STATISTICAL PROPERTIES OF PARAMETER ESTIMATORS IN THE FRACTIONAL VASICEK MODEL

S.S. Lohvinenko
Taras Shevchenko National University of Kyiv
Kyiv, UKRAINE
e-mail: stanislav.lohvinenko@gmail.com

Abstract

We study the fractional Vasicek model, described by the stochastic differential equation $dX_t = (\alpha - \beta X_t)\, dt + \gamma\, dB_t^H$, where B^H is a fractional Brownian motion. We assume that the parameters $x_0 \in \mathbb{R}$, $\gamma > 0$ and $H \in (0, 1)$ are known and consider a problem of estimating α and β. Least squares, maximum likelihood and alternative estimators are constructed, and their asymptotic properties are established.

Keywords: data science, fractional Vasicek model, stochastic differential equation

1 Introduction

The standard Vasicek model was proposed and studied by O. Vasicek [6] in 1977 for the purpose of interest rate modeling. It is described by the following stochastic differential equation

$$dX_t = (\alpha - \beta X_t)\, dt + \gamma\, dW_t, \tag{1}$$

where $\alpha, \beta, \gamma \in \mathbb{R}_+$, and W is a standard Wiener process. From the financial point of view, β corresponds to the speed of recovery, the ratio α/β is the long-term average interest rate, and γ represents the stochastic volatility. Now the Vasicek model is widely used not only in finance, but also in various scientific areas such as economics, biology, physics, chemistry, medicine and environmental studies.

In our research we deal with the fractional Vasicek model of the form

$$dX_t = (\alpha - \beta X_t)\, dt + \gamma\, dB_t^H, \tag{2}$$

where the Wiener process W is replaced with B^H, a fractional Brownian motion with Hurst index $H \in (0, 1)$. This generalization of the model (1) enables one to model processes with long-range dependence. Such processes appear in finance, hydrology, telecommunication, turbulence and image processing.

2 Model description

Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a complete probability space. Let $B^H = \{B_t^H,\ t \geq 0\}$ be a fractional Brownian motion on this probability space, that is, a centered Gaussian process with covariance function

$$\mathbb{E} B_t^H B_s^H = \frac{1}{2}\left(s^{2H} + t^{2H} - |t - s|^{2H}\right).$$

We consider the continuous (and even Hölder up to order H) modification of B_t^H that exists due to the Kolmogorov theorem.

We study the fractional Vasicek model, described by the stochastic differential equation

$$X_t = x_0 + \int_0^t (\alpha - \beta X_s)\, ds + \gamma B_t^H, \quad t \geq 0. \tag{3}$$

We assume that the parameters $x_0 \in \mathbb{R}$, $\gamma > 0$ and $H \in (0,1)$ are known. Such assumption can be made due to existence of many methods to estimate parameters γ and H (for example, see [1] and [3, Remark 2.1]). The main goal is to estimate parameters $\alpha \in \mathbb{R}$ and $\beta > 0$ by continuous observations of a trajectory of X on the interval $[0,T]$.

Following [2], for $0 < s < t \leq T$, define

$$\kappa_H = 2H\Gamma\left(3/2 - H\right)\Gamma\left(H + 1/2\right), \qquad \lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{\Gamma(3/2-H)},$$
$$k_H(t,s) = \kappa_H^{-1} s^{1/2-H}(t-s)^{1/2-H}, \qquad w_t^H = \lambda_H^{-1} t^{2-2H}.$$

Define also next stochastic processes

$$M_t^H = \int_0^t k_H(t,s)\, dB_s^H, \qquad P_H(t) = \frac{1}{\gamma}\frac{d}{dw_t^H}\int_0^t k_H(t,s) X_s\, ds,$$
$$S_t = \frac{1}{\gamma}\int_0^t k_H(t,s)\, dX_s, \qquad Q_H(t) = \frac{1}{\gamma}\frac{d}{dw_t^H}\int_0^t k_H(t,s)(\alpha - \beta X_s)\, ds = \frac{\alpha}{\gamma} - \beta P_H(t).$$

3 Main results

Let us introduce the least squares estimators of the unknown parameters:

$$\widehat{\alpha}_T^{(1)} = \frac{(X_T - X_0)\int_0^T X_t^2\, dt - \int_0^T X_t\, dX_t \int_0^T X_t\, dt}{T\int_0^T X_t^2\, dt - \left(\int_0^T X_t\, dt\right)^2}, \tag{4}$$

$$\widehat{\beta}_T^{(1)} = \frac{(X_T - X_0)\int_0^T X_t\, dt - T\int_0^T X_t\, dX_t}{T\int_0^T X_t^2\, dt - \left(\int_0^T X_t\, dt\right)^2}. \tag{5}$$

Theorem 1 ([5, Theorem 2.1]). *Let $H \in [\frac{1}{2}, 1)$. Then the estimators $\widehat{\alpha}_T^{(1)}$ and $\widehat{\beta}_T^{(1)}$ are strongly consistent.*

Since the discretization and simulation of $\widehat{\alpha}_T^{(1)}$ and $\widehat{\beta}_T^{(1)}$ when $H \neq 1/2$ is quite difficult, we introduce alternative estimators:

$$\widehat{\beta}_T^{(2)} = \left(\frac{1}{\gamma^2 H\Gamma\left(2H\right)T^2}\left(T\int_0^T X_t^2\, dt - \left(\int_0^T X_t\, dt\right)^2\right)\right)^{-\frac{1}{2H}}, \tag{6}$$

$$\widehat{\alpha}_T^{(2)} = \frac{\widehat{\beta}_T^{(2)}}{T}\int_0^T X_t\, dt. \tag{7}$$

Theorem 2 ([5, Theorem 2.2]). *Let $H \in (0,1)$. Then the estimators $\widehat{\alpha}_T^{(2)}$ and $\widehat{\beta}_T^{(2)}$ are strongly consistent.*

In applications usually the observations cannot be continuous. The estimators $\widehat{\alpha}_T^{(2)}$ and $\widehat{\beta}_T^{(2)}$ can be discretized as follows.

Let $h > 0$. Assume that a trajectory of X is observed at times $t_k = kh$, $k = 0, 1, \dots, n$. Define

$$\widehat{\beta}_n^{(3)} = \left(\frac{1}{\gamma^2 H \Gamma(2H) n^2} \left(n \sum_{k=0}^{n-1} X_{kh}^2 - \left(\sum_{k=0}^{n-1} X_{kh} \right)^2 \right) \right)^{-\frac{1}{2H}}, \tag{8}$$

$$\widehat{\alpha}_n^{(3)} = \frac{\widehat{\beta}_n^{(3)}}{n} \sum_{k=0}^{n-1} X_{kh}. \tag{9}$$

Theorem 3 ([5, Theorem 2.3]). *Let $H \in (0,1)$. Then the estimators $\widehat{\alpha}_n^{(3)}$ and $\widehat{\beta}_n^{(3)}$ are strongly consistent.*

Applying the analog of the Girsanov formula for a fractional Brownian motion (see [2, Theorem 3]), we obtain next likelihood ratio:

$$\begin{aligned} \Lambda_H(T) &= \exp\left\{ \int_0^T Q_H(t)\, dS_t - \frac{1}{2} \int_0^T (Q_H(t))^2 \, dw_t^H \right\} \\ &= \exp\left\{ \frac{\alpha}{\gamma} S_T - \beta \int_0^T P_H(t)\, dS_t - \frac{\alpha^2}{2\gamma^2} w_T^H \right. \\ &\quad \left. + \frac{\alpha\beta}{\gamma} \int_0^T P_H(t)\, dw_t^H - \frac{\beta^2}{2} \int_0^T (P_H(t))^2 \, dw_t^H \right\}. \end{aligned} \tag{10}$$

Now we can construct maximum likelihood estimators.

Theorem 4 ([4, Theorem 3.1]). *Let $H > 1/2$ and β is known. The MLE for α is*

$$\widehat{\alpha}_T^{(4)} = \frac{S_T + \beta \int_0^T P_H(t)\, dw_t^H}{w_T^H} \gamma. \tag{11}$$

It is unbiased, strongly consistent and normal:

$$T^{1-H} \left(\widehat{\alpha}_T^{(4)} - \alpha \right) \stackrel{d}{=} \mathcal{N}\left(0, \lambda_H \gamma^2\right).$$

Theorem 5 ([4, Theorem 3.2]). *Let $H > 1/2$ and α is known. The MLE for β is*

$$\widehat{\beta}_T^{(5)} = \frac{\frac{\alpha}{\gamma} \int_0^T P_H(t)\, dw_t^H - \int_0^T P_H(t)\, dS_t}{\int_0^T (P_H(t))^2 \, dw_t^H}. \tag{12}$$

It is strongly consistent and asymptotically normal:

$$\sqrt{T} \left(\widehat{\beta}_T^{(5)} - \beta \right) \xrightarrow{d} \mathcal{N}(0, 2\beta).$$

Theorem 6 ([4, Theorem 3.4]). *Let $H > 1/2$. The MLEs for α and β equal*

$$\begin{aligned}\widehat{\alpha}_T^{(6)} &= \frac{\int_0^T P_H(t)\,dS_t \int_0^T P_H(t)\,dw_t^H - S_T \int_0^T (P_H(t))^2\,dw_t^H}{\left(\int_0^T P_H(t)\,dw_t^H\right)^2 - w_T^H \int_0^T (P_H(t))^2\,dw_t^H}\,\gamma,\\ \widehat{\beta}_T^{(6)} &= \frac{w_T^H \int_0^T P_H(t)\,dS_t - S_T \int_0^T P_H(t)\,dw_t^H}{\left(\int_0^T P_H(t)\,dw_t^H\right)^2 - w_T^H \int_0^T (P_H(t))^2\,dw_t^H}.\end{aligned} \tag{13}$$

They are consistent and asymptotically normal:

$$T^{1-H}\left(\widehat{\alpha}_T^{(6)} - \alpha\right) \xrightarrow{d} \mathcal{N}(0, \lambda_H \gamma^2), \qquad \sqrt{T}\left(\widehat{\beta}_T^{(6)} - \beta\right) \xrightarrow{d} \mathcal{N}(0, 2\beta).$$

Theorem 7 ([3, Theorem 4.2]). *Let $H > 1/2$. The vector maximum likelihood estimator $\left(\widehat{\alpha}_T^{(6)}, \widehat{\beta}_T^{(6)}\right)$ for vector parameter (α, β) is asymptotically normal:*

$$\begin{bmatrix} T^{1-H}\left(\widehat{\alpha}_T^{(6)} - \alpha\right) \\ \sqrt{T}\left(\widehat{\beta}_T^{(6)} - \beta\right) \end{bmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \lambda_H \gamma^2 & 0 \\ 0 & 2\beta \end{bmatrix}\right), \quad T \to \infty, \tag{14}$$

hence estimators $\widehat{\alpha}_T^{(6)}$ and $\widehat{\beta}_T^{(6)}$ are asymptotically independent.

4 Acknowledgements

I would like to thank K. Ralchenko for constant support throughout whole my research.

References

[1] Berzin C., Latour A., León J.R. (2014). *Inference on the Hurst Parameter and the Variance of Diffusions Driven by Fractional Brownian Motion*. Springer.

[2] Kleptsyna M., Le Breton A., Roubaud M.-C. (2000). Parameter estimation and optimal filtering for fractional type stochastic systems. *Stat. Inference Stoch. Process*. Vol. **3**, pp. 173-182.

[3] Lohvinenko S.S., Ralchenko K.V. (2018). Asymptotic distribution of maximum likelihood estimator in fractional Vasicek model. *Teor. Imovir. Mat. Stat.* Vol. **99**, pp. 134-151 (In Ukrainian).

[4] Lohvinenko S.S., Ralchenko K.V. (2017). Maximum likelihood estimation in the fractional Vasicek model. *Lithuanian J. Statist.* Vol. **56(1)**, pp. 77-87.

[5] Lohvinenko S.S., Ralchenko K.V., Zhuchenko O.M. (2016). Asymptotic properties of parameter estimators in fractional Vasicek model. *Lithuanian J. Statist.* Vol. **55(1)**, pp. 102-111.

[6] Vasicek O. (1977). An equilibrium characterization of the term structure. *J. Finance Econ.* Vol. **5(2)**, pp. 177-188.