APPROXIMATE FORMULAS FOR EXPECTATION OF FUNCTIONALS FROM SOLUTION TO LINEAR SKOROHOD STOCHASTIC DIFFERENTIAL EQUATION

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Abstract

Approximate formulas for evaluation of mathematical expectation of nonlinear functionals of solution to linear stochastic differential equation of Skorohod are constructed. The formulas are exact for functional polynomial of second and third degree and converge to the exact value of mathematical expectation.
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1 Introduction

An approximate calculation of mathematical expectation of nonlinear functionals from solutions to stochastic differential equations is an urgent and, in the general case, extremely difficult task. This is due to the large computational complexity of the algorithms in which the approximations of the trajectories of solutions of stochastic equations and the approximation of integrable functionals from the solutions must be connected, which would ensure the approximation error sufficient for the convergence of the method. In [1,2], an approach to solving this problem for some types of the Ito equations by martingales was considered. The solution of the problem is simplified in cases when a solution of the stochastic equation can be found explicitly, the corresponding approximations are obtained for some kinds of the linear Ito equations in [3-6]. This paper is devoted to the construction of approximate formulas for calculating the expectations of functionals of solution to the linear Skorohod equation.

2 The results

Let us consider a stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(s) X_s \delta W_s, \tag{1}$$

where $X_0 = g\Big(\int_0^1 a(\tau)dW_\tau\Big)$; W_t, $t \in [0,1]$, is canonical Wiener process defined on probability space $\Omega = C_0([0,1])$, $W_t(\omega) = \omega(t)$; $\sigma(s)$, $g(u)$, $a(\tau)$ are deterministic

functions,
$\int\limits_0^1 \sigma^2(s)ds < \infty$; $g(u)$ is differentiable necessary number of times, $a(\tau) \in L_2([0,1])$,
$\int\limits_0^1 a(\tau)dW_\tau$ is stochastic Ito-Wiener integral.

The integral in right part of (1) is in the Skorohod sense because X_0 not adapted to the underlying filtration.

It is known [7,8] that the only solution to (1) is given by

$$X_t = X_0(T_t^{-\sigma}) \exp\Big\{ \int\limits_0^t \sigma(s)dW_s - \frac{1}{2}\int\limits_0^t \sigma^2(s)ds \Big\},$$

where $T_t^{-\sigma}$ is transformation on Ω defined by $(T_t^{-\sigma})(s) = \omega(s) - \int\limits_0^{t\wedge s} \sigma(\tau)d\tau$.
In our case we get

$$X_t = g\Big(\int\limits_0^1 a(\tau)dW_\tau - \int\limits_0^t a(\tau)\sigma(\tau)d\tau \Big) \exp\Big\{ \int\limits_0^t \sigma(s)dW_s - \frac{1}{2}\int\limits_0^t \sigma^2(s)ds \Big\}.$$

so we can evaluate the moments

$$E[X_t] = E\Big[g\Big(\int\limits_0^1 a(\tau)dW_\tau \Big) \Big],$$

$$E[X_{t_1}X_{t_2}] = \exp\Big\{ \int\limits_0^{t_1\wedge t_2} \sigma^2(\tau)d\tau \Big\} \times$$

$$E\Big[g\Big(\int\limits_0^1 a(\tau)dW_\tau + \int\limits_0^{t_1} a(\tau)\sigma(\tau)d\tau \Big) g\Big(\int\limits_0^1 a(\tau)dW_\tau + \int\limits_0^{t_2} a(\tau)\sigma(\tau)d\tau \Big] \equiv B(t_1,t_2);$$

$$E\Big[\prod_{k=1}^{3} X_{t_k} \Big] = \prod_{\{t_i,t_j\}} \exp\Big\{ \int\limits_0^{t_i\wedge t_j} \sigma^2(\tau)d\tau \Big\} \times$$

$$E\Big[\prod_{\{t_i,t_j\}} g_1\Big(\int\limits_0^1 a(\tau)dW_\tau; t_i, t_j \Big) \Big] \equiv C(t_1,t_2,t_3),$$

where $g_1\Big(\int\limits_0^1 a(\tau)dW_\tau; t_i, t_j \Big) = g\Big(\int\limits_0^1 a(\tau)dW_\tau + \int\limits_0^{t_i} a(\tau)\sigma(\tau)d\tau + \int\limits_0^{t_j} a(\tau)\sigma(\tau)d\tau \Big)$;
a couple $\{t_i,t_j\}$ in the product runs $\{t_1,t_2\},\{t_1,t_3\},\{t_2,t_3\}$.

Note that in calculating the moments we used the Girsanov transformation on the Wiener space.

Our main result is the next approximate formulas

$$I(F) \equiv E[F(X_{(\cdot)})] \approx I_n(F) - J_n(F) + J(F),$$

where

$I_n(F) = E\Big[F\Big(g\Big(\sum\limits_{k=1}^{n} \xi_k \int\limits_0^1 a(\tau)\alpha_k(\tau)d\tau - \int\limits_0^{(\cdot)} a(\tau)\sigma(\tau)d\tau\Big)\times$

$\exp\Big\{\sum\limits_{k=1}^{n} \xi_k \int\limits_0^{(\cdot)} \sigma(\tau)\alpha_k(\tau)d\tau - \frac{1}{2}\int\limits_0^{(\cdot)} \sigma^2(\tau)d\tau\Big\}\Big)\Big]$,

$\xi_k = \int\limits_0^1 \alpha_k(\tau)dW_\tau$; $\{\alpha_k(\tau)\}, k = 1, 2, \dots$, is an orthonormal bases in $L_2([0, 1])$;

$J_n(F) = \sum\limits_{l=1}^{4} J_{n,l}(F) + \Lambda F(b_n(\cdot)) + F(0)(1 - B_n(0, 1))$,

$J_{n,1}(F) = -\int\limits_0^1\int\limits_0^1 A_{n,1}(u, v)\triangle F(1_{[0,\cdot]}(u)1_{[\cdot,1]}(v))dudv$,

$J_{n,2}(F) = \int\limits_0^1 A_{n,2}(u)\triangle F(1_{[0,\cdot]}(u))du$, $J_{n,3}(F) = -\int\limits_0^1 A_{n,3}(v)\triangle F(1_{[\cdot,1]}(v))dv$,

$J_{n,4}(F) = f_n(0)f_n(1)h_n(0, 1)G_n(0, 1)$, $A_{n,1}(u, v) = \frac{\partial^2}{\partial u\partial v}B_n(u, v)$,

$A_{n,2}(u) = \frac{\partial}{\partial u}B_n(u, 1)$, $A_{n,3}(v) = \frac{\partial}{\partial v}B_n(0, v)$, $B_n(u, v) = f_n(u)f_n(v)G_n(u, v)h_n(u, v)$,

$G_n(u, v) = E\Big[g\Big(\int\limits_0^1 a_n(\tau)dW_\tau + r_n(u, v)\Big)g\Big(\int\limits_0^1 a_n(\tau)dW_\tau + r_n(v, u)\Big)\Big]$,

$r_n(u, v) = \sum\limits_{k=1}^{n}\Big(\int\limits_0^1 a(\tau)\alpha_k(\tau)d\tau\Big)\Big(\int\limits_0^v \sigma(\tau)\alpha_k(\tau)d\tau\Big) +$

$\sum\limits_{k=1}^{n}\Big(\int\limits_0^1 a(\tau)\alpha_k(\tau)d\tau\Big)\Big(\int\limits_0^u \sigma(\tau)\alpha_k(\tau)d\tau\Big) - \int\limits_0^u a(\tau)\sigma(\tau)d\tau$,

$h_n(u, v) = \exp\Big\{\sum\limits_{k=1}^{n}\Big(\int\limits_0^u \sigma(\tau)\alpha_k(\tau)d\tau\Big)\Big(\int\limits_0^v \sigma(\tau)\alpha_k(\tau)d\tau\Big)\Big\}$,

$f_n(u) = \exp\Big\{\frac{1}{2}\sum\limits_{k=1}^{n}\Big(\int\limits_0^u \sigma(\tau)\alpha_k(\tau)d\tau\Big)^2 - \frac{1}{2}\int\limits_0^u \sigma^2(\tau)d\tau\Big\}$;

$a_n(\tau) = \sum\limits_{k=1}^{n}\int\limits_0^1 a(s)\alpha_k(s)ds\,\alpha_k(\tau)$;

$b_n(t) = E\Big[g(\int\limits_0^1 a_n(\tau)dW_\tau + \sum\limits_{k=1}^{n}\int\limits_0^1 a(\tau)\alpha_k(\tau)d\tau)\int\limits_0^t \sigma(\tau)\alpha_k(\tau)d\tau - \int\limits_0^t a(\tau)\sigma_k(\tau)\Big]f_n(t)$;

$J(F) = \sum\limits_{l=1}^{4} J_l(F) + \Lambda F(b(\cdot)) + F(0)(1 - B(0, 1))$,

where $J_1(F) = -\int\limits_0^1\int\limits_0^1 A_1(u, v)\triangle F(1_{[0,\cdot]}(u)1_{[\cdot,1]}(v))dudv$,

$J_2(F) = \int\limits_0^1 A_2(u)\triangle F(1_{[0,\cdot]}(u))du$, $J_3(F) = -\int\limits_0^1 A_3(v)\triangle F(1_{[\cdot,1]}(v))dv$,

$J_4(F) = h(0, 1)G(0, 1)$, $A_1(u, v) = \frac{\partial^2}{\partial u\partial v}B(u, v)$,

$A_2(u) = \frac{\partial}{\partial u}B(u, 1)$, $A_3(v) = \frac{\partial}{\partial v}B(0, v)$, $B(u, v) = G(u, v)h(u, v)$,

$G(u, v) = E\Big[g\Big(\int\limits_0^1 a(\tau)dW_\tau + \int\limits_0^u a(\tau)\sigma(\tau)d\tau\Big)g\Big(\int\limits_0^1 a(\tau)dW_\tau + \int\limits_0^v a(\tau)\sigma(\tau)d\tau\Big)\Big]$,

$h(u,v) = \exp\Big\{ \int\limits_0^{u\wedge v} \sigma^2(\tau)d\tau \Big)$,
$\triangle F(x) = \frac{1}{2}(F(x) + F(-x)$, $\Lambda F(x) = \frac{1}{2}(F(x) - F(-x))$.

With some restrictions on $F(X_{(\cdot)})$, $\sigma(s)$, $g(u)$, $a(\tau)$ we have $I_n(F) \to I(F)$ $J_n(F) \to J$ which implies that $I_n(F) - J_n(F) + J(F) \to I(F)$, while maintaining accuracy for functional polynomials of second degree from the solution to (1).

The same approach we apply to construction of approximate formulas exact for functional polynomial of third degree from solution to (1). Here we give the elementary formula

$$J(F(X_{(\cdot)})) = \int\limits_0^1\int\limits_0^1\int\limits_0^1 \Big(\frac{\partial^3}{\partial u_1 \partial u_2 \partial u_3} C(u_1,u_2,u_3)\Big)\Lambda F(\theta(u_1,u_2,u_3);\cdot))du_1du_2du_3 +$$

$$3\int\limits_0^1\int\limits_0^1 \Big(\frac{\partial^2}{\partial u \partial v} C(u,v,0\cdot)\Big)\Lambda F(\theta_1(u,v);\cdot))dudv +$$

$$3\int\limits_0^1 \Big(\frac{\partial}{\partial u} C(u,0,0)\cdot)\Big)\Lambda F(a1_{[1,\cdot]}(u) + a_2 + a_3))du - 5C(0,0,0)\Lambda F(0),$$

where $\theta(u_1,u_2,u_3);t) = \sum\limits_{j=1}^{3} a_j 1_{[0,t]}(u_j)$, $\theta_1(u,v);t) = a_1 1_{[0,t]}(u) + a_2 1_{[0,t]}(v) + a_3$, a_1, a_2, a_3- are the roots of the polynomial $Q_3(x) = x^3 - x^2 + \frac{1}{2}x + \frac{1}{6}$.

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