COMPARATIVE ANALYSIS OF OPTIMAL ALGORITHMS TREATMENT OF STATIONARY POISSON FLUXES

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Abstract

Two Poisson signals detectors that use the results of number of events' accounts accounting or time intervals between the neighboring events measurements are described. It is displayed the both detectors could provide the close detection characteristics.
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1 Introduction

It is in common practice to describe different random fluxes by Poisson distribution. The most applicable they are for the tasks in applied physics, radio physics, radio and optical location where they are usually named as a simplest flux of homogenous events or a stationary (constant distribution parameter) Poisson flux (SPF). Depending the sample values being analyzed, the SPF could be represented by Poisson law of accounts' distribution or the exponential law of the distribution of intervals between neighboring events. It can be assumed that both ideas have the right to practical application. Indeed, the constant intensity λ of the SPF:

$$\lambda = \frac{m}{T}, \tag{1}$$

where m is the number of events that are detected within the interval T. So, it is possible to select m or T as a variable fixing the alternative that leads to obvious equivalency of two interpretations of the same random process. If we fix T counting of events will be terminated just at the moment T [1, 2]. In the opposite case the time measurement procedure should be stopped by m events registration [3].

The theory and practice of hypotheses testing use the first idea in common. Nevertheless, the second one was grounded enough for practical use [4]. Moreover, it was offered to explore the testing of a simple hypothesis about the parameter of an exponential distribution to build an optimized detector [5] for the goals of super weak optical signals at the level of single photons that are just the events of the Poisson process. Such a procedure is considered below.

2 Main part

It is known that the decision-making algorithm consists in comparing likelihood ration or its logarithm with a definite threshold. In this case, we have the series of intervals $t_1, \dots, t_m$ between the consecutive events that are independent sample values belonging to the exponential distribution

$$f(t) = \lambda e^{-\lambda t}, \; t > 0, \; \lambda > 0. \tag{2}$$

To make a decision, a simple hypothesis H_0 that $\lambda = \lambda_0$ is verified, against a simple alternative H_1, that the distribution parameter $\lambda = \lambda_1 > \lambda_0$. The logarithm of the likelihood ratio in this case is [5]:

$$\ln l(t_1, \dots, t_m) = \sum_{k=1}^{m} \ln \frac{\lambda_1 e^{-\lambda_1 t_k}}{\lambda_0 e^{-\lambda_0 t_k}} = m \ln \frac{\lambda_1}{\lambda_0} - (\lambda_1 - \lambda_0) \sum_{k=1}^{m} t_k. \tag{3}$$

This gives us the rule of a decision selection

$$\ln \frac{\lambda_1}{\lambda_0} - (\lambda_1 - \lambda_0) \sum_{k=1}^{m} t_k \geq \ln c_0 \tag{4}$$

or

$$\frac{1}{m} \sum_{k=1}^{m} t_k \leq \frac{1}{\lambda_1 - \lambda_0} (\ln \frac{\lambda_1}{\lambda_0} - \frac{1}{m} \ln c_0) = \frac{1}{\lambda_1 - \lambda_0} \ln \frac{\lambda_1}{\lambda_0 c_0^{1/m}} = c, \tag{5}$$

where c_0 and c are the thresholds of decision making.

In accordance with (5) the rule of a decision selection for fixed in advance the size of the retrieval could be given in the next way: the hypothesis H_1 is true ($\lambda = \lambda_1$) if [6]

$$S_{0.1} = \frac{1}{m} \sum_{k=1}^{m} t_k \leq \frac{1}{\lambda_1 - \lambda_0} \ln \frac{\lambda_1}{\lambda_0 c_0^{1/m}} = c, \tag{6}$$

otherwise the hypothesis H_0 is true ($\lambda = \lambda_0$).

Thus, the algorithm for testing the hypothesis on the parameter of the exponential probability distribution is reduced to comparing the arithmetic mean of sample values with the decision rule threshold

$$c = \frac{1}{\lambda_1 - \lambda_0} \ln \frac{\lambda_1}{\lambda_0 c_0^{1/m}}, \tag{7}$$

where value of c should be stated in accordance with the selected quality criterion of the decision rule.

To find such criteria it is necessary to determine conditional probabilities of errors of the first and second kind (noting that the optimal decision-making algorithm (6) contains the sum of m independent exponentially distributed random variables). It is known that the sum of m independent exponentially distributed random variables has a χ^2 distribution with $2m$ degrees of freedom [5] and its parameter $\lambda = \frac{1}{2}$. Because

$\lambda > 0$ and the hypothesis $H_0(\lambda = \lambda_0)$ is under checking in contra the simple hypothesis H_1 $(\lambda = \lambda_1)$, in the considering case the random variable $2\lambda \sum_{k=1}^{m} t_k$ is χ^2 distributed with $2m$ degrees of freedom. Thus, the errors of the first (α) and second (β) kinds could be calculated as

$$\alpha = P\{2\lambda_0 \sum_{k=1}^{m} t_k \leq 2m\lambda_0 c | H_0\} = \frac{\Gamma(m, 2m\lambda_0 c)}{\Gamma(m)}, \tag{8}$$

$$\beta = P\{2\lambda_1 \sum_{k=1}^{m} t_k > 2m\lambda_1 c | H_1\} = 1 - \frac{\Gamma(m, 2m\lambda_1 c)}{\Gamma(m)}, \tag{9}$$

where $\Gamma(m)$ is a gamma function and $\Gamma(m, 2m\lambda_i c)$ is incomplete gamma function.

It is necessary to outline that the random variables $\sum_{k=1}^{m} t_k$ and $2\lambda \sum_{k=1}^{m} t_k$ are both distributed with the law χ^2 with $2m$ degrees of freedom but they have different parameters. And arithmetic mean is not apparently presents in (8) and (9) that could make the practical realization more complete.

For the Neumann-Pierson criterion under defined error of the first kind (8) could be conversed in the threshold ($\chi^2_{1-\alpha}$ is a percent point of χ^2 distribution with $2m$ degrees of freedom)

$$c = \frac{1}{2m\lambda_0} \chi^2_{1-\alpha}. \tag{10}$$

The threshold (10) is independent from λ_1.

If $m \gg 1$ and taking into account asymptotic normality of χ^2 distribution it is possible to represent the errors (8) and (9) in the view:

$$\alpha \approx \Phi(2\sqrt{2m\lambda_0 c} - 2\sqrt{m}), \tag{11}$$

$$\beta \approx 1 - \Phi(2\sqrt{2m\lambda_1 c} - 2\sqrt{m}), \tag{12}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$ is the integral of probability.

The power of the decision selection rule:

$$D = 1 - \beta \approx \Phi(2\sqrt{2m\lambda_1 c} - 2\sqrt{m}). \tag{13}$$

The expression (12) gives us the threshold of decision that is dependent on the known m and λ_0:

$$c \approx \frac{(\sqrt{m} - \frac{1}{2}\Phi^{-1}(1-\alpha))^2}{2m\lambda_0}. \tag{14}$$

Then, the power of the decision selection rule will be

$$D \approx \Phi(2\sqrt{m}(\sqrt{k_\lambda} - 1) - \sqrt{k_\lambda}\Phi^{-1}(1-\alpha)), \tag{15}$$

where Φ^{-1} is the reverse function of the integral of probability, and $k_\lambda = \dfrac{\lambda_1}{\lambda_0}$ is the ratio of parameters of the exponential law meeting alternative of hypothesis (distance between the hypothesis and alternative).

It is possible to find the indicators of decision making quality in different way using the asymptotic normality of arithmetic mean distribution of samples. In this case the indicators could be calculated through the integrals of probability [6] (sinistral decision). In this way the error of the first kind is

$$\alpha = P(-\infty < S_{0.1} < c) = \Phi\{\frac{c - \frac{1}{\lambda_0}}{\frac{1}{\lambda_0\sqrt{m}}}\} = \Phi\{c\lambda_0\sqrt{m} - \sqrt{m}\}. \tag{16}$$

And the threshold of decision depends on the level of the given error of the first kind, number of events, and intensity λ_0:

$$c = \frac{1}{\lambda_0}(1 - \frac{1}{\sqrt{m}}\Phi^{-1}\{1-\alpha\}). \tag{17}$$

In the same way the power of the rule of decision selection could be defined as

$$D = \Phi\{\sqrt{m}(k_\lambda - 1) - k_\lambda\Phi^{-1}\{1-\alpha\}\}. \tag{18}$$

It is necessary to stress the difference in the D expressions for χ^2 distribution (15) and for Gauss distribution (18).

Now, let's consider the optimal algorithm of decision making when in the rule (4) dividing by m abandoned [3], and the rule (4) transmit in

$$m \ln\frac{\lambda_1}{\lambda_0} - (\lambda_1 - \lambda_0)\sum_{k=1}^{m} t_k \geq \ln c_0 \tag{19}$$

$$\sum_{k=1}^{m} t_k \leq \frac{m\ln\frac{\lambda_1}{\lambda_0} - \ln c_0}{\lambda_1 - \lambda_0} = c. \tag{20}$$

According (20) the rule for fixed sample volume m is the following: the alternative H_1 is true and $\lambda = \lambda_1$ if

$$S_{0.2} = \sum_{k=1}^{m} t_k \leq c, \tag{21}$$

and hypothesis H_0 is true and $\lambda = \lambda_0$ if the opposite to (21) inequity is held. Thus, the considering algorithm of hypothesis on the parameter of the exponential distribution checking reduces to comparison of the sum of the sample values with the threshold of the decision rule.

Because the sum of independent random variables that fluctuate in accordance with exponential law with parameter λ is subordinated to gamma distribution with the parameters λ and m, the error of the first kind for $m \gg 1$ when gamma distribution runs to Gauss distribution and integrals of probability [6] are applicable (sinistral decision)

$$\alpha = P(-\infty < S_{0.2} < c) = \Phi\{\frac{c - \frac{m}{\lambda_0}}{\frac{\sqrt{m}}{\lambda_0}}\} = \Phi\{\frac{c\lambda_0}{\sqrt{m}} - \sqrt{m}\}. \tag{22}$$

The threshold of decision will be equal to

$$c = \frac{\sqrt{m}}{\lambda_0}(\sqrt{m} - \Phi^{-1}\{1-\alpha\}) = \frac{\sqrt{m}}{\lambda_0}(1 - \frac{1}{\sqrt{m}}\Phi^{-1}\{1-\alpha\}). \tag{23}$$

Its value depends on the stated error of the first kind, sample volume, and parameter of the exponential distribution. For these conditions power of decision is the same as it was earlier (18).

3 Conclusion

The comparative analysis of two algorithms displayed that for close enough hypothesis and alternative the difference of powers of the decision making rules are absent or acceptable. For big distance the difference could become inadmissible. At ones, under direct application of Gaussian approximation to gamma distribution the power of rule doesn't exceed one for Gaussian approximation of χ^2 distribution.

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