SOME RESULTS ON THE BROWNIAN MEANDER

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Abstract

Some new results on the Brownian meander are considered.
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A Brownian meander is a Brownian motion evolving under the condition that $\min_{0\leq s\leq t} B(s) > v$. If the starting point $B(0) = u$ and the critical level $\min_{0\leq s\leq t} B(s) > v$ differ one can write down the joint distributions

$$P\left\{ \bigcap_{j=1}^{n} (B^{\mu}(s_j) \in \mathrm{d}y_j) \,\Big|\, \min_{0<z<t} B^{\mu}(z) > v, B^{\mu}(0) = u \right\} \tag{1}$$

for $y_i > v, i = 1, \dots, n$, $0 < s_1 < \dots < s_j < \dots < s_n < t$, as well as

$$P\left\{ \max_{0\leq z\leq s} B^{\mu}(z) \leq x \,\Big|\, \min_{0<z<t} B^{\mu}(z) > v, B^{\mu}(0) = u \right\} \tag{2}$$

for $s < t$, $v < u < x$.

Therefore also the first passage times

$$T_x = \inf\{s < t : B(s) = x\} \tag{3}$$

under the condition that $\min_{0\leq s\leq t} B(s) > v$ and $t < t'$ can be explored.

This analysis becomes more complicated in the case where the Brownian meander has drift, that is constructed by means of a drifted Brownian motion $B^{\mu}(t)$, $t > 0$, $\mu \in \mathbb{R}$.

For this reason we need some simplifying assumptions such that $u \downarrow v$ (with eventually $v = 0$).

In this case however we need to analyze the convergence of the sequence of measures (1) and, in particular to check the tightness of these probability measures.

An important result in this context is the proof that

$$\lim_{\delta\downarrow 0} \lim_{u\downarrow v} P\left(\max_{0\leq z\leq \delta} |B^{\mu}(z) - B^{\mu}(0)| < \eta \,\Big|\, \min_{0\leq z\leq t} B^{\mu}(z) > v, B^{\mu}(0) = u \right) = 1 \tag{4}$$

for all $\eta > 0$, which permits us to control the oscillations of the meander in the neighborhood of the starting point.

The analysis of the representation of the meander M as

$$P\left\{\frac{\left|B^{\mu}\Big(T_0^{\mu}+s(t-T_0^{\mu})\Big)\right|}{\sqrt{t-T_0^{\mu}}}\in \mathrm{d}y\right\} = \frac{1}{2}\left[P\left\{M^{-\mu\sqrt{t-T_0^{\mu}}}(s)\in \mathrm{d}y\right\}+P\left\{M^{\mu\sqrt{t-T_0^{\mu}}}(s)\in \mathrm{d}y\right\}\right] \tag{5}$$

for $0 < s < 1$, as a generalization of the analogous representation for the driftless meander is also considered. The T_0^{μ} random time is defined as

$$T_0^{\mu} = \sup\{s < t : B^{\mu}(s) = 0\} \tag{6}$$

The explicit distribution of T_0^{μ} reads

$$\begin{aligned} P(T_0^{\mu}\in \mathrm{d}a)/\mathrm{d}a &= \frac{e^{-\frac{\mu^2 t}{2}}}{\pi\sqrt{a(t-a)}}+\frac{\mu^2}{2\pi}\int_a^t \frac{e^{-\frac{\mu^2 y}{2}}}{\sqrt{a(y-a)}}\,\mathrm{d}y \\ &= \mathbb{E}\left(\frac{1}{\pi\sqrt{a(W-a)}}\,I_{(W\geq a)}\right) \qquad 0<a<t\,. \end{aligned}$$

where W is a truncated r.v. with an absolutely continuous component in $(0, t)$ with density

$$f_W(w) = \frac{\mu^2}{2}e^{-\frac{\mu^2}{2}w} \qquad 0\leq w\leq t$$

and a discrete component at $w = t$ with mass

$$P(W=t) = e^{-\frac{\mu^2 t}{2}}\ .$$

References

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