INTERNALLY HOMOGENEOUS RANDOM FIELDS ANALYSIS

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Abstract

An internally homogeneous random field and the variogram are introduced, and their properties are analyzed.
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1 Introduction

A number of publications is devoted to the analysis of internally stationary random processes, e.g. [1] – [4]. The variogram is a major characteristic of internally stationary random processes. The results of its properties studies and statistical properties of its estimators are presented in [5] – [6]. Here the variogram analysis for internally homogeneous random fields is performed.

2 The variogram and an internally stationary random field

Let $X(t),\ t \in R^n$, be a real valued homogeneous random field with the mathematical expectation $m = MX(t) = 0,\ t \in R^n$, the covariance function $R(t),\ t \in R^n$, the spectral density $f(\lambda),\ \lambda \in R^n$, and the correspondent spectral function $F(\lambda),\ \lambda \in R^n$.

Definition 1. A random field $X(t),\ t \in R^n$, is called internally homogeneous, if

$$M\{X(t+h) - X(t)\} = 0,$$

$$D\{X(t+h) - X(t)\} = 2\gamma(h),$$

for all $t,\ h \in R^n$, function $2\gamma(h)$ is called the variogram, and $\gamma(h)$ is the semivariogram.

Note that a homogeneous random field is also an internally homogeneous with

$$\gamma(h) = 0,5(DX(t+h) - 2\operatorname{cov}\{X(t+h), X(t)\} + DX(t)) = R(0) - R(h).$$

Although, an internally homogeneous random field is not necessary to be homogeneous.

An internally homogeneous random field $X(t),\ t \in R^n$, that satisfies the condition

$$M\{X^2(t)\} = D = \text{ const} < \infty,$$

is also a homogeneous random field.

For the real Gaussian random field, homogeneity and internal homogeneity are equivalent.

Theorem 1. *If $R(t),\ t \in R^n$, is the covariance function of an homogeneous random field, then $R(t),\ t \in R^n$, is non-negatively defined function. Vice versa, if $R(t),\ t \in R^n$, is an even non-negatively defined function, then there exist the only one Gaussian random field with zero mean and covariance function $R(t),\ t \in R^n$.*

If $R(t),\ t \in R^n$, is an integrable covariance function, then the spectral function $F(\lambda),\ \lambda \in R^n$, is absolutely continuous, and the spectral density

$$f(\lambda) = \frac{1}{(2\pi)^n} \int\limits_{R^n} R(\tau) e^{-i(\lambda,\tau)} d\tau,$$

where (λ, τ) is the scalar product of vectors $\lambda, \tau \in R^n$.

Note that the sums, the products and the limits of the non-negatively defined functions are non-negatively defined; the sumes, the products and the limits of the covariance functions are still covariance functions.

Definition 2. The function $\gamma(t),\ t \in R^n$, is called conditionally negatively defined, if for any natural $m,\ m \geq 1$, arbitrary $t_i \in R^n,\ i = \overline{1, m}$, and any non-zero real vector $(a_1, \ldots, a_m)$, such that $\sum_{i=1}^m a_i = 0$, the inequality holds

$$\sum_{i=1}^m \sum_{j=1}^m a_i a_j \gamma(t_i - t_j) \leq 0.$$

Let $X(t),\ t \in R^n$, be real internally homogeneous random field with a finite second order moment and semivariogram $\gamma(t),\ t \in R^n$.

Theorem 2. *The semivariogram $\gamma(t),\ t \in R^n$, of an internally homogeneous random field $X(t),\ t \in R^n$, is a conditionally negatively defined function.*

Proof. From the variogram definition we have

$$\sum_{i=1}^m \sum_{j=1}^m a_i a_j \gamma(t_i - t_j) = \sum_{i=1}^m \sum_{j=1}^m a_i a_j \frac{1}{2} D\{X(t_i) - X(t_j)\} =$$

$$= \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m a_i a_j (D\{X(t_i)\} - 2R(t_i, t_j) + D\{X(t_j)\}) =$$

$$= \frac{1}{2} \left(\sum_{i=1}^m a_i D\{X(t_i)\} \sum_{j=1}^m a_j - 2 \sum_{i,j=1}^m a_i a_j R(t_i, t_j) + \sum_{i=1}^m a_i \sum_{j=1}^m a_j D\{X(t_j)\} \right).$$

As $\sum_{i=1}^m a_i = 0$, then

$$\sum_{i=1}^m \sum_{j=1}^m a_i a_j \gamma(t_i - t_j) = - \sum_{i,j=1}^m a_i a_j R(t_i, t_j) \leq 0.$$

The last inequality is valid due to Theorem 1. □

Note that

$$D\left(\sum_{i=1}^{m} a_i X(t_i)\right) = -\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j \gamma(t_i - t_j).$$

Theorem 3. *Let $\gamma_1(t), \gamma_2(t)$ be the semivariograms of internally homogeneous random fields $X_1(t), X_2(t),\ t \in R^n$, respectively. Then the function $\gamma(t) = \gamma_1(t)+\gamma_2(t),\ t \in R^n$, is also the semivariogram of an internally homogeneous random field.*

Proof of the theorem follows form Theorem 2.

Theorem 4. *Let $\gamma(t),\ t \in R^n$, be the semivariogram of an internally homogeneous random field $X(t),\ t \in R^n$. Then for any $b > 0$ the function $b\gamma(t)$ is the semivariogram of the internally homogeneous random field $\sqrt{b}X(t),\ t \in R^n$.*

Proof of the theorem is based on the Theorem 2 statement.

Theorem 5. *Let an arbitrary real function $m(t) = m,\ t \in R^n$, and an even conditionally negatively defined real function $\gamma(t),\ t \in R^n$ exist. Then there exist a probability space and a real Gaussian random field defined on it $X(t),\ t \in R^n$, so that $M\{X(t)\} = m$ and $D\{X(t + h) - X(t)\} = 2\gamma(h)$ for all $h \in R^n$.*

Proof is analogous to the proof of Theorem 1 in [1] for random processes.

Corollary 1. *The class of even conditionally negatively defined real functions coincides with the class of real Gaussian homogeneous random fields semivariograms.*

Theorem 6. *The continuous function $\gamma(t),\ t \in R^n$, is a semivariogram of an internally homogeneous random field $X(t),\ t \in R^n$, with a finite second order moment, if and only if for any $a > 0$ the function $e^{-a\,\gamma(t)},\ t \in R^n$, is non-negatively defined.*

Proof. Necessity. From Theorem 5 there exist a probability space and a real Gaussian random field $X(t),\ t \in R^n$, defined on it with $M\{X(t)\} = 0$ and $D\{Y(t) - Y(s)\} = 2\gamma(t - s)$ for all $t, s \in R^n$. Note that the field $X(t)$ is internally homogeneous.

Put $Z(s) = e^{-i\sqrt{a}\,X(s)}$ and find the correlation function for this field.

$$R_Z^0(s, s + t) = M\left\{Z(s)\overline{Z(s + t)}\right\} = M\left\{e^{i\sqrt{a}\,(X(s+t)-X(s))}\right\}. \tag{1}$$

From the characteristic function definition, and the field $X(t)$ properties, the right-hand side of (1) for any $t \in R^n$ equals

$$\Psi_{X(s+t)-X(s)}(\sqrt{a}) = e^{-\frac{a}{2}D\{X(s+t)-X(s)\}} = e^{-a\,\gamma(t)},\ a > 0.$$

Hence, for any $a > 0$ the function $e^{-a\,\gamma(t)},\ t \in R^n$, is a characteristic function. From the Bokhner–Khinchin Theorem [7], $e^{-a\,\gamma(t)}$ is non-negatively defined.

Sufficiency. Let $e^{-a\,\gamma(t)},\ a > 0,\ t \in R^n$, be a non-negatively defined function. Then from the Bokhner–Khinchin Theorem this function is a characteristic function. Further proof duplicates the sufficiency proof of Theorem 1 in [6]. □

Corollary 2. *Let $\gamma(t),\ t \in R^n$, be the semivariogram of an internally homogeneous random field $X(t),\ t \in R^n$, satisfying the condition: $M\{X^2(t)\} < \infty$ for any $t \in R^n$. Then for any $a > 0$ the function $e^{-a\,\gamma(t)},\ t \in R^n$ is a correlation function of a random field.*

Theorem 7. *If the semivariogram $\gamma(t),\ t \in R^n$, of the internally homogeneous random field $X(t),\ t \in R^n$, with a finite second order moment is a continuous function, then the following statements are equivalent:*

1. *$\gamma(t)$ is a conditionally negatively defined function;*
2. *$e^{-a\,\gamma(t)}$ is a non-negatively defined function for any $a > 0,\ t \in R^n$.*

Proof follows from Theorem 6 in this paper and Theorem 1 in [6].

3 Semivariogram asymptotics

For homogeneous random fields the covariance function $R(t)$ goes to zero at $|t| \to \infty$. That is why the semivariogram $\gamma(t) \to R(0)$, when $|t| \to \infty$.

For internally homogeneous random fields that do not have finite second moment, the semivariogram $\gamma(t) \to \infty$ at $|t| \to \infty$, and the covariance function does not exist.

Let further $\gamma(t),\ t \in R^n$, be the semivariogram of an internally homogeneous random field, that has no finite moments of the second order. Analyze the asymptotics of the semivariogram $\gamma(t)$ at $t \to \infty$.

Theorem 8. *The semivariogram $\gamma(t),\ t \in R^n$, of an internally homogeneous random field $X(t),\ t \in R^n$, can not increase at the infinity faster than the function At^2, where A is a positive constant, $t \in R^n$.*

Proof. Using the variogram definition, for any $n \in N$ we have:

$$2\gamma(t) = M\{(X(t) - X(0))^2\} = M\left\{\left(\sum_{i=1}^{n}\left[X\left(\frac{it}{n}\right) - X\left(\frac{(i-1)t}{n}\right)\right]\right)^2\right\} =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} M\left\{\left[X\left(\frac{it}{n}\right) - X\left(\frac{(i-1)t}{n}\right)\right]\ \left[X\left(\frac{jt}{n}\right) - X\left(\frac{(j-1)t}{n}\right)\right]\right\}.$$

From the Cauchy–Bunyakovsky inequality we get:

$$2\gamma(t) \le$$

$$\le \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{M\left\{\left[X\left(\frac{it}{n}\right) - X\left(\frac{(i-1)t}{n}\right)\right]^2\right\}}\sqrt{M\left\{\left[X\left(\frac{jt}{n}\right) - X\left(\frac{(j-1)t}{n}\right)\right]^2\right\}} =$$

$$= n^2 \cdot 2\gamma\left(\frac{t}{n}\right).$$

Hence,
$$\frac{\gamma(t)}{t^2} \leq \frac{\gamma(t/n)}{(t/n)^2}$$

Denote by A the maximum of the function $\gamma(t)/t^2$ with $t \geq 1$, then $\gamma(t) \leq At^2$, $t \geq 1$. From here we get the result of the Theorem. □

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