TWO-FACTOR PRODUCTIONS FUNCTIONS WITH GIVEN TOTAL ELASTICITY OF PRODUCTION

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1. Introduction. Fundamental to economic analysis is the idea of a production function. It and its allied concept, the utility function, form the twin pillars of neoclassical economics (see [1; 2]). Roughly speaking, the production functions are the mathematical formalization of the relationship between the output of a firm (industry, economy) and the inputs that have been used in obtaining it. In fact, a two-factor production function is defined as a map:

$$Y:(K,L)\to F(K,L) \ \text{ for all } \ (K,L)\in G, \tag{1}$$

where K is the quantity of capital employed, L is the quantity of labor used, Y is the quantity of output, and the nonnegative function F is a continuously differentiable function on the domain G from the first quadrant $\mathbf{R}^2_+=\{(K,L):K>0,L>0\}$. The two-factor production function (1) expresses a technological relationship. It describes the maximum output obtainable, at the existing state of technological knowledge, from given amounts of factor inputs.

At the present time, production functions apply at the level of the individual firm and the macro economy at large. At the micro level, economists use production functions to generate cost functions and input demand schedules for the firm. The famous profit-maximizing conditions of optimal factor hire derive from such microeconomics functions. At the level of the macro economy, analysts use aggregate production functions to explain the determination of factor income shares and to specify the relative contributions of technological progress and expansion of factor supplies to economic growth [3–5].

Among the family of two-factor production functions (1), the most famous is the Cobb-Douglas production function. It was introduced in 1928 by the mathematician Ch.W. Cobb and the economist P. H. Douglas in the paper «A theory of production» [6]:

$$Y:(K,L)\to AK^{\alpha}L^{\beta} \ \text{ for all } \ (K,L)\in\mathbf{R}^2_+, \tag{2}$$

where A is a positive constant which signifies the total factor productivity.

The Cobb-Douglas production model was generalized in 1961 by the economists K.J. Arrow, H.B. Chenery, B.S. Minhas, and R.M. Solow [7]. They introduced the so-called Constant Elasticity of Substitution (CES) two-factor production function (or the ACMS two-factor production function)

$$Y:(K,L)\to A(\alpha K^{\gamma}+\beta L^{\gamma})^{\rho/\gamma} \ \text{ for all } \ (K,L)\in\mathbf{R}^2_+,\ A>0,\ \alpha,\beta,\rho>0,\ \gamma\neq 0,1. \tag{3}$$

Now the Cobb-Douglas production function (2) and the CES production function (3) are widely used in economics to represent the relationship of an output to inputs. Note also that CES production function include as special case the Cobb-Douglas production function and many other famous production models, like a linear production model, a multinomial production function or Leontief function. Concerning the history of development of the theory of production functions see the papers [1; 8]. For further results concerning new production models in economic see recently articles [9–18].

For production function (1), we recall some economic-mathematical indicators:

1) the output elasticity of capital (labor) is defined as:

$$E_K(K,L)=\frac{K}{F(K,L)}\frac{\partial F(K,L)}{\partial K}\qquad\left(E_L(K,L)=\frac{L}{F(K,L)}\frac{\partial F(K,L)}{\partial L}\right);$$

2) the total elasticity of production (or elasticity of scale):

$$E(K,L)=\lim_{t\to 1}\frac{t}{F(tK,tL)}\frac{\partial F(tK,tL)}{\partial t}=E_K(K,L)+E_L(K,L);$$

3) the Hicks elasticity of substitution is [19]:

$$\sigma^H(K,L)=\frac{\dfrac{\partial F}{\partial K}\cdot\dfrac{\partial F}{\partial L}\left(K\dfrac{\partial F}{\partial K}+L\dfrac{\partial F}{\partial L}\right)}{KL\left(2\dfrac{\partial F}{\partial K}\cdot\dfrac{\partial F}{\partial L}\cdot\dfrac{\partial^2 F}{\partial K\partial L}-\left(\dfrac{\partial F}{\partial K}\right)^2\dfrac{\partial^2 F}{\partial L^2}-\left(\dfrac{\partial F}{\partial L}\right)^2\dfrac{\partial^2 F}{\partial K^2}\right)}.$$

For instance, the production functions (2) and (3) are productions functions with constant Hicks elasticity of factors substitution. The Cobb-Douglas production function (2) has unit elasticity of substitution i.e. $\sigma^H(K,L)=1$. The CES production function (3) has the Hicks elasticity of factors substitution $\sigma^H(K,L)=1/(1-\gamma)$. L. Losonczi proved [9] that a twice differentiable two-factor homogeneous production function with constant Hicks elasticity of substitution is either the Cobb-Douglas production function or the CES production function. This result complements the main propositions of the classical works [20; 21] and is consistent with known results on the classification of production functions [3]. The analogue for multi-factor production functions was proved by B.-Y. Chen in [10]. These results were recently generalized for quasi-homogeneous production functions with constant Hicks elasticity of substitution [16; 17]. Note also that A. D. Vilcu and G. E. Vilcu classified homogeneous production functions with constant elasticity of labor and capital [12]. Their classification generalized some results by C. Ioan and G. Ioan concerning to the sum production function [11].

The aim of this paper is to identify all two-factor production functions with given total elasticity of production. We formulate our main result (Theorem) for two-factor production functions and obtain partial cases of production functions with given (constant, linear, linear-fractional, exponential, etc.) total elasticity of production.

2. Main results. The following statement describes the analytical form of two-factor production functions with given elasticity of production.

Theorem [18]. Suppose $E:(K,L)\to E(K,L)$ for all $(K,L)\in G$ is the total elasticity of production for some production technology. Then this production technology can be described by one of the production functions of the form:

$$F_\phi:(K,L)\to\phi\left(\frac{K}{L}\right)\exp\left(\int\frac{E(C_1L,L)}{L}dL\right)_{|C_1=\frac{K}{L}}\quad\text{for all }(K,L)\in G,$$

where ϕ is arbitrary nonnegative continuously differentiable function on $(0;+\infty)$.

Using this Theorem, from some given total elasticities of production, we obtain the corresponding classes of production functions (see Table 1).

Suppose for some production technology we know the total elasticity of production $E:(K,L)\to\delta$ for all $(K,L)\in G$, $\delta=const$. Then the production function has the form

$$F_\phi:(K,L)\to L^\delta\phi\left(\frac{K}{L}\right)\quad\text{for all }(K,L)\in G$$

and we have the following statements:

i) if $\phi:\xi\to A\xi^\alpha$ for all $\xi\in\mathbf{R}_+$, then we obtain the Cobb-Douglas production function (2) with $\beta=\delta-\alpha$;

Table 1 – The form of production function with given total elasticity of production

No	**Total elasticity of production** ($\alpha, \beta, \delta \in \mathbf{R}$, f and g are continuous functions)	**Analytical form of production function**
1.	$E(K,L)=\delta$	$F_\phi(K,L)=L^\delta \phi\left(\frac{K}{L}\right)$
2.	$E(K,L)=\alpha K+\beta L+\delta$	$F_\phi(K,L)=L^\delta \phi\left(\frac{K}{L}\right)\exp(\alpha K+\beta L)$
3.	$E(K,L)=f(\alpha K+\beta L)$	$F_\phi(K,L)=\phi\left(\frac{K}{L}\right)\exp\left(\int \frac{f(\xi)}{\xi}d\xi\right)_{\mid \xi=\alpha K+\beta L}$
4.	$E(K,L)=f(K)+g(K)$	$F_\phi(K,L)=\phi\left(\frac{K}{L}\right)\exp\left(\int \frac{f(K)}{K}dK+\int \frac{g(L)}{L}dL\right)$
5.	$E(K,L)=f\left(\frac{K}{L}\right)$	$F_\phi(K,L)=\phi\left(\frac{K}{L}\right)\exp\left(\ln L\cdot f\left(\frac{K}{L}\right)\right)$
6.	$E(K,L)=K^\alpha f\left(\frac{K}{L}\right),\ \alpha\neq 0$	$F_\phi(K,L)=\phi\left(\frac{K}{L}\right)\exp\left(\frac{1}{\alpha}K^\alpha f\left(\frac{K}{L}\right)\right)$
7.	$E(K,L)=L^\beta f\left(\frac{K}{L}\right),\ \beta\neq 0$	$F_\phi(K,L)=\phi\left(\frac{K}{L}\right)\exp\left(\frac{1}{\beta}L^\beta f\left(\frac{K}{L}\right)\right)$
8.	$E(K,L)=f(K^\alpha L^\beta),\ \alpha\neq -\beta$	$F_\phi(K,L)=\phi\left(\frac{K}{L}\right)\exp\left(\frac{1}{\alpha+\beta}\int \frac{f(\xi)}{\xi}d\xi\right)_{\mid \xi=K^\alpha L^\beta}$

Note: Developed by the authors.

ii) if $\phi:\xi\to A(\alpha\xi^\gamma+\beta)^{\delta/\gamma}$ for all $\xi\in\mathbf{R}_+$, then we get the CES production function (3) with $\rho=\delta$;

iii) if $\phi:\xi\to A(\alpha\xi^\gamma+(1-\alpha)\beta\xi^{-\sigma(1-\gamma)})^{\delta/\gamma}$ for all $\xi\in\mathbf{R}_+$, then we have the Lu-Fletcher production function (Lu and Fletcher, 1968):

$$F:(K,L)\to A\left(\alpha K^\gamma+(1-\alpha)\beta\left(\frac{K}{L}\right)^{-\sigma(1-\gamma)}L^\gamma\right)^{\delta/\gamma} \text{ for all } (K,L)\in\mathbf{R}_+^2,$$

which for $\sigma=0,\ \beta=1$ becomes the CES production function;

iv) if $\phi:\xi\to A((1-\alpha)\xi^\gamma+\alpha\xi^{\sigma\gamma})^{\delta/\gamma}$ for all $\xi\in\mathbf{R}_+$, then we obtain the Liu-Hildebrand production function (Liu and Hildebrand, 1965):

$$F:(K,L)\to A((1-\alpha)K^\gamma+\alpha K^{\sigma\gamma}L^{(1-\sigma)\gamma})^{\delta/\gamma} \text{ for all } (K,L)\in\mathbf{R}_+^2,$$

which for $\sigma=0$ becomes the CES production function;

v) if $\phi:\xi\to A(a\xi^{\alpha+\beta}+2b\xi^\alpha+c)^{\delta/(\alpha+\beta)}$ for all $\xi\in\mathbf{R}_+$, then we have the Kadiyala production function (Kadiyala, 1972):

$$F:(K,L)\to A(aK^{\alpha+\beta}+2bK^\alpha L^\beta+cL^{\alpha+\beta})^{\delta/(\alpha+\beta)} \text{ for all } (K,L)\in\mathbf{R}_+^2.$$

where $a+2b+c=1,\ a,b,c\geq 0,\ \alpha(\alpha+\beta)>0,\ \beta(\alpha+\beta)>0$. Since $b=0$, we see that the Kadiyala function is the CES production function. If $c=0$, then the Kadiyala function generates directly the Lu-Fletcher production function. For $a=c=0$, we obtain the Cobb-Douglas production function. Finally, for $\alpha=1/(\mu\gamma)-1,\ \beta=1,\ \mu=1/(1-a)$, and $c=0$, we get the VES production function (Revankar 1971; Sato and Hoffman, 1968):

$$F:(K,L)\to AK^{(1-\mu\gamma)\delta}(L+(\mu-1)K)^{\mu\gamma\delta} \text{ for all } (K,L)\in \mathbf{R}_+^2.$$

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ESSENCE AND SPECIFIC FEATURES OF CLASSIFICATION OF MONOTOWNS IN KAZAKHSTAN

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One of the reasons for the classification of cities is their functional purpose, which determines the structure of employment, the profile of production activities of city-forming enterprises and specialization in the structure of the social division of labor. In this sense, cities can be divided into monofunctional (with a predominance of one functional specialization) and multifunctionale [1]. When dividing cities into small, medium, large, large, and millionaires, the criterion of population (city size) is used. The city-forming functions determine the functional profile of the city, its place