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МИНИМАЛЬНЫЕ ВЕЛИЧИНЫ И КОНЦЕПЦИЯ ИЗМЕРИМОСТИ В КВАНТОВОЙ ТЕОРИИ, ГРАВИТАЦИИ И ТЕРМОДИНАМИКЕ

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В настоящее время подавляющее большинство исследователей согласны с тем, что минимальная длина должна появляться в высоких (планковских) энергиях. Однако современные низкоэнергетические теории при далеких от планковских энергиях (квантовая механика, квантовая теория поля, гравитация и т. д.) являются непрерывными, т. е. минимальная длина в них нулевая. В настоящей статье представлен альтернативный подход, в котором гипотетическая минимальная длина ненулевая на всех масштабах энергий. Формулируются понятия измеримости и измеримых величин, в рамках которых отсутствуют абстрактные бесконечно малые приращения пространственно-временных координат. В результате известные низкоэнергетические теории (квантовая теория или гравитация) неизбежно заменяются дискретными теориями, очень близкими к первоначальным, но имеющими совершенно другой математический аппарат. В этом случае реальная дискретность проявляется только в высоких энергиях, близких к планковским. Аналогичное (дуальное) понятие измеримости определяется в термодинамике на основе минимальной обратной температуры. Отмечено, что с помощью введенных формулировок можно получить некоторые следствия для гравитационной термодинамики черных дыр на всех масштабах энергий. Кроме этого, построен измеримый вариант общей теории относительности и показано, что он представляет ее деформацию. В общем виде продемонстрировано, что все основные ингредиенты общей теории относительности имеют измеримые аналоги.

Ключевые слова: измеримость; квантовая теория; гравитация; термодинамика.

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MINIMAL QUANTITIES AND MEASURABILITY CONCEPTION IN QUANTUM THEORY, GRAVITY AND THERMODYNAMICS

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At the present time the majority of researchers agree that a minimal length is involved at high (Planck's) energies. But all the currently used low-energy theories (quantum mechanics and quantum field theory, gravity, etc.) are continuous, i. e. the minimal length in them is zero. This article presents an alternative approach when the hypothetical minimal length is nonzero at all the energy scales. By this approach the definition of measurability and of measurable quantities is given, within the scope of which there is no abstract infinitesimal increment of space-time coordinates. As a result, the initial low-energy continuous theory (quantum theory or general relativity) inevitably must be replaced by a discrete theory that gives very close results but operates with absolutely other mathematical apparatus. A real discreteness is exhibited only at high energies which are close to the Planck energies. A analogous concept (dual) of measurability is defined in thermodynamics on the basis of a hypothetical minimal inverse temperature. Based on this notions, some implications are obtained, in particular, for gravitational thermodynamics of black holes at all the energy scales, quantum corrections of the basic quantities in the general case. Besides, the measurable variant of General Relativity (GR) is constructed and it is shown that this variant represents its deformation. In the general form it is demonstrated that all the basic ingredients of GR have their measurable analogs.

Key words: measurability; quantum theory; gravity; thermodynamics.

Introduction

The mathematical apparatus of the present-day fundamental physical theories (Quantum Theory, Special and General Relativity, etc.) is based on the initial assumption that variations of a physical system *are independent of the existing energies*. Specifically, in the above-mentioned theories the principal mathematical instruments are the infinitesimal variations (increments) dt, dx_i, dp_i, dE, $i = 1, 2, 3$. The apparatus based on the use of these variations comes from mathematical analysis and is completely adequate for classical mechanics, where continuous space-time forms the base. But in this approach, due to the introduction of ultraviolet and infrared divergences into a Quantum Theory (QT) [1] and also due to the absence of correct passage to the high-energy (ultraviolet) region in Gravity (GR) [2], we are facing very serious problems.

The present manuscript is based on the author's works [3–8]. The main target of this papers is to construct a correct quantum theory and gravity in terms of the variations (increments) *dependent on the existent energies*, i. e., the theory should not involve above infinitesimal variations (increments). By the author's opinion, these problems are solvable but beyond the paradigm of continuous space-time.

To solve these problems, in the above-mentioned works, using the minimal length and minimal time, the author investigates a discrete space-time model, for which at low energies (far from the Planck energies) the results are to a high accuracy close to those obtained with a continuous space-time model. And at high (Planck's) energies the indicated model is fundamentally discrete, leading to principally new results. All variations in any physical system considered in such a discrete model should be dependent on the existent energies.

The primary instrument for such a discrete model is the measurability notion introduced in [4] and more precise in [7; 8].

In [5; 6] is demonstrated that a similar measurability notion (in essence dual) may be also introduced in thermodynamics on the basis of a minimal inverse temperature, leading to very interesting inferences for thermodynamics of black holes at all the energy scales.

Necessary preliminary information

It is assumed that there is a minimal (universal) unit for measurement of the length ℓ corresponding to some maximal energy $E_\ell = \frac{\hbar c}{\ell}$ and a universal unit for measurement of time τ: $\tau = \frac{\ell}{c}$. Without loss of generality, we can consider ℓ and τ at Plank's level, i. e. $\ell = \kappa l_p$, $\tau = \kappa t_p$, where the numerical constant κ is on the order of 1, i. e. $E_\ell \propto E_p$ with the corresponding proportionality factor.

Then we consider a set of all nonzero momenta

$$P = \left\{ p_{x_i} \right\},\ i = 1, 2, 3;\ \left| p_{x_i} \right| \neq 0$$

and subset (Primarily Measurable Momenta, or PMM)

$$p_{x_i} \equiv p_{N_i} = \frac{\hbar}{N_i \ell}, \tag{1}$$

where N_i is an integer number and p_{x_i} is the momentum corresponding to the coordinate x_i.

From these formula it is not unreasonable to propose the following definition.

Definition 1. Primary Measurability. 1.1. Any variation in Δx_i for the coordinates x_i and Δt of the time t is considered primarily measurable if

$$\Delta x_i = N_{\Delta x_i} \ell,\ \Delta t = N_{\Delta t} \tau, \tag{2}$$

where $N_{\Delta x_i} \neq 0$ and $N_{\Delta t} \neq 0$ are integer numbers.

1.2. Let us define any physical quantity as primary or elementary measurable when its value is consistent with formulae (1) and (2).

Then we consider formula (2) and definition 1 with the addition of the momenta $p_{x_0} \equiv p_{N_0} = \dfrac{\hbar}{N_0 \ell}$, where N_0 is an integer number corresponding to the time coordinate ($N_{\Delta t}$ in formula (2)).

For convenience, we denote Primarily Measurable Quantities satisfying definition 1 in the abbreviated form as PMQ.

It is clear that PMQ is inadequate for studies of the physical processes. To illustrate, the space-time quantities

$$\begin{aligned} \frac{\tau}{N_t} &= p_{N_t c} \frac{\ell^2}{c\hbar}, \\ \frac{\ell}{N_i} &= p_{N_i} \frac{\ell^2}{\hbar},\ i = 1, 2, 3, \end{aligned} \tag{3}$$

where p_{N_i}, $p_{N_t c}$ are PMM, up to the fundamental constants are coincident with p_{N_i}, $p_{N_t c}$ and they may be involved at any stage of the calculations but, evidently, they are not PMQ in the general case.

Note: $\ell = \kappa l_p$; $l_p^2 = G\dfrac{\hbar}{c^3}$; $\dfrac{\ell^2}{\hbar} = \dfrac{\kappa^2 G}{c^3}$. Thus, it is reasonable to use definition 2.

Definition 2. Generalized Measurability. We define any physical quantity at all energy scales as generalized measurable or, for simplicity, measurable if any of its values may be obtained in terms of PMQ specified by points 1.1, 1.2 of definition 1.

The main target of the author is to form a quantum theory and gravity only in terms of measurable quantities (or of PMQ).

Now we consider separately the two cases.

A) Low Energies, $E \ll E_p$.

Domain $P_{LE} \subset P$ (LE is abbreviation of «Low Energies») defined by the conditions

$$P_{LE} = \left\{ p_{x_i} \right\},\ i = 1, 2, 3;\ P_\ell \gg \left| p_{x_i} \right| \neq 0,$$

where $P_\ell = \dfrac{E_\ell}{c}$ – maximal momentum.

In this case Primarily Measurable Momenta takes the form

$$N_i = \frac{\hbar}{p_{x_i} \ell},$$

or

$$p_{x_i} \equiv p_{N_i} = \frac{\hbar}{N_i \ell},$$

$$\left| N_i \right| \gg 1.$$

As the energies $E \ll E_\ell$ are low, i. e. ($\left| N_i \right| \gg 1$), primary measurable momenta are sufficient to specify the whole domain of the momenta to a high accuracy P_{LE}. Of course, all the calculations of point A) also comply with the primary measurable momenta $p_{N_t c} \equiv p_{N_0}$. Because of this, in what follows we understand P_{LE} as a set of the primary measurable momenta $p_{x_\mu} = p_{N_\mu}$, $(\mu = 0, \ldots, 3)$ with $\left| N_\mu \right| \gg 1$.

Remark 1. It should be noted that, as all the experimentally involved energies E are low, they meet the condition $E \ll E_\ell$, specifically for LHC the maximal energies are ≈10 TeV = 10^4 GeV, that is by 15 orders of magnitude lower than the Planck energy ≈10^{19} GeV. But since the energy E_ℓ is on the order of the Planck energy $E_\ell \propto E_p$, in this case all the numbers N_i for the corresponding momenta will meet the condition $\min|N_i| \approx 10^{15}$. So, all the experimentally involved momenta are considered to be primary measurable momenta, i. e. P_{LE} at low energies $E \ll E_\ell$.

In this way in the proposed paradigm at low energies $E \ll E_p$ any momentum with p_{x_μ}, $\mu = 0, \ldots, 3$, takes the form $p_{x_\mu} = p_{N_\mu}$, where N_μ – integer with the property $|N_\mu| \gg 1$.

Further for the fixed point x_μ we use the notion $p_{x_\mu} = p_{N_{x_\mu}}$ or $p_{x_\mu} = p_{N_{\Delta x_\mu}}$.

Naturally, the small variation Δp_{x_μ} at the point $p_{x_\mu} = p_{N_{x_\mu}}$ of the momentum space P_{LE} is represented by the primary measurable momentum $p_{N'_{x_\mu}}$ with the property $|N'_{x_\mu}| > |N_{x_\mu}|$.

The problem is as follows: is any possibility that Δp_{x_μ} is infinitesimal? For the special point $p_{x_\mu} = p_{N_{x_\mu}}$ the answer is negative.

Indeed, the «nearest» points to $p_{N_{x_\mu}}$ are $p_{N_{x_\mu}-1}$ and $p_{N_{x_\mu}+1}$.

It is obvious that

$$\left|p_{N_{x_\mu}} - p_{N_{x_\mu}-1}\right| = \left|p_{N_{x_\mu}\left(N_{x_\mu}-1\right)}\right|,$$

$$\left|p_{N_{x_\mu}} - p_{N_{x_\mu}+1}\right| = \left|p_{N_{x_\mu}\left(N_{x_\mu}+1\right)}\right|.$$

It is easily seen that the difference $\left|p_{N_{x_\mu}\left(N_{x_\mu}+1\right)}\right| - \left|p_{N_{x_\mu}\left(N_{x_\mu}-1\right)}\right|$ for $|N_{x_\mu}| \gg 1$ is infinitesimal, i. e., to within a high accuracy, we have $\left|p_{N_{x_\mu}\left(N_{x_\mu}+1\right)}\right| = \left|p_{N_{x_\mu}\left(N_{x_\mu}-1\right)}\right|$. And a small variation of $|\Delta p_{x_\mu}|$ at the point $p_{x_\mu} = p_{N_{x_\mu}}$ has a minimum that equals $\left|p_{N_{x_\mu}\left(N_{x_\mu}+1\right)}\right|$. Clearly, with an increase in $|N_{x_\mu}|$, we can obtain no matter how small $\left|p_{N_{x_\mu}\left(N_{x_\mu}+1\right)}\right|$.

So, in the proposed paradigm at low energies $E \ll E_p$ a set of the primarily measurable P_{LE} is discrete, and in every measurement of $\mu = 0, \ldots, 3$ there is the discrete subset $P_{x_\mu} \subset P_{LE}$:

$$P_{x_\mu} \doteq \left\{\ldots, p_{N_{x_\mu}-1}, p_{N_{x_\mu}}, p_{N_{x_\mu}+1}, \ldots\right\}.$$

In this case, as compared to the canonical quantum theory, in continuous space-time we have the following substitution:

$$dp_\mu \mapsto \Delta p_{N_{x_\mu}} = p_{N_{x_\mu}} - p_{N_{x_\mu}+1} = p_{N_{x_\mu}\left(N_{x_\mu}+1\right)},$$

$$\frac{\partial}{\partial p_\mu} \mapsto \frac{\Delta}{\Delta p_\mu}, \; \frac{\partial F}{\partial p_\mu} \mapsto \frac{\Delta F\left(p_{N_{x_\mu}}\right)}{\Delta p_\mu} = \frac{F\left(p_{N_{x_\mu}}\right) - F\left(p_{N_{x_\mu}+1}\right)}{p_{N_{x_\mu}} - p_{N_{x_\mu}+1}} = \frac{F\left(p_{N_{x_\mu}}\right) - F\left(p_{N_{x_\mu}+1}\right)}{p_{N_{x_\mu}\left(N_{x_\mu}+1\right)}}. \quad (4)$$

It is clear that for sufficiently high integer values of $|N_{x_\mu}|$, formula (4) reproduces a continuous paradigm in the momentum space to any preassigned accuracy.

Similarly for sufficiently high integer values of $|N_t|$ and $N_i, \ldots, N_{x_i}$, the quantities $\frac{\tau}{N_t}, \frac{\ell}{N_{x_i}}$ may be arbitrary small.

Hence, for sufficiently high integer values of $|N_t|$ and $|N_{x_i}|$, the quantities $\frac{\tau}{N_t}, \frac{\ell}{N_{x_i}}$ are nothing but a measurable analog the infinitesimal quantities dx_i, dt, i. e. dx_μ, $\mu = 0, \ldots, 3$.

Thus, for sufficiently high integer values of $\left|N_{x_\mu}\right|$, $\mu = 0, \ldots, 3$, $\frac{\tau}{N_t}$, $\frac{\ell}{N_{x_i}}$ (which are the primarily measurable momenta P_{x_μ} up to fundamental constant) represent a measurable analog of small (and infinitesimal) space-time increments in the space-time variety $\mathcal{M} \subset \mathbb{R}^4$.

Because of this, for sufficiently high integer values of $\left|N_{x_\mu}\right|$, we have the following correspondence

$$dx_\mu \mapsto \frac{\ell}{N_{x_\mu}},$$

$$\frac{\partial}{\partial x_\mu} \mapsto \frac{\Delta}{\Delta_{N_{x_\mu}}},\ \frac{\partial F}{\partial x_\mu} \mapsto \frac{\Delta F\left(x_\mu\right)}{\Delta_{N_{x_\mu}}} = \frac{F\left(x_\mu + \frac{\ell}{N_{x_\mu}}\right) - F\left(x_\mu\right)}{\ell/N_{x_\mu}}.$$

Now we formulate the principle of **correspondence to continuous theory (CCT).** At low energies $E \ll E_p$ (or same $E \ll E_\ell$) the infinitesimal space-time quantities dx_μ, $\mu = 0, \ldots, 3$, and also infinitesimal values of the momenta dp_i, $i = 1, 2, 3$, and of the energies dE form the basic instruments («construction materials») for any theory in continuous space-time. Because of this, to construct the measurable variant of such a theory, we should find the adequate substitutes for these quantities.

It is obvious that in the first case the substitute is represented by the quantities $\frac{\ell}{N_{x_\mu}}$, where $\left|N_{x_\mu}\right|$ – arbitrary large (but finite!) integer, whereas in the second case by $p_{N_{x_i}} = \frac{\hbar}{N_{x_i}\ell}$, $i = 1, 2, 3$; $\varepsilon_{N_{x_0}} = \frac{c\hbar}{N_{x_0}\ell}$, where N_{x_μ} – integer with the above properties $\mu = 0, \ldots, 3$.

In this way in the proposed approach all the primary measurable momenta $p_{N_{x_\mu}}$, $\left|N_{x_\mu}\right| \gg 1$ are small quantities at low energies $E \ll E_\ell$ and primary measurable momenta $p_{N_{x_\mu}}$ with sufficiently large $\left|N_{x_\mu}\right| \gg 1$ being analogous to infinitesimal quantities of a continuous theory.

As, according to Remark 1, all the momenta at low energies $E \ll E_p$, to a high accuracy, may be considered to be the primary measurable momenta, we derive that at low energies the primary measurable momenta $p_{N_{x_\mu}}$ generate measurable small space-time variations and at sufficiently high $\left|N_{x_\mu}\right|$ – infinitesimal variations.

B) High Energies, $E \approx E_p$.

In this case primary measurable momenta are

$$N_i = \frac{\hbar}{p_{x_i}\ell},$$

or

$$p_{x_i} \equiv p_{N_i} = \frac{\hbar}{N_i\ell},$$

$$\left|N_i\right| \approx 1,$$

where N_i is an integer number and p_{x_i} is the momentum corresponding to the coordinate x_i. Evidently, that primary measurable momenta in this case are the discrete set.

The main difference of the case B) from the case A) is in the fact that at High Energies the primary measurable momenta are inadequate for theoretical studies at the energy scales $E \approx E_p$.

This is easily seen when we consider, e. g., the Generalized Uncertainty Principle (GUP), that is an extension of Heisenberg's Uncertainty Principle (HUP), to (Planck) high energies [9–11]

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_p^2 \frac{\Delta p}{\hbar}, \tag{5}$$

where α' is a constant on the order of 1.

Obviously, (5) leads to the minimal length ℓ on the order of the Planck length l_p

$$\Delta x_{\min} = 2\sqrt{\alpha'} l_p \equiv \ell.$$

In his earlier works the author, using simple calculations, has demonstrated that for the equality in (5) at high energies $E \approx E_p$ ($E \approx E_\ell$), the primary measurable space quantity $\Delta x = N_{\Delta x}\ell$, where $N_{\Delta x} \approx 1$ is an integer number, results in the momentum $p\left(N_{\Delta x}, GUP\right)$ [5; 6]

$$\Delta p \equiv p\left(N_{\Delta x}, GUP\right) = \frac{\hbar}{1/2\left(N_{\Delta x} + \sqrt{N_{\Delta x}^2 - 1}\right)\ell}.$$

It is clear that for $N_{\Delta x} \approx 1$ the momentum $\Delta p\left(N_{\Delta x}, GUP\right)$ is not a primary measurable momentum.

On the contrary, at low energies $E \ll E_p$ ($E \ll E_\ell$), the primary measurable space quantity $\Delta x = N_{\Delta x}\ell$, where $N_{\Delta x} \gg 1$ is an integer number, due to the validity of the limit

$$\lim_{N_{\Delta x} \to \infty} \sqrt{N_{\Delta x}^2 - 1} = N_{\Delta x},$$

leads to the momentum $\Delta p\left(N_{\Delta x}, GUP\right)$:

$$\Delta p \equiv \Delta p\left(N_{\Delta x}, HUP\right) = \frac{\hbar}{1/2\left(N_{\Delta x} + \sqrt{N_{\Delta x}^2 - 1}\right)\ell} \approx \frac{\hbar}{N_{\Delta x}\ell} = \frac{\hbar}{\Delta x}.$$

It is inferred that, for sufficiently high integer values of $N_{\Delta x}$ the momentum $\Delta p\left(N_{\Delta x}, HUP\right)$ within any high accuracy may be considered to be the primary measurable momentum. This example illustrates that primary measurable momenta are insufficient for studies in the high-energy domain $E \approx E_p$ and we should use the generalized measurable momenta.

As noted above, the main target of the author is to construct a quantum theory at all energy scales in terms of measurable quantities.

Remark 2. As long as ℓ is a minimal measurable length and τ is a minimal measurable time, values of all observable quantities should agree with this condition, i. e. their expressions should not involve the lengths $l < \ell$ and the times $t < \tau$. Because of this, values of the length $\frac{\ell}{N_i}$ and of the time $\frac{\tau}{N_i}$ from formula (3) couldnot appear in expressions for observable quantities being involved only in intermediate calculations, especially at thesummation for replacement of the infinitesimal quantities dx_i, dt on passage from a continuous theory to itsmeasurable variant.

We can assume that at low energies $E \ll E_p$ all the observable quantities are PMQ.

Space-time metrics in measurable format

according to the above-mentioned results, the measurable variant of gravity should be formulated in terms of the small measurable space-time quantities $\frac{\ell}{N_{\Delta x_\mu}}$ or same primary measurable momenta $p_{N_{\Delta x_\mu}}$.

Let us consider the case of the random metric $g_{\mu\nu} = g_{\mu\nu}(x)$ where $x \in \mathbb{R}^4$ is some point of the four-dimensional space $\mathbb{R}^4$ defined in measurable terms. The phrase «some point of the four-dimensional space $\mathbb{R}^4$ defined in measurable terms» means that all variations at the indicated point are determined in terms of measurable quantities. Specifically, as mentioned above, all small measurable variations take the from $\frac{\ell}{N_{\Delta x_\mu}} \propto p_{N_{\Delta x_\mu}}$, where $p_{N_{\Delta x_\mu}}$ are primary measurable momenta and $\left|N_{\Delta x_\mu}\right| \gg 1$.

Now, any such point $x \in \left\{x^\chi\right\} \in \mathbb{R}^4$ and any set of integer numbers $\left\{N_{\Delta x^\chi}\right\}$ dependent on the point $\left\{x^\chi\right\}$ with the property $\left|N_{\Delta x^\chi}\right| \gg 1$ may be correlated to the bundle with the base $\mathbb{R}^4$ as follows:

$$B_{N_{x^\chi}} \doteq \left\{x^\chi, \frac{\ell}{N_{\Delta x^\chi}}\right\} \mapsto \left\{x^\chi\right\}. \tag{6}$$

It is clear that $\lim_{\left|N_{\Delta x^\chi}\right| \to \infty} B_{N_{\Delta x^\chi}} = \mathbb{R}^4$.

Then as a canonically measurable prototype of the infinitesimal space-time interval square [2]

$$ds^2(x) = g_{\mu\nu}(x)dx^\mu dx^\nu$$

we take the expression

$$\Delta s^2_{\{N_{\Delta x^\chi}\}}(x) = g_{\mu\nu}(x, N_{\Delta x^\chi})\frac{\ell^2}{N_{\Delta x^\mu} N_{\Delta x^\nu}}. \quad (7)$$

Here $g_{\mu\nu}(x, N_{\Delta x^\chi})$ – metric with the property that minimal measurable variation of metric $g_{\mu\nu}(x, N_{\Delta x^\chi})$ in point (7) for coordinate χ^{th} has form

$$\Delta g_{\mu\nu}(x, N_{\Delta x^\chi})_\chi = g_{\mu\nu}\left(x + \frac{\ell}{N_{\Delta x^\chi}},\ N_{\Delta x^\chi}\right) - g_{\mu\nu}(x, N_{\Delta x^\chi}).$$

Let us denote by $\Delta_\chi g_{\mu\nu}(x, N_{\Delta x^\chi})$ quantity

$$\Delta_\chi g_{\mu\nu}(x, N_{\Delta x^\chi}) = \frac{\Delta g_{\mu\nu}(x, N_{\Delta x^\chi})_\chi}{\ell/N_{\Delta x^\chi}}.$$

It is obvious that in the case under study the quantity $\Delta g_{\mu\nu}(x, N_{\Delta x^\chi})_\chi$ is a measurable analog for the infinitesimal increment $dg_{\mu\nu}(x)$ of the χ^{th} component $(dg_{\mu\nu}(x))_\chi$ in a continuous theory, whereas the quantity $\Delta_\chi g_{\mu\nu}(x, N_{\Delta x^\chi})$ is a measurable analog of the partial derivative $\partial_\chi g_{\mu\nu}(x)$.

In this manner we obtain the formula (6) induced bundle over the metric manifold $g_{\mu\nu}(x)$:

$$B_{g,\, N_{\Delta x^\chi}} \doteq g_{\mu\nu}(x, N_{\Delta x^\chi}) \mapsto g_{\mu\nu}(x).$$

The formula (7) may be written in terms of the primary measurable momenta $\left(p_{N_{\Delta x^i}},\ p_{N_{\Delta x^0}}\right) = p_{N_{\Delta x^\chi}}$ as follows:

$$\Delta s^2_{N_{\Delta x^\chi}}(x) = \frac{\ell^4}{\hbar^2} g_{\mu\nu}(x, N_{\Delta x^\chi}) p_{N_{\Delta x^\mu}} p_{N_{\Delta x^\nu}}.$$

Considering that $\ell \infty l_P$ (i. e. $\ell = \kappa l_P$), where $\kappa = \text{const}$ is on the order of 1, in the general case to within the constant $\frac{\ell^4}{\hbar^2}$ we have

$$\Delta s^2_{N_{\Delta x^\chi}}(x) = g_{\mu\nu}(x, N_{\Delta x^\chi}) p_{N_{\Delta x^\mu}} p_{N_{\Delta x^\nu}}.$$

As follows from the previous formulae, the measurable variant of General Relativity should be defined in the bundle $B_{g,\, N_{\Delta x^\chi}}$.

Measurable form of Einstein equations at low energies and transition to high energies

Thus, we have measurable (discrete) analogs infinitesimal variations and partial derivative:

$$dg_{\mu\nu}(x) \to \Delta g_{\mu\nu}(x, N_{\Delta x^\chi})_\chi,\ \partial_\chi g_{\mu\nu}(x) \to \Delta_\chi g_{\mu\nu}(x, N_{\Delta x^\chi}).$$

In particular, the Christoffel symbols

$$\Gamma^\alpha_{\mu\nu}(x) = \frac{1}{2} g^{\alpha\beta}(x)\left(\partial_\nu g_{\beta\mu}(x) + \partial_\mu g_{\nu\beta}(x) - \partial_\beta g_{\mu\nu}(x)\right)$$

have the measurable analog

$$\Gamma^\alpha_{\mu\nu}(x, N_{x_\chi}) = \frac{1}{2} g^{\alpha\beta}(x, N_{x_\chi})\left(\Delta_\nu g_{\beta\mu}(x, N_{x_\chi}) + \Delta_\mu g_{\nu\beta}(x, N_{x_\chi}) - \Delta_\beta g_{\mu\nu}(x, N_{x_\chi})\right). \quad (8)$$

Similarly, for the Riemann tensor in a continuous theory we have:

$$R^\mu_{\nu\alpha\beta}(x) \equiv \partial_\alpha \Gamma^\mu_{\nu\beta}(x) - \partial_\beta \Gamma^\mu_{\nu\alpha}(x) + \Gamma^\mu_{\gamma\alpha}(x)\Gamma^\gamma_{\nu\beta}(x) - \Gamma^\mu_{\gamma\beta}(x)\Gamma^\gamma_{\nu\alpha}(x).$$

With the use of formula (8), we can get the corresponding measurable analog, i. e. the quantity $R^{\mu}_{\nu\alpha\beta}\left(x, N_{x_\chi}\right)$.

In a similar way we can obtain the measurable variant of Ricci tensor, $R_{\mu\nu}\left(x, N_{x_\chi}\right) \equiv R^{\alpha}_{\mu\alpha\nu}\left(x, N_{x_\chi}\right)$, and the measurable variant of Ricci scalar:

$$R\left(x, N_{x_\chi}\right) \equiv R_{\mu\nu}\left(x, N_{x_\chi}\right) g^{\mu\nu}\left(x, N_{x_\chi}\right).$$

So, for the Einstein Equations (EE) in a continuous theory

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{2} \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{9}$$

we can derive their measurable analog, for short denoted as Einstein Equations Measurable (EEM):

$$R_{\mu\nu}\left(x, N_{x_\chi}\right) - \frac{1}{2} R\left(x, N_{x_\chi}\right) g_{\mu\nu}\left(x, N_{x_\chi}\right) - \frac{1}{2} \Lambda\left(x, N_{x_\chi}\right) g_{\mu\nu}\left(x, N_{x_\chi}\right) = 8\pi G T_{\mu\nu}\left(x, N_{x_\chi}\right), \tag{10}$$

where G – Newton's gravitational constant.

For correspondence with a continuous theory, the following passage to the limit must take place for all the points x:

$$\lim_{\left|N_{x_\chi}\right| \to \infty} \Lambda\left(x, N_{x_\chi}\right) = \Lambda,$$

where the cosmological constant Λ is taken from formula (9).

Moreover, for high $\left|N_{x_\chi}\right|$, the quantity $\Lambda\left(x, N_{x_\chi}\right)$ should be practically independent of the point x, and we have

$$\Lambda\left(x, N_{x_\chi}\right) \approx \Lambda\left(x', N'_{x'_\chi}\right) \approx \Lambda, \tag{11}$$

where $x \neq x'$ and $\left|N_{x_\chi}\right| \gg 1$, $\left|N'_{x'_\chi}\right| \gg 1$.

Actually, it is clear that formula (11) reflects the fact that (EEM) given by formula represents deformation of the Einstein equations (EE) in the sense of the definition given Ludwig Faddeev in 1989 [12] with the deformation parameter N_{x_χ}, and we have

$$\lim_{\left|N_{x_\chi}\right| \to \infty} (EEM) = (EE).$$

We denote this deformation as $(EEM)\left[N_{x_\chi}\right]$. Since at low energies $E \ll E_P$ and to within the known constants we have $\frac{\ell}{N_{x_\chi}} = p_{N_{x_\chi}}$, the following deformations of (EU) are equivalent to

$$(EEM)\left[N_{x_\chi}\right] \equiv (EEM)\left[p_{N_{x_\chi}}\right].$$

So, on passage from (EE) to the measurable deformation $(EEM)\left[N_{x_\chi}\right]$ (or identically $(EEM)\left[p_{N_{x_\chi}}\right]$) we can find solutions for the gravitational equations on the metric bundle $B_{g, N_{x_\chi}} \doteq \left[g_{\mu\nu}\left(x, N_{\Delta x^\chi}\right)\right]$.

However, minimal measurable increments for the energies $E \approx E_p$ are not of the form $\frac{\ell}{N_{x_\mu}}$ because the corresponding momenta $\left\{p_{N_{x_\chi}}\right\}$ are no longer primary measurable, as indicated by the results in «Necessary Preliminary Information».

So, in the proposed paradigm the problem of the ultraviolet generalization of the low-energy measurable gravity $(EEM)\left[N_{x_\chi}\right]$ is actually reduced to the problem: what becomes with the primary measurable momenta $\left\{p_{N_{x_\chi}}\right\}$, $\left|N_{x_\chi}\right| \gg 1$ at high (Planck's) energies. In a relatively simple case of GUP in «Necessary Preliminary Information» we have the answer.

In more general case [12]

$$\Delta x \, \Delta p \geq \left(\hbar 1 + \beta (\Delta p)^2\right), \tag{12}$$

$$\Delta x_0 = 2\hbar\sqrt{\beta} \doteq \ell,$$

when (12) is equality, $\Delta p = p_{N_{\Delta x^{\mu}}}$ – generalized measurable

$$l_H\left(p_{N_{\Delta x^{\mu}}}\right) \doteq \frac{\ell^2}{\hbar} p_{N_{\Delta x^{\mu}}}, \quad \left|p_{N_{\Delta x^{\mu}}}\right| \approx 1,$$

$$\Delta s^2_{N_{\Delta x^{\chi}}}(x, q) \doteq g_{\mu\nu}\left(x, N_{x_\chi}, q\right) l_H\left(p_{N_{\Delta x^{\mu}}}\right) l_H\left(p_{N_{\Delta x^{\nu}}}\right) \doteq$$
$$\doteq \frac{\ell^4}{\hbar^2} g_{\mu\nu}\left(x, N_{x_\chi}, q\right) p_{N_{\Delta x^{\mu}}} p_{N_{\Delta x^{\nu}}}; \ \left|N_{x_\chi}\right| \approx 1,$$

$$p_{N_{x_\chi}}, \left(\left|N_{x_\chi}\right| \approx 1\right) \overset{\left|N_{x_\chi}\right| \approx 1 \to \left|N_{x_\chi}\right| \gg 1}{\Rightarrow} p_{N_{x_\chi}}, \left(\left|N_{x_\chi}\right| \gg 1\right).$$

$$l_H\left(p_{N_{x_\chi}}\right), \left(\left|N_{x_\chi}\right| \approx 1\right) \overset{\left|N_{x_\chi}\right| \approx 1 \to \left|N_{x_\chi}\right| \gg 1}{\Rightarrow} \frac{\ell}{N_{x_\chi}}, \left(\left|N_{x_\chi}\right| \gg 1\right).$$

$$\Delta_q g_{\mu\nu}\left(x, N_{x_\chi}, q\right)_\chi \doteq g_{\mu\nu}\left(x + l_H\left(p_{N_{x_\chi}}\right), N_{x_\chi}, q\right) - g_{\mu\nu}\left(x, N_{x_\chi}, q\right),$$

$$\Delta_{\chi, q} g_{\mu\nu}\left(x, N_{x_\chi}, q\right) \doteq \frac{\Delta_q g_{\mu\nu}\left(x, N_{x_\chi}, q\right)_\chi}{l_H\left(p_{N_{x_\chi}}\right)}.$$

$$EEM[q] \doteq R_{\mu\nu}\left(x, N_{x_\chi}, q\right) - \frac{1}{2} R\left(x, N_{x_\chi}, q\right) g_{\mu\nu}\left(x, N_{x_\chi}, q\right) -$$
$$- \frac{1}{2}\Lambda\left(x, N_{x_\chi}, q\right) g_{\mu\nu}\left(x, N_{x_\chi}, q\right) = 8\pi G T_{\mu\nu}\left(x, N_{x_\chi}, q\right). \tag{13}$$

$$\lim_{E \ll E_{\max}} EEM[q] = EEM, \text{ or } \lim_{\left|N_{x_\chi}\right| \gg 1} EEM[q] = EEM.$$

Gravitational thermodynamics in measurable form and black holes

In the works [5; 6] it was shown that the concept of measurability (dual) can be introduced in thermodynamics based on the minimum inverse temperature. Then the following formula for minimal unit of the inverse temperature $\tilde{\tau}$:

$$\frac{1}{T} = N_{1/T}\tilde{\tau}, \ N_{1/T} > 0 \text{ is an integer number,}$$

is the analog of the primary measurability notion into thermodynamics. Similarly, generalized measurability in thermodynamics is introduced as in definition 2.

Now let us show the applicability this results to a quantum theory of black holes. Consider the case of Schwarzschild's black hole.

It should be noted that such spaces and even considerably more general cases have been thoroughly studied from the viewpoint of gravitational thermodynamics in works of professor T. Padmanabhan (for example in [13]).

The case of a static spherically-symmetric horizon in space-time is considered, the horizon being described by the metric

$$ds^2 = -f(r)c^2dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2.$$

The horizon location will be given by a simple zero of the function $f(r)$, at the radius $r = a$.

Then at the horizon $r = a$ Einstein's field equations

$$\frac{c^4}{G}\left[\frac{1}{2}f'(a)a-\frac{1}{2}\right]=4\pi Pa^2, \tag{14}$$

where $P=T_r^r$ is the trace of the momentum-energy tensor and radial pressure. Therewith, the condition $f(a)=0$ and $f'(a)\neq 0$ must be fulfilled.

On the other hand it is known that for horizon spaces one can introduce the temperature that can be identified with an analytic continuation to imaginary time. In the case under consideration

$$k_B T=\frac{\hbar c f'(a)}{4\pi}. \tag{15}$$

It is shown that in the initial (continuous) theory the Einstein equation for horizon spaces in the differential form may be written as a thermodynamic identity (the first principle of thermodynamics)

$$\underbrace{\frac{\hbar c f'(a)}{4\pi}}_{k_B T}\underbrace{\frac{c^3}{G\hbar}d\left(\frac{1}{4}4\pi a^2\right)}_{dS}-\underbrace{\frac{1}{2}\frac{c^4 da}{G}}_{-dE}=\underbrace{Pd\left(\frac{4\pi}{3}a^3\right)}_{PdV}, \tag{16}$$

where, as noted above, T – temperature of the horizon surface; S – corresponding entropy; E – internal energy; V – space volume.

It is impossible to use (16) in the formalism under consideration because, as follows from the given results da, dS, dE, dV are not measurable quantities.

First, we assume that at low energies $E\ll E_P$, a value of the radius r at the point a is a primarily measurable quantity in the sense of definition 1 i. e. $a=a_{\text{meas}}=N_a\ell$, where $N_a\gg 1$ – integer, and the temperature T from the left-hand side of (15) is the primarily measurable measurability temperature.

Then, in terms of measurable quantities, first we can rewrite (14) as

$$\frac{c^4}{G}\left[\frac{2\pi k_B T}{\hbar c}a_{\text{meas}}-\frac{1}{2}\right]=4\pi Pa_{\text{meas}}^2. \tag{17}$$

We express $a=a_{\text{meas}}=N_a\ell$ in terms of the deformation parameter $\alpha_a=\frac{1}{N_a^2}$ and the temperature T is expressed in terms of $T_{\max}\propto T_p=\frac{E_p}{k_B}$. Then equation (17) may be given as [6]

$$\frac{c^4}{G}\left[\frac{2\pi E_p}{N_{1/T}\hbar c}l_p\alpha_a^{1/2}-\frac{1}{2}\alpha_a\right]=\frac{c^4}{G}\left[\frac{2\pi}{N_{1/T}}\alpha_a^{1/2}-\frac{1}{2}\alpha_a\right]=4\pi P\ell^2. \tag{18}$$

However (18) are Einstein's equations in low energies at low energies $E\ll E_P$. In this equation, the following substitution occurs upon transition to high energies $E\approx E_P$:

$$N_a\to\frac{1}{2}\left(N_a+\sqrt{N_a^2-1}\right),\; N_{1/T}\to\left(N_{1/T}+\sqrt{N_{1/T}^2-1}\right), \tag{19}$$

where $N_a\approx 1$, $N_{1/T}\approx 1$ are integer numbers.

Then the equation (18) and the corresponding high-energy equation obtained from it by replacement (19) will be a special case of the general formulae (10), (13). Besides, in terms of measurable quantities and formulae some important implications for gravitational thermodynamics of black holes [14] at all the energy scales have been suggested [6; 7].

It should be noted that currently a lot of works on the theory with a minimum length are published (for example [10; 11; 15] and so on). The proposed approach differs from all the others in that we use space-time and momentum variations depending on the available energies instead of abstract quantities dx_μ, dp_i, dE.

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