SOME FORMULAS OF OPERATOR INTERPOLATION ON THE SET OF RANDOM PROCESSES

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Abstract

For operators defined on the Cartesian product of random and deterministic continuous functions the interpolation polynomials of arbitrary fixed degree, coinciding at the given points with the original operator, are constructed. Formulas of the linear interpolation and their applications for the approximation of specific random operators, nonlinear with respect to the Wiener process, are considered.
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1 Introduction

Let us denote by $\Xi = \{\xi(t,\omega),\ t \in T,\ \omega \in \Omega\}$ the set of random processes, defined on the probability space $\{\Omega, \mathcal{F}, P\}$, by $C(T)$ the space of the deterministic functions $x(t),\ t \in T$, continuous on T, where T is the time interval on $\mathbb{R}^+ = [0,\infty)$.

Let the operator $F(\xi,x) = F\big(\xi(t,\omega),\ x(t)\big)$ be defined on the Cartesian product of $\Xi \times C(T)$ and $F : \Xi \times C(T) \to Y$, where Y is a set, whose elements have random or deterministic nature.

We introduce vectors of the form $r_l(\xi,x) = \{\xi - \xi_l, x - x_l\}$ and $r_{lk}(\xi,x) = \{\xi_k - \xi_l, x_k - x_l\}$, where ξ, ξ_k are elements of the set Ξ, and $x = x(t), x_k = x_k(t),\ x_l = x_l(t)$ are functions continuous on $T,\ 0 \le l,k \le n$. Let us denote by (r_l, r_{lk}) the scalar product of r_l and $r_{lk} : (r_l, r_{lk}) = (\xi - \xi_l)(\xi_k - \xi_l) + (x - x_l)(x_k - x_l)$. Correspondingly (r_{lk}, r_{lk}) is the square of vector length of $r_{lk} : (r_{lk}, r_{lk}) = (\xi_k - \xi_l)^2 + (x_k - x_l)^2$.

We associate the operator $F(\xi,x)$ and the points (ξ_k, x_k) $(k = 0,1,\ldots,n)$ with a random algebraic operator polynomial $L_n(F;\xi,x)$ of the form

$$L_n(F;\xi,x) = F(\xi_0,x_0) + \sum_{k=1}^{n} \int_0^1 l_{nk}\big(\xi(\tau),\ x(\tau)\big) d_\tau F\big(\xi_0 + \tau(\xi_k - \xi_0),\ x_0 + \tau(x_k - x_0)\big), \tag{1}$$

where the integral on the variable τ in the equality (1) is understood as the Riemann–Stieltjes integral for trajectories of random processes in this integral, and

$$l_{nk}(\xi,x) = \frac{(r_0,r_{0k})(r_1,r_{1k})\cdots(r_{k-1},r_{k-1\ k})(r_{k+1},r_{k+1\ k})\cdots(r_n,r_{nk})}{(r_{0k},r_{0k})(r_{1k},r_{1k})\cdots(r_{k-1\ k},r_{k-1\ k})(r_{k+1\ k},r_{k+1\ k})\cdots(r_{nk},r_{nk})}. \tag{2}$$

2 Operator interpolation formulas of other form

Now we consider another version of the polynomial of the form (1) for the operator $F(\xi, x)$ that is differentiable by Gateaux. We denote by $\delta F[\xi, x; h]$ the Gateaux differential of this operator at the point (ξ, x) in the direction $h = (h_0, h_1)$, where $h_0 = h_0(\omega, t)$ is a random process from the set Ξ, $h_1 = h_1(t) \in C(T)$. The Gateaux differential $\delta F[\xi, x; h]$ of the operator $F(\xi, x)$ in the direction $h = (h_0, h_1)$ is defined by the equality $\delta F[\xi, x; h] = \frac{d}{d\lambda} F(\xi + \lambda h_0,\ x + \lambda h_1)_{\lambda=0},\ \lambda \in [0, 1]$. This differential is used below only at the points $(\xi, x) = \big(\xi_0 + \tau(\xi_k - \xi_0),\ x_0 + \tau(x_k - x_0)\big)$, $\tau \in [0, 1]$ and in the directions $h_{0k} = (\xi_k - \xi_0) l_{nk}(\xi, x)$ and $h_{1k} = (x_k - x_0) l_{nk}(x)$, where

$$l_{nk}(x) = \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)} \quad (k = 1, 2, \ldots, n),$$

and here we suppose that in (2) and in the fraction $l_{nk}(x)$, none of the factors in the denominator vanish.

Let us denote by $\widetilde{L}_n(F; \xi, x)$ the operator polynomial of the following form

$$\widetilde{L}_n(F; \xi, x) = F(\xi_0, x_0) + \sum_{k=1}^{n} \int_0^1 \delta F\big[\xi_0 + \tau(\xi_k - \xi_0),\ x_0 + \tau(x_k - x_0);\ h_{0k}, h_{1k}\big] d\tau. \quad (3)$$

Theorem. *Random operator polynomials (1) and (3) are interpolational for the operator $F(\xi, x)$ and nodes (ξ_k, x_k), i.e.*

$$L_n(F; \xi_k, x_k) = \widetilde{L}_n(F; \xi_k, x_k) = F(\xi_k, x_k) \quad (k = 0, 1, \ldots, n).$$

During applied problems solving are often limited by the formulas of the linear and quadratic interpolation. In the linear case we use two nodes $\big(\xi_0(t), x_0(t)\big)$, $\big(\xi_1(t), x_1(t)\big)$ and the formulas (1) and (3) take the form correspondingly

$$L_1(F; \xi, x) = F(\xi_0, x_0) + \int_0^1 l_{11}\big(\xi(\tau),\ x(\tau)\big) d_\tau F\big[\xi_0 + \tau(\xi_1 - \xi_0),\ x_0 + \tau(x_1 - x_0)\big], \quad (4)$$

$$\widetilde{L}_1(F; \xi, x) = F(\xi_0, x_0) +$$

$$+ \int_0^1 \delta F\big[\xi_0 + \tau(\xi_1 - \xi_0),\ x_0 + \tau(x_1 - x_0);\ (\xi_1 - \xi_0) l_{11}(\xi, x),\ (x_1 - x_0) l_{11}(x)\big] d\tau, \quad (5)$$

where

$$l_{11}\big(\xi(\tau), x(\tau)\big) = \frac{\big(\xi(\tau) - \xi_0(\tau)\big)\big(\xi_1(\tau) - \xi_0(\tau)\big) + \big(x(\tau) - x_0(\tau)\big)\big(x_1(\tau) - x_0(\tau)\big)}{\big(\xi_1(\tau) - \xi_0(\tau)\big)^2 + \big(x_1(\tau) - x_0(\tau)\big)^2},$$

$$l_{11}\big(x(\tau)\big) = \frac{x(\tau) - x_0(\tau)}{x_1(\tau) - x_0(\tau)}.$$

It is obvious that for linear on $\Xi \times C(T)$ operators (4) and (5) the interpolation conditions $L_1(F; \xi_i, x_i) = \widetilde{L}_1(F; \xi_i, x_i) = F(\xi_i, x_i)$ $(i = 0, 1)$ hold true.

In the case, when the approximated operator F depends only on random process $\xi(t)$ the formulas (4) and (5) take the form

$$L_1(F;\xi) = F(\xi_0) + \int_0^1 \frac{\xi(\tau)-\xi_0(\tau)}{\xi_1(\tau)-\xi_0(\tau)} d_\tau F\big(\xi_0 + \tau(\xi_1-\xi_0)\big),$$

$$\widetilde{L}_1(F;\xi) = F(\xi_0) + \int_0^1 \delta F\big[\xi_0 + \tau(\xi_1-\xi_0); \xi-\xi_0\big] d\tau. \tag{6}$$

3 Some applications of formulas of the linear interpolation

We construct the formulas of the linear interpolation for certain types of random processes. Let

$$F(\xi,x) = X_0 e^{\sigma\xi(t)+\left(r-\frac{1}{2}\sigma^2\right)x(t)}, \tag{7}$$

where X_0 is a random variable independent on $\xi(t)=\xi(\omega,t)$; r and σ are arbitrary given numbers, $t \in T$. When $\xi(t)$ is a standard Wiener process $W(t)$ and $x(t) = t$, then $F(W,t) = X(t)$ (see, for example [2], p. 524 or [3]) is the solution of the stochastic differential equation with linear drift and linear volatility $dX(t) = rX(t)dt+$ $+\sigma X(t)dW(t),\ X(0) = X_0$.

For the random process (7) we construct the polynomial of the form (5) at the nodes $\big(\xi_0(t), x_0(t)\big)$ and $\big(\xi_1(t), x_1(t)\big)$, where $\xi_i(t)$ is a stochastic process defined on the space $\{\Omega, \mathcal{F}, P\}$, and $x_i(t)$ is a deterministic function continuous on T $(i = 0,1)$. The Gateaux differential of the operator (7) at the point (ξ, x) in the direction $h = (h_0, h_1)$ can be calculated by the formula $\delta F[\xi,x;\ h_0,h_1] = F(\xi,x)\left[\sigma h_0 + \left(r-\frac{1}{2}\sigma^2\right)h_1\right]$. In this case the integral in (5), when $h_0 = (\xi_1-\xi_0)l_{11}(\xi,x),\ h_1 = (x_1-x_0)l_{11}(x)$, can be calculated exactly, and the formula (5) can be transformed to the form

$$\widetilde{L}_1(F;\xi,x) = F(\xi_0,x_0) + \big[F(\xi_1,x_1) - F(\xi_0,x_0)\big]\times$$
$$\times\frac{\sigma\big(\xi_1(t)-\xi_0(t)\big)l_{11}(\xi,x) + \left(r-\frac{1}{2}\sigma^2\right)\big(x_1(t)-x_0(t)\big)l_{11}(x)}{\sigma\big(\xi_1(t)-\xi_0(t)\big) + \left(r-\frac{1}{2}\sigma^2\right)\big(x_1(t)-x_0(t)\big)}. \tag{8}$$

Since for the operator (7) at every fixed function $x(t)$ the Gateaux differential $\delta F[\xi,x;h_0]$ with respect to variable ξ is defined by the formula $\delta F[\xi,x;h_0] = \sigma h_0(t)F(\xi,x)$, then the integral in (6) can be also calculated exactly and this interpolation formula with the nodes ξ_0 and ξ_1 takes the form

$$\widetilde{L}_1(F;\xi,x) = F(\xi_0,x) + \big[F(\xi_1,x) - F(\xi_0,x)\big]\frac{\xi(t)-\xi_0(t)}{\xi_1(t)-\xi_0(t)}. \tag{9}$$

Interpolation formulas (8) and (9) can be used for the linear approximation of the random processes $F(\xi,x)$ of the form (7).

Similarly we are constructing the interpolation formulas of the form (8), (9) and for the operator $F(W;t) = Y(t) = X^\alpha(t), \alpha \in \mathbb{R}$, where $X(t) = X_0 e^{\sigma W(t)+\left(r-\frac{1}{2}\sigma^2\right)t}$ and X_0

is a random value or a given number. The mathematical expectation of this operator in the case of stochastic independence of the initial condition X_0 and the Wiener process $W(t)$ (see [3], [4]) is given by $E\{Y(t)\} = E\{X_0^{\alpha}\}e^{\alpha[r-\frac{1}{2}\sigma^2(1-\alpha)]t}$.

The requirements of coincidence of the mathematical expectation and the variance of the interpolated operator with the expectation and the variance of the corresponding interpolation polynomial in the approximation problem of random functions are natural. Let us illustrate the construction of such class of interpolation formulas on the simplest examples.

We consider the interpolation polynomial (6) for the random process $F(W) = W^2(t)$. We take two deterministic functions $x_0(t)$ and $x_1(t)$ as interpolation nodes. In this case

$$\widetilde{L}_1(F;W) = \widetilde{L}_1(W) = -x_0(t)x_1(t) + \big[x_0(t) + x_1(t)\big]W(t). \tag{10}$$

We choose the functions $x_0(t)$ and $x_1(t)$ such that the mathematical expectation and dispersion of both $F(W)$ and $\widetilde{L}_1(W)$ coincide. Since $E\{F(W)\} = E\{W^2(t)\} = t$ and the dispersion of this process $D\{F(W)\} = 2t^2$, then with the requirement of coincidence of the mean values and dispersions of the random processes $W^2(t)$ and $\widetilde{L}_1(W)$ the nodes $x_0(t)$ and $x_1(t)$ have to be determined from the system of equations $E\{\widetilde{L}_1(W)\} = t$, $D\{\widetilde{L}_1(W)\} = 2t^2$. From the equality (10) we obtain, that $E\{\widetilde{L}_1(W)\} = -x_0(t)x_1(t)$. Hence, $x_1(t) = -\frac{t}{x_0(t)}$ and correspondingly the formula (10) has the form $\widetilde{L}_1(W) = = t + \left[x_0(t) - \frac{t}{x_0(t)}\right] W(t)$, where $x_0(t)$ is arbitrary function which does not vanish on T. Since $E\{\widetilde{L}_1^2(W)\} = t^2 + \left[x_0(t) - \frac{t}{x_0(t)}\right]^2 t$, then for the dispersion of the process (10) we obtain the equality $D\{\widetilde{L}_1(W)\} = \left[x_0(t) - \frac{t}{x_0(t)}\right]^2 t$. Thus, $x_0(t)$ has to be determined from the equation $\left[x_0(t) - \frac{t}{x_0(t)}\right]^2 = 2t$, solving which we get two pairs of nodes $x_0(t) = \frac{\sqrt{2t}\left(1\pm\sqrt{3}\right)}{2}$, $x_1(t) = -\frac{\sqrt{2t}}{1\pm\sqrt{3}}$, and correspondingly the equality (10) at these nodes takes the following simple form

$$\widetilde{L}_1(W) = t + \sqrt{2t}W(t). \tag{11}$$

As a consequence equalities $E\{F(W)\} = E\{\widetilde{L}_1(W)\} = t$, $D\{F(W)\} = D\{W^2(t)\} = = 2t^2$, in some problems the approximate replacing of the square of the Wiener process with the linear process relative to $W(t)$ of the form (11).

The interpolation formula (1) and (2) are the basis for construction of approximation and for other random processes of the form considered here. In particular, as interpolation nodes can be used only deterministic functions.

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