DRIFT PARAMETER ESTIMATION IN GAUSSIAN REGRESSION MODEL BY CONTINUOUS AND DISCRETE OBSERVATIONS

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Abstract

The paper is devoted to the maximum likelihood estimation in the regression model of the form $X_t = \theta G(t) + B_t$, where B is a Gaussian process, $G(t)$ is a known function, and θ is an unknown drift parameter. The estimation techniques for the cases of discrete-time and continuous-time observations are presented. As examples, models with fractional Brownian motion, sub-fractional Brownian motion and two independent fractional Brownian motions are considered.
Keywords: data science, fractional Brownian motion, discrete observation, continuous observation, drift parameter

1 Introduction

We study rather general model where the noise is represented by a centered Gaussian process $B = \{B_t, t \geq 0\}$ with known covariance function, $B_0 = 0$. We assume that all finite-dimensional distributions of the process $\{B_t,\ t > 0\}$ are multivariate normal distributions with nonsingular covariance matrices. We observe the process X_t with a drift $\theta G(t)$, that is,

$$X_t = \theta G(t) + B_t,$$

where $G(t) = \int_0^t g(s)\, ds$, and $g \in L_1[0, t]$ for any $t > 0$. The paper is devoted to the estimation of the parameter θ by observations of the process X. We consider the MLEs for discrete and continuous schemes of observations. The results presented are based on the recent articles [2, 1, 3].

2 Drift parameter estimator for discrete-time observations

Let the process X be observed at the points $0 < t_1 < t_2 < \ldots < t_N$. Then the vector of increments

$$\Delta X^{(N)} = (X_{t_1},\ X_{t_2} - X_{t_1},\ \ldots,\ X_{t_N} - X_{t_{N-1}})^\top$$

is a one-to-one function of the observations. We assume in this section that the inequality $G(t_k) \neq 0$ holds at least for one k.

Evidently, vector $\Delta X^{(N)}$ has Gaussian distribution $\mathcal{N}(\theta \Delta G^{(N)}, \Gamma^{(N)})$, where

$$\Delta G^{(N)} = \big(G(t_1),\ G(t_2) - G(t_1),\ \ldots,\ G(t_N) - G(t_{N-1})\big)^\top.$$

Let $\Gamma^{(N)}$ be the covariance matrix of the vector

$$\Delta B^{(N)} = (B_{t_1},\, B_{t_2} - B_{t_1},\, \dots,\, B_{t_N} - B_{t_{N-1}})^\top.$$

Then one can take the density of the distribution of the vector $\Delta X^{(N)}$ for a given θ w. r. t. the density for $\theta = 0$ as a likelihood function:

$$L^{(N)}(\theta) = \exp\left\{\theta(\Delta G^{(N)})^\top(\Gamma^{(N)})^{-1}\Delta X^{(N)} - \frac{\theta^2}{2}(\Delta G^{(N)})^\top(\Gamma^{(N)})^{-1}\Delta G^{(N)}\right\}.$$

The corresponding MLE equals

$$\hat\theta^{(N)} = \frac{\left(\Delta G^{(N)}\right)^\top \left(\Gamma^{(N)}\right)^{-1} \Delta X^{(N)}}{\left(\Delta G^{(N)}\right)^\top \left(\Gamma^{(N)}\right)^{-1} \Delta G^{(N)}}. \tag{1}$$

Theorem 1 (Properties of the discrete-time MLE [2]). *1. The estimator $\hat\theta^{(N)}$ is unbiased and normally distributed:*

$$\hat\theta^{(N)} - \theta \simeq \mathcal{N}\left(0, \frac{1}{(\Delta G^{(N)})^\top(\Gamma^{(N)})^{-1}\Delta G^{(N)}}\right).$$

2. Assume that

$$\frac{\operatorname{var} B_t}{G^2(t)} \to 0, \quad \text{as } t \to \infty.$$

If $t_N \to \infty$, as $N \to \infty$, then the discrete-time MLE $\hat\theta^{(N)}$ converges to θ as $N \to \infty$ almost surely and in $L_2(\Omega)$.

3 Drift parameter estimator for continuous-time observations

In this section we suppose that the process X_t is observed on the whole interval $[0, T]$. We investigate MLE for the parameter θ based on these observations.

Let $\langle f,\, g\rangle = \int_0^T f(t)g(t)\, dt$. Assume that the function G and the process B satisfy the following conditions.

(A) There exists a linear self-adjoint operator $\Gamma = \Gamma_T : L_2[0, T] \to L_2[0, T]$ such that

$$\operatorname{cov}(X_s, X_t) = \mathsf{E}\, B_s B_t = \int_0^t \Gamma_T\, \mathbb{1}_{[0,s]}(u)\, du = \langle \Gamma_T\, \mathbb{1}_{[0,s]},\, \mathbb{1}_{[0,t]}\rangle.$$

(B) The drift function G is not identically zero, and in its representation $G(t) = \int_0^t g(s)\, ds$ the function $g \in L_2[0, T]$.

(C) There exists a function $h_T \in L_2[0, T]$ such that $g = \Gamma h_T$.

Note that under assumption (A) the covariance between integrals of deterministic functions $f \in L_2[0,T]$ and $g \in L_2[0,T]$ w. r. t. the process B equals

$$\mathsf{E} \int_0^T f(s)\, dB_s \int_0^T g(t)\, dB_t = \langle \Gamma_T f,\, g\rangle.$$

Theorem 2 (Likelihood function and continuous-time MLE [2])**.** *Let T be fixed, assumptions (A)–(C) hold. Then one can choose*

$$L(\theta) = \exp\left\{\theta \int_0^T h_T(s)\, dX_s - \frac{\theta^2}{2} \int_0^T g(s) h_T(s)\, ds\right\} \tag{2}$$

as a likelihood function. The MLE equals

$$\hat\theta_T = \frac{\int_0^T h_T(s)\, dX_s}{\int_0^T g(s) h_T(s)\, ds}. \tag{3}$$

It is unbiased and normally distributed:

$$\hat\theta_T - \theta \simeq \mathcal{N}\left(0, \frac{1}{\int_0^T g(s) h_T(s)\, ds}\right).$$

Theorem 3 (Consistency of the continuous-time MLE [2])**.** *Assume that assumptions (A)–(C) hold for all $T > 0$. If, additionally,*

$$\liminf_{t\to\infty} \frac{\operatorname{var} B_t}{G(t)^2} = 0,$$

then the estimator $\hat\theta_T$ converges to θ as $T \to \infty$ almost surely and in mean square.

Theorem 4 (Relations between discrete and continuous MLEs [2])**.** *Let the assumptions of Theorem 2 hold. Construct the estimator $\hat\theta^{(N)}$ from* (1) *by observations $X_{Tk/N}$, $k = 1, \dots, N$. Then*

1) the estimator $\hat\theta^{(N)}$ converges to $\hat\theta_T$ in mean square, as $N \to \infty$,

2) the estimator $\hat\theta^{(2^n)}$ converges to $\hat\theta_T$ almost surely, as $n \to \infty$.

4 Application of estimators to various models

4.1 Model with fractional Brownian motion and power drift

Let $0 < H < 1$ and $\alpha > -1$. Consider the process

$$X_t = \theta t^{\alpha+1} + B_t^H, \tag{4}$$

where $B^H = \left\{B_t^H, t \ge 0\right\}$ is a fractional Brownian motion with Hurst index H.

Theorem 5 ([2])**.** *If $\alpha > H - 1$, the model* (4) *satisfies the conditions of Theorem 1. The estimator $\hat\theta^{(N)}$ in the model* (4) *is L_2-consistent and strongly consistent (provided that* $\lim_{N\to\infty} t_N = +\infty$*). If $\alpha > 2H - \frac{3}{2}$, the conditions of Theorems 2, 3 and 4, are satisfied. The estimator $\hat\theta_T$ is L_2-consistent and strongly consistent. For fixed T, it can be approximated by discrete-sample estimator in mean-square sense.*

4.2 Model with subfractional Brownian motion

Let $0 < H < 1$. Consider the model

$$X_t = \theta t + \widetilde{B}_t^H, \tag{5}$$

where $\widetilde{B}^H = \left\{ \widetilde{B}_t^H, t \geq 0 \right\}$ is a subfractional Brownian motion with Hurst parameter H.

Theorem 6 ([2]). *Under condition $t_N \to +\infty$ as $N \to \infty$, the estimator $\hat{\theta}^{(N)}$ in the model* (5) *is L_2-consistent and strongly consistent. If $\frac{1}{2} < H < \frac{3}{4}$, then the random process $\widetilde{B}^H$ satisfies Theorems 2, 3, and 4. As the result, $L(\theta)$ defined in* (2) *is the likelihood function in the model* (5)*, and $\hat{\theta}_T$ defined in* (3) *is the MLE. The estimator is L_2-consistent and strongly consistent. For fixed T, it can be approximated by discrete-sample estimator in mean-square sense.*

4.3 The model with two independent fractional Brownian motions

Consider the following model:

$$X_t = \theta t + B_t^{H_1} + B_t^{H_2}, \tag{6}$$

where B^{H_1} and B^{H_2} are two independent fractional Brownian motion with Hurst indices $H_1, H_2 \in (\frac{1}{2}, 1)$.

Theorem 7 ([3]). *Under condition $t_N \to +\infty$ as $N \to \infty$, the estimator $\hat{\theta}^{(N)}$ in the model* (6) *is L_2-consistent and strongly consistent. If $H_1 \in (1/2, 3/4]$ and $H_2 \in (H_1, 1)$, then the random process $B^{H_1} + B^{H_2}$ satisfies Theorems 2, 3, and 4. As the result, $L(\theta)$ defined in* (2) *is the likelihood function in the model* (6)*, and $\hat{\theta}_T$ is the maximum likelihood estimator. The estimator is L_2-consistent and strongly consistent. For fixed T, it can be approximated by discrete-sample estimator in mean-square sense.*

References

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