NON-ASYMPTOTIC CONFIDENCE ESTIMATION OF THE AUTOREGRESSIVE PARAMETER IN AR(1) PROCESS WITH AN UNKNOWN NOISE VARIANCE

S.E. Vorobeychikov, Yu.B. Burkatovskaya
Tomsk State University, Tomsk Polytechnic University
Tomsk, RUSSIA
e-mail: sev@mail.tsu.ru, tracey@tpu.ru

Abstract

The paper considers the problem of estimating the autoregressive parameter in the first-order autoregressive with Gaussian noises, when the noise variance is unknown. We propose the non-asymptotic technique for compensating the unknown variance, and then, for constructing an estimator. The results of Monte-Carlo simulations are given.
Keywords: data science, confidence estimation, autoregression

1 Introduction

The problem of estimation with prescribed accuracy of the parameter of first-order autoregressive process was considered in [1]. An approach on the base of sequential analysis with a special choice of stopping time was proposed. The mean square accuracy of the estimator was determined by the parameter of the procedure. To construct this estimator, one needs to know the variance of the noises. In paper [3], authors proposed a two-stage procedure to construct the estimator of an unknown parameter if the noise variance is unknown. The first stage is used to obtain the upper bound of the variance. It should be noted that if the absolute value of the autoregressive parameter is close to unity then the estimate [3] exceeds manifold the variance. It implies increasing of estimation time.

In [2], a modification of the sequential estimation procedure ([1]) was proposed. It allows one to obtain an estimator of the autoregressive parameter with non-asymptotic Gaussian distribution. We propose to use this estimator to construct a modified two-stage estimation procedure for AR(1) process with unknown noise variance.

2 Problem statement

Consider the first-order autoregressive model AR(1) defined as follows:

$$x_k = \theta x_{k-1} + b\varepsilon_k,\ \varepsilon_k \text{ i.i.d. } \mathcal{N}(0,1),\ k = 1, 2, \ldots \tag{1}$$

where θ and b are unknown real parameters. The problem is to construct an estimator for θ with a prescribed mean-square deviation on the basis of observations $\{x_k\}$.

3 Two-stage sequential point estimator

We propose a modified two-stage procedure to estimate parameter θ in model (1). At the first stage, we construct the following statistics to compensate the unknown noise variance

$$\Gamma_l(h) = \frac{h}{2(l-2)} \sum_{i=1}^{l} \left(\hat{\theta}_{2i}(h) - \hat{\theta}_{2i-1}(h) \right)^2 . \tag{2}$$

We use here as $\{\hat{\theta}_j(h)\}$ the improved sequential point estimates proposed in [2]. These estimates represent a special modification of the least squares (maximum likelihood) estimates. For each $h > 0$ we introduce the sequence of stopping instances

$$\tau_j = \tau_j(h) = \inf \left\{ n \geq 1 : \sum_{k=\tau_{j-1}+1}^{n} x_{k-1}^2 \geq h \right\}, \quad \tau_0 = 0, \tag{3}$$

and define the sequence of sequential estimates by the formula

$$\hat{\theta}_j(h) = \frac{1}{\tilde{h}_j} \sum_{k=\tau_{j-1}+1}^{\tau_j} \sqrt{\beta_k} x_{k-1} x_k, \tag{4}$$

where $\beta_k = 1$ if $k < \tau_j$ and $\beta_{\tau_j} = \alpha_{\tau_j}$, α_{τ_j} is the correction factor, $0 < \alpha_{\tau_j} \leq 1$, uniquely defined by the equation

$$\sum_{k=\tau_{j-1}+1}^{\tau_j - 1} x_{k-1}^2 + \alpha_{\tau_j} x_{\tau_j - 1}^2 = h,$$

and

$$\tilde{h}_j = \sum_{k=\tau_{j-1}+1}^{\tau_j} \sqrt{\beta_k} x_{k-1}^2 .$$

According to [2],

$$m_j(h) = \frac{\tilde{h}_j}{\sqrt{h}} (\hat{\theta}_j(h) - \theta)$$

has Gaussian distribution $N(0, b^2)$, which, together with the inequality $\tilde{h}_j \geq h$ let one to construct the confidence interval for $\hat{\theta}_j(h) - \theta$ if b^2 is known. Besides, $\{m_j(h)\}$ are independent. It allows us to use $\Gamma_l(h)$ as an estimator for b^2 in model (1).

At the second stage, we construct an estimator for parameter θ. First, we introduce a stopping time

$$\tau = \tau(H) = \inf \left\{ n \geq 1 : \sum_{k=\tau_{2l}+1}^{n} \frac{x_{k-1}^2}{\Gamma_l(h)} \geq H \right\} \tag{5}$$

and define a sequential estimator by the following formula

$$\hat{\theta}(h, l, H) = \frac{1}{\tilde{H}} \sum_{k=\tau_{2l}+1}^{\tau} \sqrt{\beta_k} \frac{x_{k-1} x_k}{\Gamma_l(h)}, \tag{6}$$

where $\beta_k = 1$ if $k < \tau_j$ and $\beta_\tau = \alpha_\tau$, α_τ is the correction factor, $0 < \alpha_\tau \leq 1$, uniquely defined by the equation

$$\sum_{k=\tau_{2l}+1}^{\tau-1} \frac{x_{k-1}^2}{\Gamma_l(h)} + \alpha_\tau \frac{x_{\tau-1}^2}{\Gamma_l(h)} = H,$$

and

$$\tilde{H} = \sum_{k=\tau_{2l}+1}^{\tau} \sqrt{\beta_k} \frac{x_{k-1}^2}{\Gamma_l(h)}.$$

Note that, at he first stage, the parameter h can be small compared with H. As for the parameter l, according to [3], it should be not less that 3, to provide the limited expectation of the multiplier $1/\Gamma_l(h)$. However, we recommend to take $l \geq 10$, which makes estimator (2) more stable, even if we use small values of h.

Theorem 1. *The stopping instant (5) is finite with the probability one; the mean square deviation of estimator (6) is bounded from above*

$$E\left(\hat{\theta}(h, l, H) - \theta\right)^2 \leq \frac{1}{H}. \tag{7}$$

4 Simulation results

In this section, we report and discuss the results of Monte Carlo experiments. Selected data obtained by the simulations are tabulated in Table 1 . For our study, we set $\theta = 0.1, 0.3, 0.5, 0.7, 0.9, 0.99$. For each θ, 100 replications were run. The quantities recorded in Table 1 are: h – threshold in the sequential sampling rule at the first stage; H – threshold in the sequential sampling rule at the second stage; θ – the autoregressive parameter; Γ – the mean estimator for the parameter b^2 obtained at the first stage; $\tilde{\theta}$ – the mean estimator for the parameter θ obtained at the second stage; $\tilde{\sigma}^2$ – the mean square deviation for $\tilde{\theta}$; N_1 and N_2 – the mean numbers of observations at the first and at the second stages, correspondingly. The noise variance $b^2 = 0.81$ in all cases. We also compared our results with the estimator described in [3], here D – the mean estimator for the parameter b^2 obtained at the first stage; $\hat{\theta}$ – the mean estimator for the parameter θ obtained at the second stage; $\hat{\sigma}^2$ – the mean square deviation for $\tilde{\theta}$; T – the mean number of observations at the second stage; at the first stage, the number of observation was always taken equal to N_1. The threshold parameter of the procedure is equal to H.

The simulation demonstrates, that, for both procedures, the estimators of θ are in good agreement with the real value of the parameter; the mean square deviation is about $1/H$, as Theorem 1 states. But the estimators of the noise variance b^2 behave differently: for our algorithm, they are in the interval $[0.78, 1.1]$, while the real value is 0.81; for the algorithm described in [3], the interval is $[0.85, 180.7]$, so, the estimator exceeds the real value more than 200 times if the autoregressive parameter is close to the bound of the stability region. It implies the grows of the number of observations in the same proportion. If the autoregressive parameter is close to zero then estimator [3]

Table 1: Parameter estimation for AR(1) (the noise variance 0.81)

h	H	θ	Γ	$\hat{\theta}$	$\hat{\sigma}^2$	N_1	N_2	D	$\hat{\theta}$	$\hat{\sigma}^2$	T
50	500	0.1	0.848	0.102	0.0016	1239	516	0.923	0.113	0.0008	522
50	500	0.3	0.969	0.297	0.0020	1174	540	0.915	0.297	0.0011	535
50	500	0.5	1.069	0.492	0.0025	965	501	1.082	0.499	0.0014	505
50	500	0.7	1.085	0.706	0.0020	675	341	1.603	0.704	0.0010	501
50	500	0.9	1.068	0.894	0.0026	259	128	4.459	0.899	0.0003	534
50	500	0.99	0.782	0.998	0.0017	40	12	180.7	0.989	3.9612	2380
100	1000	0.1	1.069	0.095	0.0010	3467	1306	0.822	0.102	0.0010	1000
100	1000	0.3	1.065	0.300	0.0009	2275	1205	0.887	0.304	0.0010	1003
100	1000	0.5	1.010	0.497	0.0011	1876	939	1.081	0.499	0.0011	1008
100	1000	0.7	0.968	0.701	0.0011	1279	616	1.592	0.703	0.0005	998
100	1000	0.9	0.990	0.897	0.0012	505	240	4.465	0.901	0.0002	1047
100	1000	0.99	0.765	0.998	0.0008	65	25	147.4	0.990	1.4631	3622

slightly outperforms estimator (6); the number of observations T is less then N_2 for about 25-30 per cent (1000 vs 1306).

So, our procedure can be used for the estimation of the autoregressive parameter in AR(1). It should be noted that it can be applied even if the process is not stable, unlike estimator [3].

Acknowledgement

This study was supported by The Ministry of Education and Science of the Russian Federation, Goszadanie No 2.3208.2017/4.6

References

[1] Borisov V.Z., Konev V.V. (1977). On sequential parameter estimation in discrete time processes. *Automat. Remote Control*. Vol. **38**, pp. 58-64.

[2] Konev V.V., Vorobeychikov S.E. (2017). Non-asymptotic confidence estimation of the parameters in stochastic regression models with Gaussian noises. *Sequential Analysis – Design Methods and Applications*. Vol. **36**, pp. 55-75.

[3] Dmitrienko A.A., Konev V.V. (1994). On a Guaranteed Estimation of Autoiregressive Parameters for an Unknown Variance of Noise. *Automat. Remote Control*. Vol. **55**, pp. 218-228.