FRACTIONAL STOCHASTIC VOLATILITY: F-ORNSTEIN–UHLENBECK AND F-CIR PROCESSES

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Abstract

We consider fractional Ornstein–Uhlenbeck process as well as fractional CIR-process with Hurst index $H \in (0, 1)$, and several approaches to the exact and approximate option pricing of the asset price model that is described by the geometric linear model with stochastic volatility, where volatility is driven by fractional Ornstein–Uhlenbeck process. We assume that the Wiener process driving the asset price and the fractional Brownian motion driving stochastic volatility are correlated. We consider three possible levels of representation and approximation of option price, with the corresponding rate of convergence of discretized option price to the original one.

We can rigorously treat the class of discontinuous payoff functions of polynomial growth. As an example, our model allows to analyze linear combinations of digital and call options. Moreover, we provide rigorous estimates for the rates of convergence of option prices for polynomial discontinuous payoffs f and Hölder volatility coefficients, a crucial feature considering settings for which exact pricing is not possible.

Keywords: F-Ornstein–Uhlenbeck process, F-CIR process, stochastic volatility, data science

1 Model with stochastic volatility driven by fractional Ornstein–Uhlenbeck process

These results are common with K. Ralchenko, V. Piterbarg, V. Bezborodov, L. Di Persio, A. Yurchenko-Titarenko, S. Kuchuk-Jatsenko and partially are published in [1]-[4]. We consider a financial market, characterized by a finite maturity time T, and composed by a risk free bond, or bank account, $\beta = \{\beta_t, t \in [0, T]\}$, whose dynamic reads as $\beta_t = e^{\rho t}$, where $\rho \in \mathbb{R}^+$ represents the risk free interest rate, and a risky asset $S = \{S_t, t \in [0, T]\}$ whose stochastic price dynamic is defined over the probability space $\{\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, \mathrm{P}\}$, by the following system of stochastic differential equations

$$dS_t = bS_t dt + \sigma(Y_t)S_t dW_t, \tag{1}$$

$$dY_t = -\alpha Y_t dt + dB_t^H,\ t \in [0, T]. \tag{2}$$

Here $W = \{W_t, t \in [0, T]\}$ is a standard Wiener process, $b \in \mathbb{R}, \alpha \in \mathbb{R}^+$, are constants, while $Y = \{Y_t, t \in [0, T]\}$ characterizes the stochastic volatility term of our model, being the argument of the function σ. The process Y is Ornstein-Uhlenbeck, driven by a fractional Brownian motion $B^H = \{B_t^H, t \in [0, T]\}$, of Hurst parameter $H \in (0, 1)$, assumed to be correlated with W.

We assume that payoff function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following conditions:

(**A**)

(i) f is a measurable function of polynomial growth,

$$f(x) \leq C_f(1 + x^p), \quad x \geq 0,$$

for some constants $C_f > 0$ and $p > 0$.

(ii) Function f is locally Riemann integrable, possibly, having discontinuities of the first kind.

Moreover we assume that the function $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(**B**) there exists $C_\sigma > 0$ such that

(i) σ is bounded away from 0, $\sigma(x) \geq \sigma_{\min} > 0$;

(ii) σ has moderate polynomial growth, i.e., there exists $q \in (0, 1)$ such that

$$\sigma(x) \leq C_\sigma(1 + |x|^q), \ x \in \mathbb{R};$$

(iii) σ is uniformly Hölder continuous, so that there exists $r \in (0, 1]$ such that

$$|\sigma(x) - \sigma(y)| \leq C_\sigma |x - y|^r, \ x, y \in \mathbb{R};$$

(iv) σ is differentiable a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$, and its derivative is of polynomial growth: there exists $q' > 0$ such that

$$|\sigma'(x)| \leq C_\sigma(1 + |x|^{q'}),$$

a.e. w.r.t. the Lebesgue measure on $\mathbb{R}$.

Lemma 1. (i) *Equation* (2) *has a unique solution of the form*

$$Y_t = Y_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} dB_s^H.$$

Moreover, for any $\alpha > 0$ *and any* $\beta < 2$

$$\mathrm{E} \exp\{\alpha \sup_{t \in [0,T]} |Y_t|^\beta\} < \infty.$$

(ii) *Equation* (1) *has a unique solution of the form*

$$S_t = S_0 \exp\left\{bt + \int_0^t \sigma(Y_s) dW_s - \frac{1}{2} \int_0^t \sigma^2(Y_s) ds\right\}.$$

Moreover, for any $m \in \mathbb{Z}$ *we have* $\mathrm{E}(S_T)^m < \infty$*, and for any* $m > 0$ *it holds* $\mathrm{E}(f(S_T))^m < \infty$.

According to [5], fBm admits a compact interval representation via some Wiener process B, specifically,

$$B_t^H = \int_0^t k(t,s)dB_s, \quad k(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}} du \mathbb{1}_{s<t},$$

with $c_H = (H-\frac{1}{2}) \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} \right)^{1/2}$. Denote also

$$X(t) = \log S(t) = \log S_0 + bt - \frac{1}{2}\int_0^t \sigma^2(Y_s)ds + \int_0^t \sigma(Y_s)dW_s.$$

Lemma 2. *(i) The stochastic derivatives of the fBm B^H equal to*

$$D_u^W B_t^H = 0, \;\; D_u^B B_t^H = k(t,u).$$

(ii) The stochastic derivatives of Y equal to

$$D_u^W Y_t = 0, \; D_u^B Y_t = c_H e^{-\alpha t} u^{1/2-H} \int_u^t e^{\alpha s} s^{H-1/2}(s-u)^{H-3/2} ds \mathbb{1}_{u<t}.$$

(iii) The stochastic derivatives of X equal to

$$\begin{aligned} &D_u^W X_t = \sigma(Y_u)\mathbb{1}_{u<t}, \\ &D_u^B X_t = \left(-\int_0^t \sigma(Y_s)\sigma'(Y_s) D_u^B Y_s ds + \int_0^t \sigma'(Y_s) D_u^B Y_s dW_s \right) \mathbb{1}_{u<t}. \end{aligned}$$

Lemma 3. *The laws of S_T and X_T are absolutely continuous with respect to the Lebesgue measure.*

Let us introduce the following notations: $g(y) = f(e^y)$, $F(x) = \int_0^x f(z)dz$ and let $G(y) = \int_0^y g(z)dz$, $x \geq 0$, $y \in \mathbb{R}$. Also, let

$$Z_T = \int_0^T \sigma^{-1}(Y_u)dW_u. \tag{3}$$

Note that Z_T is well defined because of condition (**B**), (i).

Theorem 1. *Under conditions* (**A**) *and* (**B**) *the option price* $\mathrm{E}f(S_T) = \mathrm{E}g(X_T)$ *can be represented as*

$$\mathrm{E}f(S_T) = \mathrm{E}\left(\frac{F(S_T)}{S_T}\left(1 + \frac{Z_T}{T}\right)\right).$$

Alternatively,

$$\mathrm{E}g(X_T) = \frac{1}{T}\mathrm{E}\left(G(X_T)Z_T\right).$$

Consider the first approach to the numerical approximation of the solution for the option pricing problem. Consider equidistant partition of the interval $[0,T]$: $t_i = t_i(n) = \frac{iT}{n}$, $i = 0,1,2,\dots,n$. Then we define the discretizations of Wiener process W and fractional Brownian motion B^H:

$$\Delta W_i = W\left(t_{i+1}\right) - W\left(t_i\right), \quad \Delta B_i^H = B^H\left(t_{i+1}\right) - B^H\left(t_i\right), \quad i = 0,1,2,\dots,n.$$

Discretized processes Y and X, corresponding to a given partition have the form

$$Y_{t_j}^n = Y_0 e^{-\alpha t_j} + e^{-\alpha t_{j-1}} \sum_{i=0}^{j-1} e^{\alpha t_i} \Delta B_i^H,$$

$$X_{t_j}^n = X_0 + bt_j - \frac{1}{2n} \sum_{i=0}^{j-1} \sigma^2\left(Y_{t_i}^n\right) + \sum_{i=0}^{j-1} \sigma\left(Y_{t_i}^n\right) \Delta W_i$$

$$= X_0 + bt_j - \frac{1}{2}\int_0^{t_j} \sigma^2(Y_s^n) ds + \int_0^{t_j} \sigma(Y_s^n) dW_s, \quad j = 0,\dots,n,$$

where we put $Y_s^n = Y_{t_i}^n$ for $s \in \left[t_i, t_{i+1}\right)$. Concerning the discretization of the term Z_T from (3), it has a form $Z_T^n = \int_0^T \frac{1}{\sigma(Y_s^n)} dW_s$. Eventually we define $S_{t_j}^n = \exp\left\{X_{t_j}^n\right\}$.

Theorem 2. *Let conditions* (**A**) *and* (**B**) *hold. There exists a constant* C *not depending on* n *such that*

$$\left| \mathrm{E} f(S_T) - \mathrm{E}\left(\frac{F(S_T^n)}{S_T^n} \left(1 + \frac{Z_T^n}{T}\right)\right)\right| \le C n^{-rH}.$$

Let us introduce the following notations: let the covariance matrix reads as follows $C_{X,Z} = \begin{pmatrix} \sigma_Y^2 & T \\ T & \sigma_Z^2 \end{pmatrix}$, and let $\sigma_Y^2 = \int_0^T \sigma^2(Y_s) ds$, $m_Y = X_0 + bT - \frac{1}{2}\sigma_Y^2$, $\sigma_Z^2 = \int_0^T \sigma^{-2}(Y_s) ds$.

We assume additionally that the following assumption is fulfilled.

(**C**) $\Delta = \sigma_Y^2 \sigma_Z^2 - T^2 > 0$ with probability 1.

Theorem 3. *Under conditions* (**A**)–(**C**) *the following equality holds:*

$$\begin{aligned} \mathrm{E} g(X_T) &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} G(x) \mathrm{E}\left(\frac{(x - m_Y)}{\sigma_Y^3} \exp\left\{ - \frac{(x - m_Y)^2}{2\sigma_Y^2} \right\}\right) dx \\ &= (2\pi)^{-\frac{1}{2}} \mathrm{E}\left((\sigma_Y)^{-1} \int_{\mathbb{R}} G((x + m_Y)\sigma_Y) x e^{-\frac{x^2}{2}} dx \right). \end{aligned} \tag{4}$$

Let $\sigma_{Y,n} = \int_0^T \sigma^2(Y_s^n) ds$, $m_{Y,n} = X_0 + bT - \frac{1}{2}\sigma_{Y,n}^2$.

Theorem 4. *Under conditions* (**A**), (**B**), *and* (**C**) *we have*

$$\left| \mathrm{E} g(X_T) - (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} G(x) \mathrm{E}\left(\frac{(x - m_{Y,n})}{\sigma_{Y,n}^3} \exp\left\{ - \frac{(x - m_{Y,n})^2}{2\sigma_{Y,n}^2} \right\}\right) dx \right| \le C n^{-rH}.$$

Applying Theorem 3 and equality (4), we clearly see that the option price depends on the random variable $\sigma_Y^2 = \int_0^T \sigma^2(Y_s)ds$. Therefore it is natural to derive the density of this random variable. Since σ_Y^2 depends on the whole trajectory of the fBm B^H on $[0,T]$, we apply Malliavin calculus in an attempt to find the density.

Now we introduce additional assumptions on the function σ.

(**D**) The function $\sigma \in C^{(2)}(\mathbb{R})$, its derivative σ' is strictly nonnegative, $\sigma'(x) > 0$, $x \in \mathbb{R}$, and σ', σ'' are of polynomial growth.

Lemma 4. *Under assumptions* (**B**) *and* (**D**) *the stochastic process*

$$\frac{D^B\sigma_{_Y}^2}{||D^B\sigma_{_Y}^2||_H^2} = \left\{\frac{D_t^B\sigma_{_Y}^2}{||D^B\sigma_{_Y}^2||_H^2},\ t\in[0,T]\right\}$$

belongs to the domain $Dom\,\delta$ of the Skorokhod integral δ.

Denote $\eta = (||D^B\sigma_{_Y}^2||_H^2)^{-1}$, $l(u,s) = c_H e^{-\alpha s}\int_u^s e^{\alpha v} v^{H-1/2}(v-u)^{H-3/2}dv$, $\kappa(y) = \sigma(y)\sigma'(y)$.

Theorem 5. (*i*) *The density $p_{\sigma_{_Y}^2}$ of the random variable $\sigma_{_Y}^2$ is bounded, continuous and given by the following formulas*

$$p_{\sigma_{_Y}^2}(u) = \mathrm{E}\left[\mathbb{1}_{\sigma_{_Y}^2>u}\,\delta\left(\frac{D^B\sigma_{_Y}^2}{||D^B\sigma_{_Y}^2||_H^2}\right)\right], \tag{5}$$

where the Skorokhod integral is in fact reduced to a Wiener integral,

$$\delta\left(\frac{D^B\sigma_{_Y}^2}{||D^B\sigma_{_Y}^2||_H^2}\right) = 2\eta\int_0^T \kappa(Y_s)\left(\int_0^s u^{1/2-H}l(u,s)dB_u\right)ds - \int_0^T D_u^B\eta D_u^B(\sigma_{_Y}^2)du.$$

(*ii*) *The option price $\mathrm{E}g(X_T)$ can be represented as the integral with respect to the density $p_{\sigma_{_Y}^2}(u)$ defined by* (5) *as follows:*

$$\mathrm{E}g(X_T) = (2\pi)^{-\frac{1}{2}}T\int_{\mathbb{R}} G(x)\int_{\mathbb{R}} \frac{(x+u/2-X_0-bT)}{u^3} \times \exp\left\{-\frac{(x+u/2-X_0-bT)^2}{2u^2}\right\}p_{\sigma_{_Y}^2}(u)du.$$

2 Fractional CIR. Case $k = 0$

Consider the stochastic differential equation of the following form:

$$dX_t = \tilde{a}X_t dt + \tilde{\sigma}\sqrt{X_t}dB_t^H, \quad t\geq 0, \quad \tilde{a}\in\mathbb{R}, \quad X_0, \tilde{\sigma} > 0, \tag{6}$$

$B^H = \{B^H, t\geq 0\}$ is a fractional Brownian motion with $H > 2/3$.

It is known that if $H > 2/3$, the equation (6) has a unique solution until the first moment of reaching zero, and the integral $\int_0^t \sqrt{X_s}dB_s^H$ exists as a pathwise Riemann-Stieltjes sums limit. Denote $\tau_0 := \inf\{t > 0 : X_t = 0\}$ and consider the trajectories of the process $\{X_t, t \geq 0\}$ on $[0, \tau_0)$. After substitution $Y_t = \sqrt{X_t}$ and using the Ito formula for integrals with respect to fractional Brownian motion, we obtain:

$$dY_t = \frac{dX_t}{2\sqrt{X_t}} = \frac{\tilde{a}X_t dt}{2\sqrt{X_t}} + \frac{\tilde{\sigma}}{2} dB_t^H.$$

Denoting $a = \tilde{a}/2$, $\sigma = \tilde{\sigma}/2$, we get

$$dY_t = aY_t dt + \sigma dB_t^H$$

with the initial condition $Y_0 = \sqrt{X_0}$.

So, in the case of $H > 2/3$, the solution $\{X_t, t \in [0, \tau_0)\}$ of the equation (6) is the square of the fractional Ornstein–Uhlenbeck process until it reaches zero.

Let $H \in (0, 1)$ be an arbitrary Hurst index, $\{Y_t, t \geq 0\}$ be a fractional Ornstein-Uhlenbeck process, i.e. the solution of the SDE

$$dY_t = aY_t dt + \sigma dB_t^H, \quad t \geq 0, \quad a \in \mathbb{R}, \sigma > 0,$$

and τ be the first moment of reaching zero by the latter.

Definition 1. The fractional Cox–Ingersoll–Ross process (with zero "mean" parameter) is the process $\{X_t, t \geq 0\}$ such that for all $t \geq 0$, $\omega \in \Omega$:

$$X_t(\omega) = Y_t^2(\omega) 1_{\{t < \tau(\omega)\}}.$$

Theorem 6. *Let τ be the first moment of zero hitting by the fractional Ornstein–Uhlenbeck process with parameters $a \in \mathbb{R}$ and $\sigma > 0$. Then, for $0 \leq t \leq \tau$, the corresponding fractional CIR process satisfies the following SDE:*

$$dX_t = 2aX_t dt + 2\sigma\sqrt{X_t} \circ dB_t^H,$$

where $X_0 = Y_0^2 > 0$ and the integral with respect to the fractional Brownian motion is defined as the pathwise Stratonovich integral.

The next natural question regarding the fractional CIR process is finiteness of its zero hitting time moment. It is obvious that it coincides with the respective moment of the corresponding fractional Ornstein–Uhlenbeck process $\{Y_t, t \geq 0\}$.

Let $\{Y_t, t \geq 0\}$ be a fractional Ornstein–Uhlenbeck process, i.e. the solution of the SDE

$$dY_t = aY_t dt + \sigma dB_t^H, \quad t \geq 0, \quad a \in \mathbb{R}, \sigma > 0,$$

and τ be the first moment of reaching zero by the latter.

Y can be written explicitly as

$$Y_t = e^{at}\left(Y_0 + \sigma \int_0^t e^{-as} dB_s^H\right),$$

where the integral with respect to fractional Brownian motion is the limit of Riemann-Stieltjes sums and can be defined by integration by parts:

$$\int_0^t e^{-as} dB_s^H = e^{-at} B_t^H + a \int_0^t e^{-as} B_s^H ds.$$

Proposition 1. *Let $t \geq s \geq 0$. Then covariance function $R_H(t,s)$ of the fractional Ornstein–Uhlenbeck process Y can be represented in the following form:*

$$R_H(t,s) = \frac{H\sigma^2}{2}\bigg(- e^{at-as}\int_0^{t-s} e^{-az}z^{2H-1}dz + e^{-at+as}\int_{t-s}^{t} e^{az}z^{2H-1}dz$$
$$-e^{at+as}\int_s^t e^{-az}z^{2H-1}dz + e^{at-as}\int_0^s e^{az}z^{2H-1}dz + 2e^{at+as}\int_0^t e^{-az}z^{2H-1}dz\bigg).$$

Let τ be the first moment of zero hitting by the fractional Ornstein-Uhlenbeck process (and consequently by the corresponding fractional CIR process with zero "mean" parameter).

Theorem 7. *(1) If $a \leq 0$, then $\mathbb{P}(\tau < \infty) = 1$.*

(2) If $a > 0$, then $\mathbb{P}(\tau < \infty) \in (0,1)$, and we have the upper bound

$$\mathbb{P}(\tau < \infty) \leq C_1 \left(\frac{Y_0}{\sigma}\right)^{\frac{1}{H}-2} \exp\left(-\frac{a^{2H}Y_0^2}{\sigma^2\Gamma(2H+1)}\right),$$

where $C_1 > 0$ is a constant.

3 Fractional CIR. Case $k > 0$

Consider the process $Y = \{Y_t, t \geq 0\}$ that satisfies the following SDE until its first zero hitting:

$$dY_t = \frac{1}{2}\left(\frac{k}{Y_t} - aY_t\right)dt + \frac{\sigma}{2}dB_t^H, \quad Y_0 > 0, \tag{7}$$

where $a, k \in \mathbb{R}$, $\sigma > 0$ and $\{B_t^H, t \geq 0\}$ is a fractional Brownian motion with the Hurst parameter $H \in (0,1)$.

Definition 2. Let $H \in (0,1)$ be an arbitrary Hurst index, $\{Y_t, t \geq 0\}$ be the process that satisfies the equation (7) and τ be the first moment of reaching zero by the latter.

The fractional Cox-Ingersoll-Ross process is the process $\{X_t, t \geq 0\}$ such that for all $t \geq 0, \omega \in \Omega$:

$$X_t(\omega) = Y_t^2(\omega)1_{\{t<\tau(\omega)\}}.$$

Similarly to the case $k = 0$, the definition of the fractional CIR process is natural as the following theorem holds:

Theorem 8. *Let τ be the first moment of hitting zero by Y. For $0 \leq t \leq \tau$ the fractional CIR process satisfies the following SDE:*

$$dX_t = (k - aX_t)dt + \sigma\sqrt{X_t} \circ dB_t^H,$$

where $X_0 = Y_0^2 > 0$ and the integral with respect to the fractional Brownian motion is defined as the pathwise Stratonovich integral.

Just as in the case $k = 0$, let us consider the question of finiteness of the zero hitting time moment by the fractional CIR process.

Theorem 9. *Let $k > 0, H > 1/2$. Then the process $\{Y_t, t \geq 0\}$, defined by the equation* (7) *(and consequently the corresponding fractional CIR process), is strictly positive a.s.*

Let $\{B_t^H, t \geq 0\}$ be the fractional Brownian motion with $H < 1/2$ and let $a \in \mathbb{R}$, $\sigma > 0$ be fixed. Consider the set of processes

$$\mathbb{Y} := \{Y^{(k)} = \{Y_t^{(k)}, t \geq 0\}, k > 0\},$$

such that

$$Y_t^{(k)}(\omega) = \begin{cases} Y_0 + \frac{1}{2}\int_0^t \left(\frac{k}{Y_s^{(k)}(\omega)} - aY_s^{(k)}(\omega)\right) ds + \frac{\sigma}{2} dB_t^H(\omega), & \text{if } t < \tau^{(k)}(\omega) \\ 0, & \text{if } t \geq \tau^{(k)}(\omega) \end{cases},$$

where $\tau^{(k)} := \inf\{t \geq 0 | Y_t^{(k)} = 0\}$.

Theorem 10. *For all $T > 0$, $\mathbb{P}(\tau^{(k)} > T) \to 1$, $k \to \infty$.*

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