibilities in controlling the structure and the intensity of convective process by applied magnetic field. This is especially important under zero-gravity conditions, when the gravitational mechanism is absent.

The most common and investigated model of a thermomagnetic convection is a model for homogeneous, non-conducting and incompressible magnetic fluid without heat sources in the temperature equation and with the linear state equations [8–12]. The system of the steady-state convective equations for this model under the Boussinesq approximation for the density and the non-inductive approximation for the magnetic field takes the form

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nu\nabla^2\mathbf{v} + \frac{1}{\rho_0}\left(-\nabla p + \rho\mathbf{g} + \mu_0 M\nabla H\right), \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0,\ \mathbf{v}\cdot\nabla T = a\nabla^2 T; \tag{2}$$

$$\rho = \rho_0\left[1-\beta\left(T-T_0\right)\right],\ M = M\left(T_0, H_0\right) - K\left(T-T_0\right) + \frac{\partial M\left(T_0, H_0\right)}{\partial H}\left(H-H_0\right),$$

$$\rho_0 = \rho\left(T_0\right),\ \beta = -\frac{1}{\rho_0}\frac{\partial\rho\left(T_0\right)}{\partial T},\ K = -\frac{\partial M\left(T_0, H_0\right)}{\partial T},$$

where $\mathbf{v}$ is the velocity vector of the convective motion; T is the absolute temperature of the fluid; p is the pressure; H is the given value of the magnetic-field intensity; ρ is the fluid density; $\mathbf{g}$ is the gravitational acceleration vector; $M = M\left(T, H\right)$ is the magnetization of the fluid for the uniform distribution of magnetic particles; $\mu_0 = 4\pi\cdot 10^{-7}$ H/m is the magnetic constant; T_0 and H_0 are the characteristic values of the temperature and the field intensity in the fluid bulk; ν, a and β are the coefficients of the kinematic viscosity, the thermal conductivity and the volumetric thermal expansion of the fluid; K is the pyromagnetic coefficient. The last two terms in equation (1) define the gravitational and magnetic mechanisms of convection, respectively.

The idea of the non-inductive approximation consists in neglecting the influence of the fluid on the external magnetic field. The validity of the non-inductive approximation is shown in [8–11] for a wide class of thermomagnetic convection problems.

A Cartesian coordinate system x_1, x_2, x_3 with the coordinate orts $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is introduced. We set in equations (1), (2) that $\mathbf{v} = \mathbf{v}\left(\upsilon_1, \upsilon_2, 0\right)$, $\upsilon_1 = \upsilon_1\left(x_1, x_2\right)$, $\upsilon_2 = \upsilon_2\left(x_1, x_2\right)$, $T = T\left(x_1, x_2\right)$, $\rho = \rho\left(x_1, x_2\right)$, $H = H\left(x_1, x_2\right)$, $\mathbf{g} = \mathbf{g}\left(g_1, g_2, 0\right)$ assuming that the convective problem is two-dimensional. Let us define a stream function $\psi\left(x_1, x_2\right)$ and a vorticity $\omega\left(x_1, x_2\right)$ associated with the velocity components by relations

$$\upsilon_1 = \frac{\partial\psi}{\partial x_2},\ \upsilon_2 = -\frac{\partial\psi}{\partial x_1},\ \omega = \frac{\partial\upsilon_2}{\partial x_1} - \frac{\partial\upsilon_1}{\partial x_2}. \tag{3}$$

The continuity equation $\nabla\cdot\mathbf{v} = 0$ is automatically satisfied in these variables. We obtain the vector equation for the vorticity by applying the rotor operator to motion equation (1) and taking into account (3):

$$\nabla\times\left[\mathbf{v}\times\left(\omega\mathbf{k}\right)\right] = \nu\nabla\times\left[\nabla\times\left(\omega\mathbf{k}\right)\right] + \beta\nabla\times\left(T\mathbf{g}\right) + \left(\frac{\mu_0 K}{\rho_0}\right)\nabla T\times\nabla H.$$

Thus, equations (1), (2) in 2D case are transformed into a system of three scalar equations for the temperature T, the stream function ψ and the vorticity ω:

$$\begin{gathered}\frac{\partial\psi}{\partial x_2}\frac{\partial T}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial T}{\partial x_2} = a\left(\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2}\right),\ \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \omega = 0,\\ \frac{\partial\psi}{\partial x_2}\frac{\partial\omega}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial\omega}{\partial x_2} = \nu\left(\frac{\partial^2\omega}{\partial x_1^2} + \frac{\partial^2\omega}{\partial x_2^2}\right) - \beta\left(\frac{\partial T}{\partial x_1}g_2 - \frac{\partial T}{\partial x_2}g_1\right) + \frac{\mu_0 K}{\rho_0}\left(\frac{\partial T}{\partial x_2}\frac{\partial H}{\partial x_1} - \frac{\partial T}{\partial x_1}\frac{\partial H}{\partial x_2}\right).\end{gathered} \tag{4}$$

Let Ox_2-axis be the vertical axis in the Cartesian coordinate system x_1, x_2, in which case $g_1 = 0$, $g_2 = -g$. Let T_0 and $T_1 = T_0 + \Delta T$ define the given minimum and maximum values of the temperature on the walls. We introduce dimensionless variables by choosing the characteristic size of the computational domain l as the length scale, the kinematic viscosity ν as the scale for the stream function, the relation νl^2 as the scale for the vorticity, the relation νl as the scale for the velocity, the temperature difference ΔT as the scale for the temperature, and the value γl as the scale for the magnetic field intensity where γ is a characteristic value of the field gradient. For convenience, we denote the dimensionless variables in the same way as the dimensional ones, and write system (4) in new variables (see [12]):

$$\begin{gathered}\mathbf{v}\cdot\nabla T = \frac{1}{\mathrm{Pr}}\nabla^2 T,\ \nabla^2\psi + \omega = 0,\ \mathbf{v}\cdot\nabla\omega = \nabla^2\omega + f,\ f = \mathrm{Gr}\frac{\partial T}{\partial x_1} + \mathrm{Gr}_m\frac{\partial(H,T)}{\partial(x_1,x_2)};\\ \upsilon_1 = \frac{\partial\psi}{\partial x_2},\ \upsilon_2 = -\frac{\partial\psi}{\partial x_1};\ \mathrm{Pr} = \frac{\nu}{a},\ \mathrm{Gr} = \frac{\beta g l^3\Delta T}{\nu^2},\ \mathrm{Gr}_m = \frac{\mu_0 K l^3\Delta T\gamma}{\rho_0\nu^2},\end{gathered} \tag{5}$$

where Pr is the Prandtl number; Gr is the Grashof number and Gr_m is the magnetic Grashof number. Equations (5) at $\mathrm{Gr}_m = 0$ describe the process of natural (gravitational) convection.

Equation of particle diffusion in magnetic fluid

The magnetic fluid is a stable colloidal suspension of ferromagnetic nanoparticles in a nonmagnetic carrier liquid. A particle size is of the order of 10 nm = 10^{-8} m and they are in a Brownian motion in the carrier liquid. Due to the magnetic properties of particles, not only the Brownian motion but also the diffusion of particles under the action of a non-uniform magnetic field (magnetophoresis) occurs in the magnetic fluid. The particles are distributed in the fluid bulk as a result of the competition between these two mechanisms.

The steady-state diffusion equation for magnetic particles in a magnetic fluid in the presence of a convective motion takes the form [1; 2; 13]:

$$\begin{gathered}\nabla^2 C - \left(\frac{1}{D}\mathbf{v} + \alpha\right)\cdot\nabla C - qC = 0,\\ q = \nabla\cdot\alpha,\ \alpha = \mathcal{L}(\xi)\nabla\xi,\ \xi = \frac{\mu_0 m}{kT}H,\ \mathcal{L}(\xi) = \coth(\xi) - \frac{1}{\xi} > 0,\end{gathered} \tag{6}$$

where C is the volume particle concentration in the colloid; D is the diffusion coefficient; $\mathcal{L}(\xi)$ is the Langevin function; m is the magnetic moment of a particle; $k = 1.380\,656\,8\cdot 10^{-23}$ J/K is the Boltzmann constant; T is the particle temperature.

The magnetization M is a function of the field intensity and the particles concentration, i. e. $M = M(H, C)$, for isothermal magnetic fluids. Under the condition $M(H, C) \ll H$, the Maxwell equations are of the form $\nabla\times\mathbf{H} = 0$, $\nabla\cdot\mathbf{H} = 0$. In 2D case of Cartesian coordinates, it follows that

$$\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2} = 0,\ \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} = 0, \tag{7}$$

where H_1, H_2 are the components of the intensity vector $\mathbf{H}$.

From the point of view of stability of the difference scheme, it is important to show that the coefficient q in equation (6) takes only positive values. We prove this taking into account (7). Consider first

$$\begin{aligned}|\nabla H|^2 &= \left(\frac{\partial H}{\partial x_1}\right)^2 + \left(\frac{\partial H}{\partial x_2}\right)^2 = \frac{1}{H^2}\left[\left(H_1\frac{\partial H_1}{\partial x_1} + H_2\frac{\partial H_2}{\partial x_1}\right)^2 + \left(H_1\frac{\partial H_1}{\partial x_2} + H_2\frac{\partial H_2}{\partial x_2}\right)^2\right] =\\ &= \frac{1}{H^2}\left[H_1^2\left(\frac{\partial H_1}{\partial x_1}\right)^2 + H_2^2\left(\frac{\partial H_2}{\partial x_1}\right)^2 + H_1^2\left(\frac{\partial H_1}{\partial x_2}\right)^2 + H_2^2\left(\frac{\partial H_2}{\partial x_2}\right)^2\right] +\\ &+ \frac{2}{H^2}H_1H_2\underbrace{\left(\frac{\partial H_1}{\partial x_1}\frac{\partial H_2}{\partial x_1} + \frac{\partial H_1}{\partial x_2}\frac{\partial H_2}{\partial x_2}\right)}_{\text{equal 0 by virtue of (7)}} = \left(\frac{\partial H_1}{\partial x_1}\right)^2 + \left(\frac{\partial H_2}{\partial x_1}\right)^2 \overset{(7)}{=} \left(\frac{\partial H_2}{\partial x_2}\right)^2 + \left(\frac{\partial H_1}{\partial x_2}\right)^2.\end{aligned} \tag{8}$$

Then

$$\nabla^2 H = \frac{\partial}{\partial x_1}\left(\frac{1}{2H}\frac{\partial\left(H_1^2 + H_2^2\right)}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{2H}\frac{\partial\left(H_1^2 + H_2^2\right)}{\partial x_2}\right) =$$

$$= \frac{1}{H}\left[\left(\frac{\partial H_1}{\partial x_1}\right)^2 + \left(\frac{\partial H_2}{\partial x_1}\right)^2 + \left(\frac{\partial H_1}{\partial x_2}\right)^2 + \left(\frac{\partial H_2}{\partial x_2}\right)^2 + H_1\nabla^2 H_1 + H_2\nabla^2 H_2\right] -$$

$$- \frac{1}{2H^2}\left(\frac{\partial H}{\partial x_1}\frac{\partial\left(H^2\right)}{\partial x_1} + \frac{\partial H}{\partial x_2}\frac{\partial\left(H^2\right)}{\partial x_2}\right) \overset{(7),\,(8)}{=} \frac{1}{H}\left|\nabla H\right|^2 > 0 \text{ if } \nabla H \neq 0.$$

Hence

$$\nabla^2 \xi = \frac{1}{\xi}\left|\nabla\xi\right|^2 > 0 \text{ if } \nabla\xi \neq 0. \tag{9}$$

Taking into account (9), we obtain

$$q = \nabla\cdot\alpha = \nabla\cdot\left(\mathcal{L}(\xi)\nabla\xi\right) = \nabla\mathcal{L}(\xi)\cdot\nabla\xi + \mathcal{L}(\xi)\nabla^2\xi = \frac{d}{d\xi}\left(\xi\mathcal{L}(\xi)\right)\frac{1}{\xi}\left|\nabla\xi\right|^2 \geq 0.$$

Thus, we get that $q \geq 0$ in concentration equation (6). Moreover, we have $q \equiv 0$ if and only if $\nabla H \equiv 0$, i. e. when the magnetic field is uniform or absent.

Difference scheme of high order accuracy

Let us consider a two-dimensional steady-state convection – diffusion equation

$$\sum_{\alpha=1}^{2}\mathcal{L}_\alpha^{(k,\,\upsilon)}u - q(x)u = -f(x),\ x = (x_1,\,x_2) \in G, \tag{10}$$

where $\mathcal{L}_\alpha^{(k,\,\upsilon)}u = \mathcal{L}_\alpha u - \upsilon_\alpha(x)k(x)\dfrac{\partial u}{\partial x_\alpha}$, $\mathcal{L}_\alpha u = \dfrac{\partial}{\partial x_\alpha}\left(k(x)\dfrac{\partial u}{\partial x_\alpha}\right)$, $k(x) > 0$, $q(x) \geq 0$, $u = u(x)$, is the unknown function satisfying equation (10); k, q, υ_1, υ_2, and f are the given functions; x_1, x_2 are the space coordinates. All functions are assumed to be sufficiently smooth. The first term in the differential operator $\mathcal{L}_\alpha^{(k,\,\upsilon)}u$ is the diffusion term, the second one is the convective term. Note that each of equations (4)–(6) can be written in form (10).

Scheme construction. We construct the finite-difference scheme for equation (10) which has the fourth order of approximation on the minimal nine-point stencil of a uniform mesh and satisfies the maximum principle. Note, that for $q \equiv 0$ the high order scheme is presented in [12].

We approximate the differential operators $\mathcal{L}_\alpha^{(k,\,\upsilon)}$, $\alpha = 1,\,2$, by monotone difference operators $\Lambda_\alpha^{(a,\,b)}$ of the form

$$\Lambda_\alpha^{(a,\,b)}u = æ_\alpha\left(a_\alpha u_{\bar{x}_\alpha}\right)_{x_\alpha} - b_\alpha^+ a_\alpha u_{\bar{x}_\alpha} - b_\alpha^- a_\alpha^{(+1_\alpha)}u_{x_\alpha} =$$

$$= \left(1 + æ_\alpha R_\alpha^4\right)\left(a_\alpha u_{\bar{x}_\alpha}\right)_{x_\alpha} - \frac{1}{2}b_\alpha\left(a_\alpha u_{\bar{x}_\alpha} + a_\alpha^{(+1_\alpha)}u_{x_\alpha}\right) \tag{11}$$

with the coefficients

$$a_\alpha = \frac{6}{\left(\dfrac{1}{k^{(-1_\alpha)}} + \dfrac{4}{k^{(-0.5_\alpha)}} + \dfrac{1}{k}\right)} > 0,\ b_\alpha = \upsilon_\alpha + O\left(h^4\right),\ h = \sqrt{h_1^2 + h_2^2},$$

$$æ_\alpha = \frac{1}{1 + R_\alpha + R_\alpha^2 + R_\alpha^3} > 0,\ R_\alpha = \frac{1}{2}h_\alpha\left|b_\alpha\right| > 0, \tag{12}$$

$$b_\alpha^+ = \frac{1}{2}\left(b_\alpha + \left|b_\alpha\right|\right) \geq 0,\ b_\alpha^- = \frac{1}{2}\left(b_\alpha - \left|b_\alpha\right|\right) \leq 0.$$

Here h_1 and h_2 are steps of a uniform mesh relative to variables x_1 and x_2, respectively. The standard non-index notations are used for the left and right difference derivatives and for the function values at the peripheral points of the stencil:

$$u_{\bar{x}_\alpha} = \frac{u - u^{(-1_\alpha)}}{h_\alpha},\ u_{x_\alpha} = \frac{u^{(+1_\alpha)} - u}{h_\alpha},\ \alpha = 1, 2,$$

$$u = u(x),\ u^{(\pm 1_1)} = u(x_1 \pm h_1,\ x_2),\ u^{(\pm 1_2)} = u(x_1,\ x_2 \pm h_2),$$

$$k = k(x),\ k^{(\pm 0.5_1)} = k(x_1 \pm 0.5h_1,\ x_2),\ k^{(\pm 0.5_2)} = k(x_1,\ x_2 \pm 0.5h_2),$$

where $x = (x_1, x_2)$ is the central node of the stencil.

The finite-difference operators $\Lambda_\alpha^{(a,b)}u$ approximate the corresponding differential operators $L_\alpha^{(k,\upsilon)}u$ with the second order. We note that operators (11) are analogous to the operators of the well-known monotone scheme of the second order described in the book of A. A. Samarskii [3], but we define the scheme coefficients a_α, b_α and $æ_\alpha$ in a different way.

Under assumptions (12) for the coefficients a_α, the following asymptotic expansions at the center node of the mesh stencil are valid:

$$\begin{aligned} a_\alpha u_{\bar{x}_\alpha} &= k\frac{\partial u}{\partial x_\alpha} - \frac{1}{2}h_\alpha L_\alpha u + \frac{1}{6}h_\alpha^2\sqrt{k}\,\frac{\partial}{\partial x_\alpha}\left(\frac{1}{\sqrt{k}}L_\alpha u\right) - \frac{1}{24}h_\alpha^3 L_\alpha\left(\frac{1}{k}L_\alpha u\right) + O\left(h_\alpha^4\right), \\ a_\alpha^{(+1_\alpha)}u_{x_\alpha} &= k\frac{\partial u}{\partial x_\alpha} + \frac{1}{2}h_\alpha L_\alpha u + \frac{1}{6}h_\alpha^2\sqrt{k}\,\frac{\partial}{\partial x_\alpha}\left(\frac{1}{\sqrt{k}}L_\alpha u\right) + \frac{1}{24}h_\alpha^3 L_\alpha\left(\frac{1}{k}L_\alpha u\right) + O\left(h_\alpha^4\right),\ \alpha = 1, 2. \end{aligned} \tag{13}$$

Taking into account (13) we get the following relation

$$\sum_{\alpha=1}^{2} L_\alpha^{(k,\upsilon)}u = \sum_{\alpha=1}^{2}\Lambda_\alpha^{(a,b)}u + \frac{h^2}{12}Eu + O\left(h^4\right), \tag{14}$$

connecting the differential and difference operators for any sufficiently smooth function $u(x_1, x_2)$, where

$$Eu = \sum_{\alpha=1}^{2}\left\{\delta_\alpha^2\left[-L_\alpha^{(k,\upsilon)}\left(\frac{1}{k}L_\alpha^{(k,\upsilon)}u\right) + p_\alpha L_\alpha u - r_\alpha k\frac{\partial u}{\partial x_\alpha}\right]\right\},$$

$$p_\alpha = \upsilon_\alpha^2 + \frac{\upsilon_\alpha}{k}\frac{\partial k}{\partial x_\alpha} - 2\frac{\partial \upsilon_\alpha}{\partial x_\alpha},\ r_\alpha = L_\alpha^{(k,\upsilon)}\left(\frac{\upsilon_\alpha}{k}\right),\ \delta_\alpha = \frac{h_\alpha}{h}.$$

Following the conventional methodology of increasing the approximation order on the minimal stencil, we modify the operator Eu, by expressing $L_1^{(k,\upsilon)}u = -L_2^{(k,\upsilon)}u + qu - f$, $L_2^{(k,\upsilon)}u = -L_1^{(k,\upsilon)}u + qu - f$ from equation (10) and substituting them into a term with the derivatives of the order 3–4. We exclude in this way the derivatives of a high order, which are not suitable for difference approximation on the minimum stencil. In addition we introduce in the operator Eu some regularization parameters $\sigma_0 = \sigma_0(x) \geq 0$ and $\sigma_1 = \sigma_1(x)$ by adding a term, which is identically zero on the solution of equation (10) $u = u(x)$.

Due to these changes we get

$$\begin{aligned} Eu = \sum_{\alpha=1}^{2}\left\{\delta_\alpha^2\left[L_\alpha^{(k,\upsilon)}\left(\frac{1}{k}\left(L_\beta^{(k,\upsilon)}u - qu + f\right)\right) + p_\alpha L_\alpha u - r_\alpha k\frac{\partial u}{\partial x_\alpha}\right]\right\} + \\ + (\sigma_0 + \sigma_1)\underbrace{\left(\sum_{\alpha=1}^{2}L_\alpha^{(k,\upsilon)}u - qu + f\right)}_{\text{equal } 0},\ \beta \neq \alpha. \end{aligned}$$

The introduced regularization parameters allow regulating the basic properties of the difference scheme providing the maximum-principle conditions and keeping the fourth order of approximation.

By simple manipulations, the operator Eu is reduced to the final form

$$Eu = \sum_{\alpha=1}^{2}\left\{\delta_\alpha^2\left[L_\alpha^{(k,\upsilon)}\left(\frac{1}{k}L_\beta^{(k,\upsilon)}u\right) + p_\alpha L_\alpha u - r_\alpha k\frac{\partial u}{\partial x_\alpha} - \frac{q}{k}L_\alpha^{(k,\upsilon)}u - \right.\right.$$

$$\left.\left. - 2k\frac{\partial\left(\frac{q}{k}\right)}{\partial x_\alpha}\frac{\partial u}{\partial x_\alpha} - L_\alpha^{(k,\upsilon)}\left(\frac{q}{k}\right)u\right] + \left(\sigma_0+\sigma_1\right)L_\alpha^{(k,\upsilon)}u\right\} +$$

$$+ \sum_{\alpha=1}^{2}\delta_\alpha^2 L_\alpha^{(k,\upsilon)}\left(\frac{f}{k}\right) + \left(\sigma_0+\sigma_1\right)\left(f-qu\right) =$$

$$= \sum_{\alpha=1}^{2}\left\{\delta_\alpha^2\left[L_\alpha^{(k,\upsilon)}\left(\frac{1}{k}L_\beta^{(k,\upsilon)}u\right) + p_\alpha L_\alpha u - \left(r_\alpha + 2\frac{\partial\left(\frac{q}{k}\right)}{\partial x_\alpha}\right)k\frac{\partial u}{\partial x_\alpha} + \right.\right.$$

$$\left.\left. + \frac{q}{k}\left(L_\beta^{(k,\upsilon)}u - qu + f\right)\right] + \left(\sigma_0+\sigma_1\right)L_\alpha^{(k,\upsilon)}u\right\} + \sum_{\alpha=1}^{2}\delta_\alpha^2 L_\alpha^{(k,\upsilon)}\left(\frac{f}{k}\right) -$$

$$- \sum_{\alpha=1}^{2}\delta_\alpha^2 L_\alpha^{(k,\upsilon)}\left(\frac{q}{k}\right)u + \left(\sigma_0+\sigma_1\right)\left(f-qu\right).$$

Thus, we get for the main part of the approximation error

$$\frac{h^2}{12}Eu = \sum_{\alpha=1}^{2}\left(\tilde{k}_\alpha L_\alpha^{(k,\tilde{\upsilon})}u + \frac{h^2}{12}\sigma_0 L_\alpha^{(k,\upsilon)}u\right) + \frac{h_1^2}{12}L_1^{(k,\upsilon)}\left(\frac{1}{k}L_2^{(k,\upsilon)}u\right) + \frac{h_2^2}{12}L_2^{(k,\upsilon)}\left(\frac{1}{k}L_1^{(k,\upsilon)}u\right) - \tilde{q}u + \tilde{f}, \tag{15}$$

where

$$\tilde{k}_\alpha = 1 + \frac{h^2}{12}\tilde{p}_\alpha,\ \ \tilde{p}_\alpha = \delta_\alpha^2 p_\alpha + \delta_\beta^2\frac{q}{k} + \sigma_1,\ \ p_\alpha = \upsilon_\alpha^2 + \frac{\upsilon_\alpha}{k}\frac{\partial k}{\partial x_\alpha} - 2\frac{\partial\upsilon_\alpha}{\partial x_\alpha},$$

$$\tilde{\upsilon}_\alpha = \frac{1}{\tilde{k}_\alpha}\left(\upsilon_\alpha + \frac{h^2}{12}\tilde{r}_\alpha\right),\ \ \tilde{r}_\alpha = \delta_\alpha^2\left(r_\alpha + 2\frac{\partial\left(\frac{q}{k}\right)}{\partial x_\alpha}\right) + \left(\delta_\beta^2\frac{q}{k} + \sigma_1\right)\upsilon_\alpha,\ \ r_\alpha = L_\alpha^{(k,\upsilon)}\left(\frac{\upsilon_\alpha}{k}\right), \tag{16}$$

$$\tilde{q} = q + \frac{h^2}{12}\left[\sum_{\alpha=1}^{2}\delta_\alpha^2 L_\alpha^{(k,\upsilon)}\left(\frac{q}{k}\right) + \left(\frac{q}{k} + \sigma_0 + \sigma_1\right)q\right],$$

$$\tilde{f} = f + \frac{h^2}{12}\left[\sum_{\alpha=1}^{2}\delta_\alpha^2 L_\alpha^{(k,\upsilon)}\left(\frac{f}{k}\right) + \left(\frac{q}{k} + \sigma_0 + \sigma_1\right)f\right].$$

We choose the regularization parameter σ_1 in expression (15) from the conditions $\tilde{k}_\alpha \geq 1$ and $\tilde{q} \geq 0$ for $\sigma_0 \geq 0$. A feasible value of the parameter σ_1 is determined from these inequalities:

$$\sigma_1 = -\min\left[\min_\alpha\left(\delta_\alpha^2 p_\alpha\right),\ \frac{1}{q}\sum_{\alpha=1}^{2}\delta_\alpha^2 L_\alpha^{(k,\upsilon)}\left(\frac{q}{k}\right)\right] = O(1). \tag{17}$$

Taking into account representation (15) for the main part of the approximation error, the scheme of high approximation order can be written in the following form

$$\sum_{\alpha=1}^{2}\left(\tilde{a}_\alpha\Lambda_\alpha^{(a,\tilde{b})}y+\frac{h^2}{12}\sigma_0\Lambda_\alpha^{(a,b)}y\right)+\frac{h_1^2}{12}\Lambda_1^{(a,b)}\left(\frac{1}{k}\Lambda_2^{(a,b)}y\right)+\frac{h_2^2}{12}\Lambda_2^{(a,b)}\left(\frac{1}{k}\Lambda_1^{(a,b)}y\right)-\tilde{d}y+\tilde{\varphi}=0, \tag{18}$$

where $y=y(x)$ is the solution of the difference problem; $x=(x_1,x_2)$ is the internal mesh node,

$$\Lambda_\alpha^{(a,\tilde{b})}y=\tilde{æ}_\alpha\left(a_\alpha y_{\bar{x}_\alpha}\right)_{x_\alpha}-\tilde{b}_\alpha^{+}a_\alpha y_{\bar{x}_\alpha}-\tilde{b}_\alpha^{-}a_\alpha^{(+1_\alpha)}y_{x_\alpha},\ \tilde{æ}_\alpha=\frac{1}{1+\tilde{R}_\alpha+\tilde{R}_\alpha^2+\tilde{R}_\alpha^3},\ \tilde{R}_\alpha=\frac{1}{2}h_\alpha\left|\tilde{b}_\alpha\right|,$$
$$\tilde{a}_\alpha=\tilde{k}_\alpha+O\left(h^4\right)\geq 1,\ \tilde{b}_\alpha=\tilde{\upsilon}_\alpha+O\left(h^4\right),\ \tilde{d}=\tilde{q}+O\left(h^4\right)\geq 0,\ \tilde{\varphi}=\tilde{f}+O\left(h^4\right). \tag{19}$$

Obviously, scheme (18) is defined on the minimum nine-point stencil.

Approximation order. Let us consider the approximation error for scheme (18):

$$\nu=\sum_{\alpha=1}^{2}\left(\tilde{a}_\alpha\Lambda_\alpha^{(a,\tilde{b})}u-L_\alpha^{(k,\upsilon)}u+\frac{h^2}{12}\sigma_0\Lambda_\alpha^{(a,b)}u\right)+$$
$$+\frac{h_1^2}{12}\Lambda_1^{(a,b)}\left(\frac{1}{k}\Lambda_2^{(a,b)}u\right)+\frac{h_2^2}{12}\Lambda_2^{(a,b)}\left(\frac{1}{k}\Lambda_1^{(a,b)}u\right)-\tilde{d}u+\tilde{\varphi}+qu-f,$$

where $u=u(x)$ is the solution of differential equation (10). Taking into account (12), (14) and (19), we have

$$\nu=\sum_{\alpha=1}^{2}\left[\tilde{k}_\alpha\left(\Lambda_\alpha^{(k,\tilde{\upsilon})}u-L_\alpha^{(k,\tilde{\upsilon})}u\right)+\frac{h^2}{12}\tilde{p}_\alpha L_\alpha u-\frac{h^2}{12}\tilde{r}_\alpha k\frac{\partial u}{\partial x_\alpha}+\frac{h^2}{12}\sigma_0\Lambda_\alpha^{(k,\upsilon)}u\right]+$$
$$+\frac{h_1^2}{12}L_1^{(k,\upsilon)}\left(\frac{1}{k}L_2^{(k,\upsilon)}u\right)+\frac{h_2^2}{12}L_2^{(k,\upsilon)}\left(\frac{1}{k}L_1^{(k,\upsilon)}u\right)-(\tilde{q}-q)u+\tilde{f}-f+$$
$$+\sigma_0O\left(h^4\right)+O\left(h^4\right)=\sum_{\alpha=1}^{2}\left[\tilde{k}_\alpha\left(\Lambda_\alpha^{(k,\tilde{\upsilon})}u-L_\alpha^{(k,\tilde{\upsilon})}u\right)\right]+\frac{h^2}{12}Eu+\sigma_0O\left(h^4\right)+O\left(h^4\right)=$$
$$=\frac{h^2}{12}\sum_{\alpha=1}^{2}\left[\tilde{p}_\alpha\left(\Lambda_\alpha^{(k,\tilde{\upsilon})}u-L_\alpha^{(k,\tilde{\upsilon})}u\right)\right]+\sigma_0O\left(h^4\right)+O\left(h^4\right)=\sigma_0O\left(h^4\right)+O\left(h^4\right). \tag{20}$$

It follows that scheme (18) has the fourth order approximation for $\sigma_0=O(1)$. A concrete value of the parameter σ_0 is determined from the monotonicity conditions of the difference scheme.

Stability and convergence. We investigate the stability of scheme (18) using the maximum principle [3]. For this purpose, scheme (18) is rewritten in the canonical form of the maximum principle:

$$Cy=\sum_{\alpha=1}^{2}\left(A_\alpha y^{(-1_\alpha)}+B_\alpha y^{(+1_\alpha)}\right)+A_{12}y^{(-1_1,-1_2)}+B_{12}y^{(+1_1,+1_2)}+D_{12}y^{(-1_1,+1_2)}+D_{21}y^{(+1_1,-1_2)}+\tilde{\varphi},$$

where

$$A_\alpha=\frac{1}{12h_\alpha^2}\eta_\alpha\left(\sigma_0h^2-A_\alpha^*\right),\ B_\alpha=\frac{1}{12h_\alpha^2}\xi_\alpha\left(\sigma_0h^2-B_\alpha^*\right),$$
$$A_{12}=\frac{1}{12}\left[\frac{1}{h_2^2}\eta_1\left(\frac{\eta_2}{k}\right)^{(-1_1)}+\frac{1}{h_1^2}\eta_2\left(\frac{\eta_1}{k}\right)^{(-1_2)}\right]>0, \tag{21}$$
$$B_{12}=\frac{1}{12}\left[\frac{1}{h_2^2}\xi_1\left(\frac{\xi_2}{k}\right)^{(+1_1)}+\frac{1}{h_1^2}\xi_2\left(\frac{\xi_1}{k}\right)^{(+1_2)}\right]>0,$$

$$D_{\alpha\beta} = \frac{1}{12}\left[\frac{1}{h_\beta^2}\eta_\alpha\left(\frac{\xi_\beta}{k}\right)^{(-1_\alpha)} + \frac{1}{h_\alpha^2}\xi_\beta\left(\frac{\eta_\alpha}{k}\right)^{(+1_\beta)}\right] > 0,$$

$$C = \sum_{\alpha=1}^{2}\left(A_\alpha + B_\alpha\right) + A_{12} + B_{12} + D_{12} + D_{21} + \tilde{d};$$

$$A_\alpha^* = -12\frac{\tilde{\eta}_\alpha\tilde{a}_\alpha}{\eta_\alpha} + \frac{h_\alpha^2}{h_\beta^2}\left(\frac{\eta_\beta + \xi_\beta}{k}\right)^{(-1_\alpha)} + \frac{\eta_\beta + \xi_\beta}{k},\ B_\alpha^* = -12\frac{\tilde{\xi}_\alpha\tilde{a}_\alpha}{\xi_\alpha} + \frac{h_\alpha^2}{h_\beta^2}\left(\frac{\eta_\beta + \xi_\beta}{k}\right)^{(+1_\alpha)} + \frac{\eta_\beta + \xi_\beta}{k},$$

$$\eta_\alpha = a_\alpha\left(æ_\alpha + h_\alpha b_\alpha^+\right) > 0,\ \xi_\alpha = a_\alpha^{(+1_\alpha)}\left(æ_\alpha - h_\alpha b_\alpha^-\right) > 0,$$

$$\tilde{\eta}_\alpha = a_\alpha\left(\tilde{æ}_\alpha + h_\alpha\tilde{b}_\alpha^+\right) > 0,\ \tilde{\xi}_\alpha = a_\alpha^{(+1_\alpha)}\left(\tilde{æ}_\alpha - h_\alpha\tilde{b}_\alpha^-\right) > 0.$$

The coefficients A_α and B_α correspond to left, right, lower and upper peripheral nodes of the stencil relative to the central node. Their signs depend on the choice of the regularization parameter σ_0. The angular coefficients A_{12}, B_{12}, D_{12} и D_{21} are positive for any mesh steps regardless of the regularization parameters.

From the requirement that the coefficients A_α and B_α for $\alpha = 1, 2$ are non-negative, we get the sufficient condition under which scheme (18) satisfies the maximum principle:

$$\sigma_0 = \begin{cases} \frac{1}{h^2}\max\left(A_\alpha^*, B_\alpha^*\right), \text{ if } \max\limits_\alpha\left(A_\alpha^*, B_\alpha^*\right) > 0, \\ 0, \text{ if } \max\limits_\alpha\left(A_\alpha^*, B_\alpha^*\right) \le 0. \end{cases} \tag{22}$$

Analysis of the coefficients A_α^*, B_α^* shows that they can be positive on coarse meshes. In this case we have $\sigma_0 = O\left(h^{-2}\right)$ and $\nu = \sigma_0 O\left(h^4\right) + O\left(h^4\right) = O\left(h^2\right)$ according to (20), (22). The use of formula (22) may seem unreasonable due to the threat of a decrease in the order of approximation. However, it follows from formulas (21) that $A_\alpha^*, B_\alpha^* \underset{h\to 0}{\to} -10 + 2\left(\frac{h_\alpha}{h_\beta}\right)^2 < 0$ if $\frac{h_\alpha}{h_\beta} < \sqrt{5}$, $\beta \ne \alpha$. Consequently, if the mesh steps are related by the condition

$$\frac{1}{\sqrt{5}} < \frac{h_1}{h_2} < \sqrt{5}, \tag{23}$$

all coefficients A_α^*, B_α^* should become negative for sufficiently small steps h_1, h_2, thereby providing $\sigma_0 = 0$ and therefore the approximation error $\nu = O\left(h^4\right)$. For instance, for the test problem in the following section all coefficients A_α^*, B_α^* become negative on meshes with the step $h \le \frac{1}{53}$ at the Grashof number $\mathrm{Gr} = 10^6$ and on meshes with the step $h \le \frac{1}{135}$ at the Grashof number $\mathrm{Gr} = 10^7$.

Thus, scheme (18) with coefficients (16), (17), (19), (22) subject to constraint (23) satisfies the maximum principle and has the fourth order of approximation. This means that scheme (18), supplemented by difference boundary conditions with the same approximation and stabilization properties, converges with the rate of $O\left(h^4\right)$ as $h \to 0$, i. e. is of fourth order of accuracy.

It should be noted that condition (23) relates the mesh steps but does not limit their values. It agrees with the convergence condition of the high order accuracy scheme for the two-dimensional Poisson equation [3] which corresponds to $k \equiv 1$, $q \equiv 0$, $\upsilon_1 \equiv 0$, $\upsilon_2 \equiv 0$.

Scheme testing

Scheme (18) has been tested on the well-known problem of a natural convection in a horizontal channel of a square cross-section with a uniform heating of the right vertical wall [12; 14; 15]. The problem geometry and the boundary conditions for the temperature are shown in fig. 1.

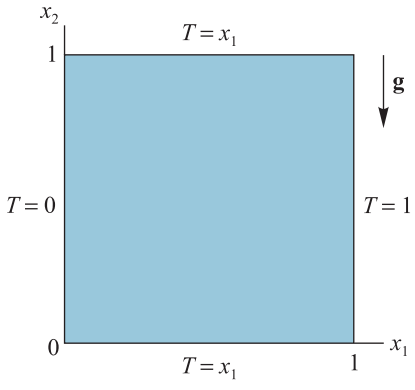

Fig. 1. Illustration of the test problem statement

The dimensionless mathematical model of the test problem is defined by equations (5) with respect to the temperature $T(x_1, x_2)$, the stream function $\psi(x_1, x_2)$ and the vorticity $\omega(x_1, x_2)$ at $\mathrm{Gr}_m = 0$ with the boundary temperature conditions: $T(0, x_2) = 0$, $T(1, x_2) = 1$, $T(x_1, 0) = T(x_1, 1) = x_1$.

The test computations were carried out for the Prandtl number Pr = 1 and the Grashof number in the range $\mathrm{Gr} \le 5 \cdot 10^7$ corresponding to the laminar mode of convection. A square mesh was used with a step $h = h_1 = h_2$ and the number of partitions $10 \le N = \dfrac{1}{h} \le 1000$ on each side of the square domain. Note that the numerical solution for $N = 1000$ requires to solve a system with more than 3 million of nonlinear difference equations. An approximate condition of the fourth order was applied for the vorticity on the boundary [12; 16]. The realization of the difference scheme was carried out by a relaxation method described in [12; 17].

Figures 2 and 3 illustrate the temperature distribution (left) and the flow pattern (right) obtained with scheme (18) on the square mesh with the step $h = \dfrac{1}{500}$ for the Grashof numbers $\mathrm{Gr} = 10^6$ and $\mathrm{Gr} = 5 \cdot 10^7$. The last of them is close to the critical value at which a turbulization of a laminar flow begins. The resulting thermoconvective structures are characterized by a formed boundary layer, in which the dominant velocity and temperature gradients are concentrated. Due to this, an extensive stagnation zone is formed in the central part of the domain with a constant vertical gradient of the temperature $|\nabla T| = \left|\dfrac{\partial T}{\partial x_2}\right| = 0.656$.

Figure 4 and table below show the dependences of the maximum values of the stream function and vorticity on the number of the mesh partition, which are obtained by applying fourth order scheme (18) and the second order monotone Samarskii scheme [3] to the test problem. The upper numbers in the cells of table correspond to the second order scheme, the lower numbers – to the fourth order scheme. The comparison of the simulations results shows that the fourth order scheme has significant advantages in the rate of convergence as $N \to \infty$. For example, the solution, obtained by the fourth order scheme for $N = 100$, is not inferior in accuracy to the solution,

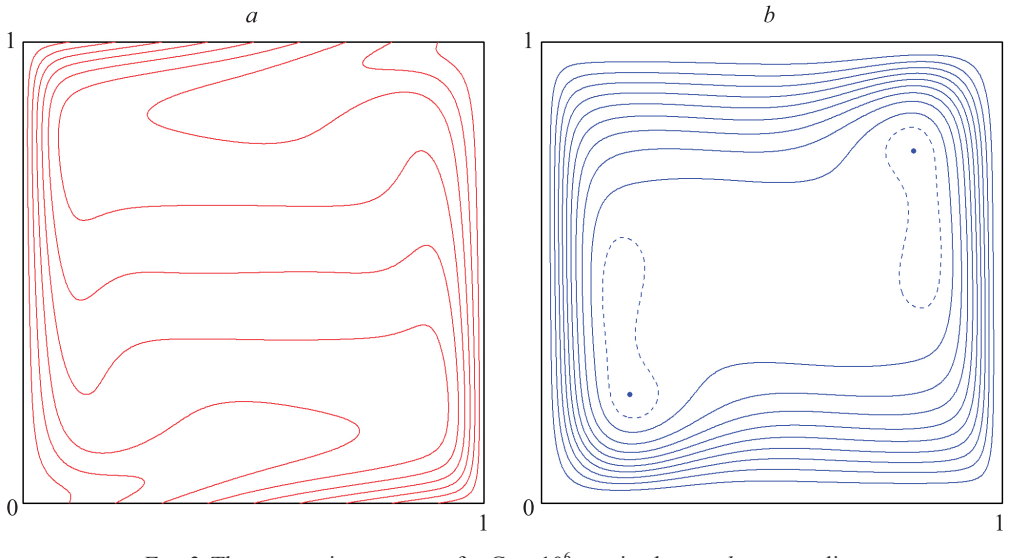

Fig. 2. The convection structure for $\mathrm{Gr} = 10^6$: *a* – isotherms; *b* – streamlines

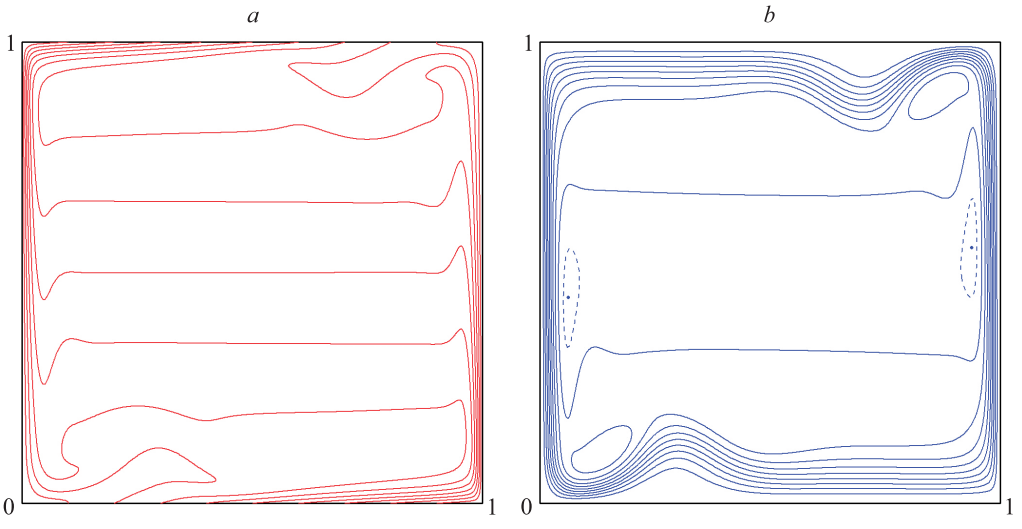

Fig. 3. The convection structure for $\mathrm{Gr} = 5 \cdot 10^7$: *a* – isotherms; *b* – streamlines

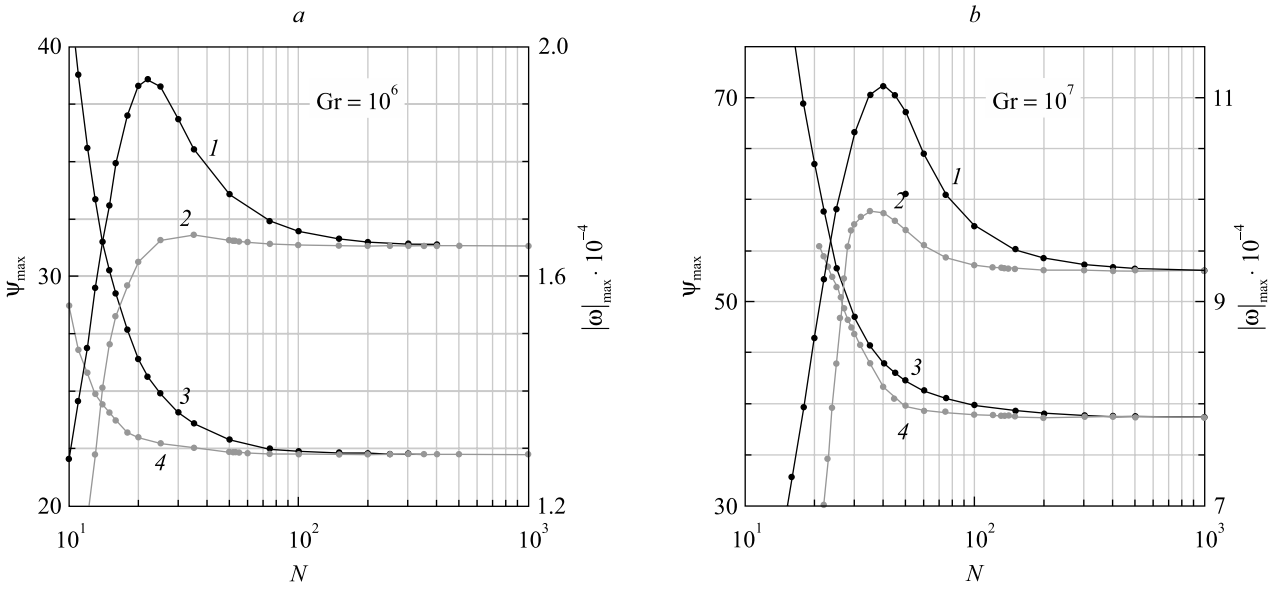

Fig. 4. The dependence of the maximum values of vorticity $|\omega|_{\max}$ (lines *1*, *2*) and stream function $\psi_{\max}$ (lines *3*, *4*) on the mesh number $N = \dfrac{1}{h}$ for $\mathrm{Gr} = 10^6$ (*a*) and $\mathrm{Gr} = 10^7$ (*b*):
1, *3* – the monotone second-order scheme (Samarskii); *2*, *4* – fourth order scheme (18)

obtained by the second order scheme for $N = 300$. It means a nine-fold decrease in size of the system of nonlinear difference equations as well as decrease in the number of iterations to solve this system with the same accuracy. However, the gain in time, expected due to the nine-fold decrease in the number of nodes as well as due to higher convergence rate of iteration process for larger mesh steps, is somewhat compensated by the time difference for the one iteration, which is a 4–5 times higher for scheme (18) than for the scheme of the second order.

Maximum values of the stream function ($\psi_{\max}$) and vorticity $|\omega|_{\max}$ depending on the mesh step (h) for $\mathrm{Gr} = 10^7$

h	$\frac{1}{20}$	$\frac{1}{50}$	$\frac{1}{100}$	$\frac{1}{200}$	$\frac{1}{300}$	$\frac{1}{400}$	$\frac{1}{500}$	$\frac{1}{1000}$
$\psi_{\max}$	63.505 –	42.241 39.840	39.874 38.946	39.072 38.734	38.879 38.701	38.800 38.689	38.760 38.685	38.704 38.683
$\lvert\omega\rvert_{\max}$	86 445.7 –	108 591.1 97 033.3	97 442.5 93 580.4	94 287.3 93 132.4	93 624.6 93 065.3	93 377.2 93 044.7	93 257.3 93 036.5	93 088.9 93 028.7

The test computations show that the constructed scheme of the higher approximation order becomes effective at the Rayleigh numbers Ra = GrPr corresponding to the developed laminar convection. Although the high order scheme significantly complicates a computational algorithm, it could have significant advantages over monotone schemes of the first and the second order [3; 12; 14; 18] for the Rayleigh numbers close to the beginning of a convective flow turbulization because it allows to get numerical solutions with a high accuracy on relatively coarse meshes.

Conclusion

The finite-difference scheme of high order accuracy for the two-dimensional steady-state convection – diffusion equation is constructed. The scheme defined on the minimal stencil of a uniform mesh, has the fourth order of approximation and satisfies the maximum principle for any mesh steps satisfied the condition $\frac{1}{\sqrt{5}} < \frac{h_1}{h_2} < \sqrt{5}$. The scheme is focused on solving a wide range of applied problems of convection – diffusion such as the gravitational convection, a thermomagnetic convection and a diffusion of particles in magnetic fluids. The high approximation and stabilization properties, compared with other methods, provide a higher accuracy with less calculation cost. It is especially important for modeling of convection and diffusion processes in developed boundary layers with the large gradients of velocity, temperature and particle concentration. The proposed scheme is tested on the well-known problem of the high-intensity gravitational convection in the horizontal channel of a square cross-section with the uniform heating from the side. A detailed comparison with the monotone Samarskii scheme of the second order [3] is performed on the sequences of square meshes with the number of partitions from 10 to 1000 on each side of the square domain in the whole range of the Rayleigh numbers $\mathrm{Ra} \le 5 \cdot 10^7$, corresponding to the laminar convection mode. A significant advantage of the fourth order scheme in the convergence rate is shown for the decreasing mesh step.

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