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Factorization of Triangular Matrix-Functions of an Arbitrary Order

L. Primachuk[1] **and S. Rogosin**[1*]

(Submitted by A. M. Elizarov)

[1] *Belarusian State University, pr. Nezavisimosti 4, Minsk, 220030 Belarus*

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Abstract—An efficient method for factorization of square triangular matrix-functions of arbitrary order is proposed. It generalizes the method by G. N. Chebotarev.

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1. INTRODUCTION

Let Γ be a simple smooth closed curve on the complex plane $\mathbb{C}$ dividing $\mathbb{C}$ into two domains $D^+ \ni 0$ and $D^- \ni \infty$. By the factorization of a non-singular continuous complex-valued matrix-function $G \in (\mathcal{C}(\Gamma))^{n\times n}$ it is understood the determination of matrices $G^{\pm}$ analytic in $D^{\pm}$, respectively, together with their inverses $(G^{\pm})^{-1}$, and of the diagonal matrix $\Lambda(t) = \mathrm{diag}\ \{t^{\kappa_1}, \ldots, t^{\kappa_n}\}$, $\kappa_1, \ldots, \kappa_n \in \mathbb{Z}$, such that the following representation of the matrix G

$$G(t) = G^+(t)\Lambda(t)G^-(t), \quad t \in \Gamma, \tag{1}$$

holds on Γ. The representation (1) is called the *left (continuous or standard) factorization*. A similar representation $G(t) = G^-(t)\Lambda(t)G^+(t)$ is called the *right (continuous or standard) factorization*. If the left (right) factorization exists, then the integer numbers $\kappa_1, \ldots, \kappa_n$, which are called *partial indices*, are determined uniquely up to their order. In particular, there exists a constant nonsingular transformation of *factors* G^+, G^-, such that $\kappa_1 \geq \ldots \geq \kappa_n$. The factors G^+, G^- are not determined uniquely (in fact, they are found up to multiplication on special non-singular polynomial matrices, see [6]).

Factorization of matrix-functions was first considered in connection with the solution of the vector-matrix Riemann (or Riemann–Hilbert) boundary value problem (see, e.g., [3, 4]). Such problem is to determine vectors Φ^+, Φ^-, analytic in $D^+, D^- \setminus \{\infty\}$, respectively, satisfying the following linear relation:

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in \Gamma, \tag{2}$$

where $G(t), t \in \Gamma$, is a given non-singular matrix-function. The problem (2) was formulated by Riemann in his work on the construction of complex differential equations with a given monodromy group (see, e.g., [12]). In [2, 7], the solution to the Riemann vector-matrix boundary value problem (2) has been reduced to the solution of the equivalent system of singular integral equations. Note, that as opposed to the one-dimensional case (see [4]), in the vector-matrix situation it is not always possible to represent the solution to (2) in so called closed form. Analogously, it is not found a general constructive approach which allows to have a finite algorithm for determination of factors and partial indices in the factorization problem. Such approach has been proposed only for special types of matrices such as, e.g., for factorization of meromorphic matrix-functions in [1] or for factorization of triangle matrix-functions

[*] E-mail: rogosin@bsu.by

of the second order in [10]. A number of other constructive results has been described in the survey paper [8].

In this article, it is proposed an algorithm for solution of the factorization problem for triangle matrix-functions of arbitrary order generalizing Chebotarev's method [10]. The main idea of the approach in [10] is in a justification of the equivalence of the factorization problem to the construction of the matrix solution $X^{\pm}(z)$ of the problem (2), having a normal form at infinity (so called canonical matrix). This means that the sum of orders of its columns is equal to the order at infinity of the determinant of $X^{-}(z)$. Remind that the order of a column at a point is the minimum of the orders of its elements at this point, but the order of a function is equal to the order of its zero or minus order of its pole. If $X^{\pm}(z)$ is the canonical matrix, then the orders of columns of the matrix $X^{-}(z)$ at infinity coincide with the partial indices κ_j of G, and the factors can be determined as follows: $G^{+} = X^{+}$, $G^{-} = \Lambda^{(-1)}(X^{-})^{(-1)}$, where $\Lambda(z) = \mathrm{diag}\,\{z^{\kappa_1}, \ldots, z^{\kappa_n}\}$. For further convenience we briefly describe here Chebotarev's algorithm. Let

$$A(t) = \begin{pmatrix} \zeta_1(t) & 0 \\ a(t) & \zeta_2(t) \end{pmatrix}.$$

Denote $\kappa_j = ind_{\Gamma}\zeta_j(t)$ and let $x_j^{\pm}(z)$ be canonical functions for the homogeneous Riemann boundary value problems with coefficients $\zeta_j(t)$, respectively (see [4]), $j = 1, 2$. Then piece-wise analytic matrix

$$X^{\pm}(t) = \begin{pmatrix} x_1^{\pm}(t) & 0 \\ x_2^{\pm}(t)\phi^{\pm}(t) & x_2^{\pm}(t) \end{pmatrix}, \quad \phi^{\pm}(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{a(\tau)x_1^{-}(\tau)d\tau}{x_2^{+}(\tau)(\tau - z)}, \quad z \in D^{\pm},$$

satisfies the following boundary condition $X^{+}(t) = A(t)X^{-}(t)$. Let $\mu \geq 1$ be the order of the function $\phi^{-}(t)$ at infinity. The orders of non-zero elements of the matrix $X^{-}(z)$ at infinity can be characterized by the following table

$$\begin{pmatrix} \kappa_1 & - \\ \kappa_2 + \mu & \kappa_2 \end{pmatrix}.$$

If $\kappa_1 \leq \kappa_2 + \mu$, then the matrix $X^{-}(z)$ has the normal form at infinity, i.e. it is the canonical matrix. Thus, partial indices of the matrix $A(t)$ are equal (κ_1, κ_2).

In the case $\kappa_1 > \kappa_2 + \mu$ G. N. Chebotarev proposed the following method of construction of the canonical matrix.

Let the function $1/\phi^{-}(z)$ admits the following expansion in (generally speaking infinite) continued fraction

$$\frac{1}{\phi^{-}(z)} = q^{\gamma_0}(z) + \frac{1}{q^{\gamma_1}(z) + \frac{1}{q^{\gamma_2}(z)+\ldots}},$$

where $q^{\gamma_i}(z)$ are polynomials of orders γ_i, respectively ($\gamma_0 = \mu$). Denote $\mu_1 = \gamma_0 + \gamma_1$, $\mu_2 = \gamma_0 + \gamma_1 + \gamma_2$, ..., $\mu_n = \gamma_0 + \gamma_1 + \ldots + \gamma_n$, ...

Proposition 1 (G. N. Chebotarev). *If* $\mu, \mu + \mu_1, \mu_1 + \mu_2, \ldots, \mu_{i-1} + \mu_i < \kappa_1 - \kappa_2$, but $\mu_i + \mu_{i+1} \geq \kappa_1 - \kappa_2$, *then the partial indices of the matrix* $A(t)$ *are equal* $(\kappa_1 - \mu_i, \kappa_2 + \mu_i)$, *wherein the canonical matrix has the form*

$$\widetilde{X}^{+}(z) = \begin{pmatrix} x_1^{+}(z) & 0 \\ x_2^{+}(z)\phi^{+}(z) & x_2^{+}(z) \end{pmatrix} P(z), \quad \widetilde{X}^{-}(z) = \begin{pmatrix} x_1^{-}(z)q^{\mu_{i-1}}(z) & x_1^{-}(z)q^{\mu_i}(z) \\ x_2^{-}(z)r_{\mu_i}(z) & x_2^{-}(z)r_{\mu_{i+1}}(z) \end{pmatrix}.$$

Here $P(z)$ *is the polynomial matrix with the unit determinant, i.e.* $\det P(z) \equiv 1$, *and the functions* $r_{\mu_i}(z), r_{\mu_{i+1}}(z)$ *are analytic in* D^{-}. *They are constructed by using expansion of* $1/\phi^{-}(z)$ *in the continued fraction.*

The proof is based on the elementary transformations of the columns of the matrix $X^{-}(z)$ (see [10]).

2. AUXILIARY LEMMA

In order to avoid additional technical difficulties, we consider, in what follows, matrix-functions with Hölder continuous entries. Tough the proposed method could be realized for wider classes of matrix-functions. Chebotarev's method is extended here to the triangular matrix-functions of arbitrary order. An inductive consideration which allows to obtain such an extension is based on the following auxiliary statement.

Lemma 1. *Let Γ be a simple smooth closed contour, and $B(t), t \in \Gamma$ be a non-singular Hölder continuous square matrix-function of the order n having the following form:*

$$B(t) = \begin{pmatrix} A(t) & \mathbf{0} \\ b_1(t) \dots b_{n-1}(t) & c(t) \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3}$$

Suppose that the non-singular square matrix-function $A(t)$ of the order $n-1$ admits factorization $A(t) = A^+(t)\Lambda(t)A^-(t)$, where $\Lambda(t) = \text{diag } \{t^{\kappa_1}, \dots, t^{\kappa_{n-1}}\}$. Then the matrix-function $B(t)$ possesses factorization if the following matrix does:

$$\begin{pmatrix} \Lambda(t) & \mathbf{0} \\ (\mathbf{b}(t)|\mathbf{Y}_1^-(t)) \dots (\mathbf{b}(t)|\mathbf{Y}_{n-1}^-(t)) & c(t) \end{pmatrix}.$$

Here $\mathbf{b}(t) = (b_1(t), \dots, b_{n-1}(t))$ is the row of first $n-1$ entries of the lowest row of $B(t)$, $\mathbf{Y}_j^-(t) = (y_{1j}^-(t), \dots, y_{n-1,j}^-(t))^T$ is the j-th column of the matrix-function $Y^-(t) = (X^-(t))^{(-1)}$, and

$$(\mathbf{b}(t)|\mathbf{Y}_j^-(t)) = \sum_{k=1}^{n-1} b_k(t) y_{kj}^-(t).$$

$\triangleleft$ Really, in the above conditions the matrix-function $B(t)$ can be represented in the form

$$B(t) = \begin{pmatrix} X^+(t) & \mathbf{0} \\ 0 \dots 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda(t) & \mathbf{0} \\ (\mathbf{b}(t)|\mathbf{X}_1^-(t)) \dots (\mathbf{b}(t)|\mathbf{X}_{n-1}^-(t)) & c(t) \end{pmatrix} \begin{pmatrix} X^-(t) & \mathbf{0} \\ 0 \dots 0 & 1 \end{pmatrix}.$$

By this representation the conclusion of the lemma follows. $\triangleright$

Therefore, if a matrix-function of n-th order has the form (3) and satisfies the conditions of Lemma 1 (i.e. contains the block $A(t)$ of order $n-1$ admitting factorization), then its factorization is reduced to the factorization of triangular matrix of n-th order having the above described special form. This is a base for an inductive approach, in particular in the case when the matrix $A(t)$ is a triangular one.

3. SOLUTION TO THE FACTORIZATION PROBLEM FOR A TRIANGULAR MATRIX-FUNCTION OF THE THIRD ORDER

Let $A(t), t \in \Gamma$, be a non-singular triangular matrix-function of the 2-nd order with Hölder continuous entries. It is known [10] that such a matrix possesses factorization, i.e. admits the following representation $A(t) = A^+(t)\Lambda(t)A^-(t)$, where $\Lambda(t) = \text{diag}\{t^{\kappa_1}, t^{\kappa_2}\}$. Partial indices of $A(t)$ are equal κ_1, κ_2, and besides (see, e.g., [2]) $\kappa_1 + \kappa_2 = \text{ind}_\Gamma \det A(t)$. By Lemma 1 the factorization of the matrix-function

$$B_3(t) = \begin{pmatrix} A(t) & \mathbf{0} \\ b_1(t) b_2(t) & c(t) \end{pmatrix}$$

is reduced to the factorization of a non-singular 3-rd order triangular matrix function $D_3(t)$ of special kind. Let us write such a matrix in a slightly more general form and suppose that its angular element is equal $d_{33}(t) = c(t) \equiv 1$ (without loss of generality due non-singularity of $D_3(t)$):

$$D_3(t) = \begin{pmatrix} \zeta_1(t) & 0 & 0 \\ 0 & \zeta_2(t) & 0 \\ a_1(t) & a_2(t) & 1 \end{pmatrix}.$$

All entries of this matrix are Hölder continuous on Γ. Hence (see, e.g., [4]) the functions $\zeta_j(t), j = 1, 2$, can be factorized in the form

$$x_j^+(t) = \zeta_j(t) x_j^-(t). \tag{4}$$

Let the orders at the infinity of the functions $x_j^-(z)$, $j = 1, 2$, be equal $\kappa_j = \operatorname{ind} \zeta_j(t)$. Denote

$$\phi_j^\pm(z) := \frac{1}{2\pi i} \int\limits_\Gamma \frac{a_j(\tau) x_j^-(\tau)}{\tau - z} d\tau, \quad j = 1, 2.$$

Then the analytic in $D^\pm$ matrix-functions

$$X_3^\pm(z) = \begin{pmatrix} x_1^\pm(z) & 0 & 0 \\ 0 & x_2^\pm(z) & 0 \\ \phi_1^\pm(z) & \phi_2^\pm(z) & 1 \end{pmatrix}$$

satisfy the following boundary condition $X_3^+(t) = D_3(t) X_3^-(t)$, $t \in \Gamma$. Let $\gamma_1 \geq 1$, $\gamma_2 \geq 1$ be the orders of the functions $\phi_1^-(z), \phi_2^-(z)$ at infinity. Note that $\operatorname{ind} \det D_3(t) = \kappa_1 + \kappa_2$. In these notations the orders of columns of the matrix-function $X_3^-(z)$ at the infinity are equal, respectively: $(\min\{\kappa_1, \gamma_1\}, \min\{\kappa_2, \gamma_2\}, 0)$. If

$$\kappa_1 \leq \gamma_1, \quad \kappa_2 \leq \gamma_2, \tag{5}$$

then $X_3^-(z)$ has the normal form at the infinity and thus $X_3^\pm(z)$ is the canonical matrix. In this case the partial indices of $D_3(z)$ are equal $(\kappa_1, \kappa_2, 0)$. If at least one of inequalities (5) is violated, then $X_3^-(z)$ does not have the normal form at the infinity.

Let us consider the possible situations: 1) $\kappa_1 \leq \gamma_1$, $\kappa_2 > \gamma_2$; 2) $\kappa_1 > \gamma_1$, $\kappa_2 \leq \gamma_2$; 3) $\kappa_1 > \gamma_1$, $\kappa_2 > \gamma_2$. In the cases 1) and 2) one can apply directly Chebotarev's method [10], since in these cases it is necessary to carry out the elementary transformations only with the second and the third columns (in the case 1)) or with the first and the third columns (in the case 2)). Let us expand the functions $1/\phi_j^-(z)$ in the (generally infinite) continued fraction

$$\frac{1}{\phi_j^-(z)} = q^{\lambda_{j,0}}(z) + \cfrac{1}{q^{\lambda_{j,1}}(z) + \frac{1}{q^{\lambda_{j,2}}(z) + \ldots}},$$

where $\lambda_{j,i}$ is the order of the polynomial $q^{\lambda_{j,i}}(z)$, $\lambda_{j,0} = \gamma_j$. Denote $\mu_{j,k} = \lambda_{j,0} + \lambda_{j,1} + \ldots + \lambda_{j,k}$, $j = 1, 2$. Then, according to Proposition 1, in the cases 1) and 2) the partial indices of the matrix-function $D_3(z)$ are equal $(\kappa_1, \kappa_2 - \mu_{2,n}, \mu_{2,n})$, $(\kappa_1 - \mu_{1,l}, \kappa_2, \mu_{1,l})$, respectively. Here the parameters $\mu_{2,n}, \mu_{1,l}$ are determined as in Chebotarev's algorithm.

Now return to the case 3), i.e. when $\kappa_1 > \gamma_1$, $\kappa_2 > \gamma_2$. The following situations are possible

$$\begin{array}{ll} \text{I} & a)\ \kappa_1 - \gamma_1 \leq \kappa_2 - \gamma_2, \quad \gamma_1 \leq \gamma_2, \quad b)\ \kappa_1 - \gamma_1 \leq \kappa_2 - \gamma_2, \quad \gamma_2 \leq \gamma_1, \\ \text{II} & a)\ \kappa_1 - \gamma_1 \geq \kappa_2 - \gamma_2, \quad \gamma_1 \leq \gamma_2, \quad b)\ \kappa_1 - \gamma_1 \geq \kappa_2 - \gamma_2, \quad \gamma_2 \leq \gamma_1. \end{array}$$

Let us consider the case Ia).

The fraction $\phi_1^-(z)/\phi_2^-(z)$ can be represented in a neighborhood of the infinity in the form

$$\phi_1^-(z)/\phi_2^-(z) = q^{\gamma_2 - \gamma_1}(z) + r_1(z),$$

where $q^{\gamma_2-\gamma_1}(z)$ is a polynomial of the order $\gamma_2-\gamma_1$, and a function $r_1(z)$, analytic in a neighborhood of the infinity, has the order ν_1 at the infinity. Analogously, $1/\phi_2^-(z)=q^{\gamma_2}(z)+\tilde r_1(z)$, where $q^{\gamma_2}(z)$ is a polynomial of the order γ_2, and a function $\tilde r_1(z)$, analytic in a neighborhood of the infinity, has the order $\tilde\nu_1$ at the infinity.

We proceed with the following elementary transformation of the matrix-function $X_3^-(z)$: multiply the second column on $q^{\gamma_2-\gamma_1}(z)$ and subtracted it from the first one, then multiply the second column on $q^{\gamma_2}(z)$ and subtracted it from the third one. Then we get the matrix-function

$$X_{3,1}^-(z)=\begin{pmatrix} x_1^-(z) & 0 & 0\\ -x_2^-(z)q^{\gamma_2-\gamma_1}(z) & x_2^-(z) & -x_2^-(z)q^{\gamma_2}(z)\\ \phi_2^-(z)r_1(z) & \phi_2^-(z) & \phi_2^-(z)\tilde r_1(z)\end{pmatrix}.$$

The order at the infinity of its first column is determined either by the order of the first element $x_1^-(z)$, or by the order of the third element $\phi_2^-(z)r_1(z)$. Similarly, the order at the infinity of the third column is determined either by the order of its second element $-x_2^-(z)q^{\gamma_2}(z)$, or by the order of the third element $\phi_2^-(z)\tilde r_1(z)$. The following sets of inequalities are possible

$$\begin{aligned} &\mathrm{I}a1)\quad \kappa_1\le\gamma_2+\nu_1,\quad \kappa_2-\gamma_2\le\gamma_2+\tilde\nu_1,\\ &\mathrm{I}a2)\quad \kappa_1\le\gamma_2+\nu_1,\quad \gamma_2+\tilde\nu_1<\kappa_2-\gamma_2,\\ &\mathrm{I}a3)\quad \gamma_2+\nu_1<\kappa_1,\quad \kappa_2-\gamma_2\le\gamma_2+\tilde\nu_1,\\ &\mathrm{I}a4)\quad \gamma_2+\nu_1<\kappa_1,\quad \gamma_2+\tilde\nu_1<\kappa_2-\gamma_2. \end{aligned}\tag{6}$$

In the case Ia1) the matrix-function $X_{3,1}^-(z)$ has the normal form at the infinity and partial indices of $D_3(z)$ are equal $(\kappa_1,\gamma_2,\kappa_2-\gamma_2)$. In the case I$a$2) by taking into account the representation of the expression $1/\phi_2^-(z)$ in the continued fraction and proceeding with elementary transformations of the first and third columns according to Chebotarev's method, we also obtain the canonical matrix. Wherein the partial indices of $D_3(z)$ are equal $(\kappa_1,\kappa_2-\gamma_2-\tilde\gamma^m,\gamma_2+\tilde\gamma^m)$, where $\tilde\gamma^m=\tilde\nu_1+\tilde\nu_2+\ldots+\tilde\nu_m$, and $\gamma_2+\tilde\gamma^m\le\kappa_2-\gamma_2-\tilde\gamma^m\le\gamma_2+\tilde\gamma^{m+1}$. In the case I$a$3) by taking into account the representation of the expression $\phi_1^-(z)/\phi_2^-(z)$ in the continued fraction and proceeding with elementary transformations of the last two columns according to Chebotarev's method, we also obtain the canonical matrix. The partial indices are then equal $\big(2\gamma_2+\gamma^l,\kappa_1-\gamma_2-\gamma^l,\kappa_2-\gamma_2\big)$, where $\gamma^l=\nu_1+\nu_2+\ldots+\nu_l$, and $2\gamma_2+\gamma^l\le\kappa_1-\gamma_2-\gamma^l\le2\gamma_2+\gamma^{l+1}$.

If the inequalities Ia4) are satisfied, then we have to consider the cases:

$$\begin{aligned} &\kappa_1-\gamma_2-\nu_1\le\kappa_2-2\gamma_2-\tilde\nu_1:\quad A)\ \tilde\nu_1\le\nu_1;\quad B)\ \nu_1\le\tilde\nu_1;\\ &\kappa_2-2\gamma_2-\tilde\nu_1\le\kappa_1-\gamma_2-\nu_1:\quad A)\ \tilde\nu_1\le\nu_1;\quad B)\ \nu_1\le\tilde\nu_1. \end{aligned}$$

In both situations we continue elementary transformations by Chebotarev's method, obtain finally the canonical matrix and calculate the partial indices.

Consider now the case Ib), i.e. $\kappa_1-\gamma_1\le\kappa_2-\gamma_2$, $\gamma_2\le\gamma_1$. Then

$$\frac{\phi_2^-(z)}{\phi_1^-(z)}=q^{\gamma_1-\gamma_2}(z)+s_1(z),\qquad \frac{1}{\phi_2^-(z)}=q^{\gamma_2}(z)+\tilde r_1(z),$$

where $s_1(z),\tilde r_1(z)$ are functions analytic in a neighborhood of the infinity having at the infinity the orders $\mu_1,\tilde\mu_1=\tilde\nu_1$, respectively.

Let us carry out the following elementary transformations: multiply the second column on q^{γ_2} and subtract it from the third one, then multiply the first column on $q^{\gamma_1-\gamma_2}$ and subtract it from the second one. We obtain the following matrix:

$$X_{3,1}^-(z)=\begin{pmatrix} x_1^-(z) & -x_1^-(z)q^{\gamma_1-\gamma_2}(z) & 0\\ 0 & x_2^-(z) & -x_2^-(z)q^{\gamma_2}(z)\\ \phi_1^-(z) & \phi_1^-(z)s_1(z) & \phi_2^-(z)\tilde r_1(z)\end{pmatrix}.$$

The order at the infinity of the second column of this matrix is determined either by the order of the first element $-x_1^-(z)q^{\gamma_1-\gamma_2}(z)$, or by the order of the third element $\phi_1^-(z)s_1(z)$. Similarly, the order at the infinity of the third column is defined either by the order of the second element $-x_2^-(z)q^{\gamma_2}(z)$, or by the order of the third element $\phi_2^-(z)\tilde{r}_1(z)$. Further, it is sufficient to examine the situations analogous to those considered in the cases (6).

Let us specify the conditions under which the canonical matrix for $D_3(t)$ and its partial indices are determined by a modification of the above described approach.[1)]

Let the following inequality $\kappa_1-\gamma_1\le\kappa_2-\gamma_2$ holds. The canonical matrix for the matrix of the 2nd order $\begin{pmatrix}\zeta_2(t) & 0\\ a_2(t) & 1\end{pmatrix}$ has the form

$$\tilde{X}^+(z)=\begin{pmatrix}x_2^+(z) & 0\\ x_2^+(z)\phi_2^+(z) & x_2^+(z)\end{pmatrix}P(z),\quad \tilde{X}^-(z)=\begin{pmatrix}x_2^-(z)q^{\gamma_{i-1}^2}(z) & x_2^-(z)q^{\gamma_i^2}(z)\\ r_{\gamma_i^2}(z) & r_{\gamma_{i+1}^2}(z)\end{pmatrix}.$$

Then the pair of matrices

$$\tilde{X}_{3,2}^+(z)=\begin{pmatrix}x_1^+(z) & 0 & 0\\ 0 & \begin{pmatrix}x_2^+(z) & 0\\ x_2^+(z)\phi_2^+(z) & x_2^+(z)\end{pmatrix}P(z)\\ \phi_1^+(z) & & \end{pmatrix},$$

$$\tilde{X}_{3,2}^-(z)=\begin{pmatrix}x_1^-(z) & 0 & 0\\ 0 & x_2^-(z)q^{\gamma_{i-1}^2}(z) & x_2^-(z)q^{\gamma_i^2}(z)\\ \phi_1^-(z) & r_{\gamma_i^2}(z) & r_{\gamma_{i+1}^2}(z)\end{pmatrix}$$

satisfies the following boundary condition: $\tilde{X}_{3,2}^+(t)=D_3(t)\tilde{X}_{3,2}^-(t)$. Now, to obtain the canonical matrix for $D_3(t)$ it is sufficient to transform the matrix-function $\tilde{X}_{3,2}^-(z)$ to the normal form at infinity.

Let the following conditions are valid: $\kappa_1-\gamma_1\le\kappa_2-\gamma_{i-1}^2-\gamma_i^2$, and either a) $\gamma_1\le\gamma_i^2$, or b) $\gamma_i^2\le\gamma_1$.

If $\gamma_1\le\gamma_i^2$, then $\phi_1^-(z)/r_{\gamma_i^2}(z)=q^{\gamma_i^2-\gamma_1}(z)+\tilde{r}_1(z)$, where $\tilde{r}_1(z)$ has the order $\tilde{\mu}_1\ge 1$ at the infinity. We multiply the second column of $\tilde{X}_{3,1}^-(z)$ on $q^{\gamma_i^2-\gamma_1}(z)$ and subtract it from the first one. We get the matrix

$$\tilde{X}_{3,3}^-(z)=\begin{pmatrix}x_1^-(z) & 0 & 0\\ -x_2^-(z)q^{\gamma_{i-1}^2}(z)q^{\gamma_i^2-\gamma_1}(z) & x_2^-(z)q^{\gamma_{i-1}^2}(z) & x_2^-(z)q^{\gamma_i^2}(z)\\ \tilde{r}_1(z)r_{\gamma_i^2}(z) & r_{\gamma_i^2}(z) & r_{\gamma_{i+1}^2}(z)\end{pmatrix}.$$

The order of the first column is determined either by the first or by the third elements.

If $\kappa_1\le\tilde{\mu}_1+\gamma_i^2$, then the matrix-function $\tilde{X}_{3,3}^-(z)$ has a normal form at infinity and partial indices of $D_3(t)$ are equal $\left(\kappa_1,\gamma_i^2,\kappa_2-\gamma_i^2\right)$.

If $\le\tilde{\mu}_1+\gamma_i^2\le\kappa_1$, then by using representation of $\phi_1^-(z)/r_{\gamma_i^2}(z)$ in continued fraction, after finite number of steps we construct the canonical matrix and obtain the partial indices of $D_3(t)$ in the form $\left(\gamma_i^2+\tilde{\mu}_l,\kappa_1-\tilde{\mu}_l,\kappa_2-\gamma_i^2\right)$.

Remark 1. *The condition $\kappa_1-\gamma_1\le\kappa_2-\gamma_{i-1}^2-\gamma_i^2$ could be valid only if $\kappa_1-\gamma_1<\kappa_2-\gamma_2$.*

[1)] By this modified method we obtain the final result faster. Anyway, it should be noted that such modification can be applied only under certain restrictions on the parameters of the considered matrix.

If $\gamma_i^2 \leq \gamma_1$, then we have to consider the expression $r_{\gamma_i^2}(z)/\phi_1^-(z)$ and proceed with the similar considerations. In this case the partial indices of $D_3(t)$ are equal $\left(\tilde{\mu}_l, \kappa_1 - \tilde{\mu}_l + \gamma_i^2, \kappa_2 - \gamma_i^2\right)$.

If $\kappa_2 - \gamma_2 \leq \kappa_1 - \gamma_1$, then we have to construct first the canonical matrix for the following matrix-function

$$\begin{pmatrix} \zeta_1(z) & 0 & 0 \\ 0 & 1 & 0 \\ a_1(t) & 0 & 1 \end{pmatrix},$$

and then to repeat the transformations similar to those above described.

4. ON FACTORIZATION OF TRIANGULAR MATRIX-FUNCTIONS OF ARBITRARY ORDER

Extending Chebotarev's method to higher dimension we first examine the above treated case $n = 3$. It can be noted that in the above algorithm only relations between orders at infinity γ_i of functions ϕ_i and orders κ_i of diagonal elements, as well as relations between parameters $\kappa_i - \gamma_i$, play the role. If we take into account an inductive consideration, then together with the above noticed it could allow us to extend algorithm for the triangular matrix-functions of the arbitrary order.

Let us consider an n-th order ($n \geq 3$) non-singular triangular matrix-function of the form

$$D_n(t) = \begin{pmatrix} \zeta_1(t) & 0 & \dots & 0 \\ 0 & \zeta_2(t) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_1(t) & a_2(t) & \dots & 1 \end{pmatrix}.$$

Factorizing the diagonal elements $\zeta_j(t)$ (see [4]) we come to the conclusion that analytic in $D^{\pm}$ matrices

$$X_n^{\pm}(z) = \begin{pmatrix} x_1^{\pm}(z) & 0 & \dots & 0 \\ 0 & x_2^{\pm}(z) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \phi_1^{\pm}(z) & \phi_2^{\pm}(z) & \dots & 1 \end{pmatrix},$$

with $\phi_j^{\pm}(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{a_j(t) x_j^-(t) dt}{(t-z)}, z \in D^{\pm}$, satisfy the boundary condition $X_n^+(t) = D_n(t) X_n^-(t)$, $t \in \Gamma$. Let the indices of the functions $\zeta_j(t)$ be equal κ_j, and the orders at the infinity of the functions $\phi_j(z)$, analytic in D^-, be equal $\gamma_j \geq 1$, $j = 1, \dots, n-1$.

The following situations are possible:

1. $\kappa_1 \leq \gamma_1, \dots, \kappa_{n-1} \leq \gamma_{n-1}$;
2. $\exists j : 1 \leq j < n-1$, such that exactly j inequalities have opposite sign;
3. $\kappa_1 > \gamma_1, \dots, \kappa_{n-1} > \gamma_{n-1}$.

In the first case the matrix-function $X_n^-(z)$ has the normal form at the infinity. Thus, $X_n^{pm}(z)$ is the canonical matrix. Therefore, the partial indices are equal $(\kappa_1, \dots, \kappa_{n-1}, 0)$.

In the second case we have to use j columns and the last column in order to transform the matrix of order $j + 1 < n$ to the normal form at the infinity. This is possible due to the inductive assumption.

In the third case we propose the following algorithm.

Every column of the matrix $X_n^-(z)$ is characterized by the pair of numbers (γ_j, κ_j), $j = 1, \dots, n$. In the last column we have only one non-zero element. Hence we put $\gamma_n = \kappa_n = 0$. For shortness we will identify the columns with their characteristic pairs.

We split the set of all columns according to the following rule:

- let $\tilde{\gamma}_1 = \max\limits_{1\le j\le n} \gamma_j$; we fix the column $(\tilde{\gamma}_1, \tilde{\kappa}_1)$ and denote its number by $\tilde{j}_1$;

- denote by E_1 the following set (class) of columns: $E_1 = \{(\gamma_j, \kappa_j)\,|\kappa_j - \gamma_j \le \tilde{\kappa}_1 - \tilde{\gamma}_1\}$;

- let $\tilde{\gamma}_2 = \max\limits_{(\gamma_j,\kappa_j)\notin E_1} \gamma_j$ and $\tilde{\gamma}_2 \le \tilde{\gamma}_1$; we fix the column $(\tilde{\gamma}_2, \tilde{\kappa}_2)$;

- determine the set (class) of columns E_2 as follows: $E_2 = \{(\gamma_j, \kappa_j)\,|\tilde{\kappa}_1 - \tilde{\gamma}_1 < \kappa_j - \gamma_j \le \tilde{\kappa}_2 - \tilde{\gamma}_2\}$.

By continuing this process we obtain the partition of the set of all columns into the classes $E_1, E_2, \ldots, E_l$.

Remark 2. *It can happen that each class consists of the only one element.*

The columns $(\tilde{\gamma}_1, \tilde{\kappa}_1), (\tilde{\gamma}_2, \tilde{\kappa}_2), \ldots, (\tilde{\gamma}_l, \tilde{\kappa}_l)$ satisfy the inequalities $\tilde{\kappa}_1 - \tilde{\gamma}_1 < \tilde{\kappa}_2 - \tilde{\gamma}_2 < \ldots < \tilde{\kappa}_l - \tilde{\gamma}_l$, and $\tilde{\gamma}_l \le \ldots \le \tilde{\gamma}_2 \le \tilde{\gamma}_1$. We proceed with elementary transformations of the matrix-function $X_n^-(z)$ according to the following scheme.

1-st step. The columns of the class E_1 we transform by using the column $(\tilde{\gamma}_1, \tilde{\kappa}_1)$. If $(\gamma_i, \kappa_i) \in E_1$, then the expression $\phi_i^-(z)/\phi_{\tilde{j}_1}^-(z)$ can be written in a neighborhood of the infinity in the form

$$\phi_i^-(z)/\phi_{\tilde{j}_1}^-(z) = q^{\tilde{\gamma}_1 - \gamma_i}(z) + r_i^1(z),$$

where the function $r_i^1(z)$ is analytic in a neighborhood of the infinity and has at the infinity the order $\lambda_i^1 \ge 1$. Multiplying the column $(\tilde{\gamma}_1, \tilde{\kappa}_1)$ on $q^{\tilde{\gamma}_1 - \gamma_i}(z)$ and subtracting from the column (γ_i, κ_i), we obtain that the later contain three non-zero elements $x_i^-(z)$, $-x_{\tilde{j}_1}^-(z)q^{\tilde{\gamma}_1 - \gamma_i}(z)$, $\phi_{\tilde{j}_1}^-(z)r_i^1(z)$. The order of this column is determined either by the order of the element $x_i^-(z)$ or by the order of the element $\phi_{\tilde{j}_1}^-(z)r_i^1(z)$. Thus it is equal either κ_i or $\tilde{\gamma}_1 + \lambda_i^1$. The similar transformations we apply to all columns from E_1.

2-nd step. Analogous transformations we apply to all classes $E_2, \ldots, E_l$. Now it is necessary to check remaining columns $(\tilde{\gamma}_1, \tilde{\kappa}_1), (\tilde{\gamma}_2, \tilde{\kappa}_2), \ldots, (\tilde{\gamma}_l, \tilde{\kappa}_l)$. The scheme of transformation is the following: $(\tilde{\gamma}_1, \tilde{\kappa}_1) \leftarrow (\tilde{\gamma}_2, \tilde{\kappa}_2) \leftarrow \ldots \leftarrow (\tilde{\gamma}_l, \tilde{\kappa}_l) \leftarrow (0, 0)$. Let us briefly describe the details of this scheme.

The fraction $1/\phi_{\tilde{j}_l}^-(z)$ can be written in the form $1/\phi_{\tilde{j}_l}^-(z) = q^{\tilde{\gamma}_l}(z) + r_l^1(z)$, where the order at the infinity of the function $r_l^1(z)$ is equal $\lambda_l^1 \ge 1$. We multiply the column $(\tilde{\gamma}_l, \tilde{\kappa}_l)$ on $q^{\tilde{\gamma}_l}(z)$ and subtract it from the last column. Non-zero elements in the transformed column are $-x_{\tilde{j}_l}^-(z)q^{\tilde{\gamma}_l}(z)$ and $\phi_{\tilde{j}_l}^-(z)r_l^1(z)$. Their orders at the infinity are equal $\tilde{\kappa}_l - \tilde{\gamma}_l$ and $\tilde{\gamma}_l + \lambda_l^1$, respectively.

The fraction $\phi_{\tilde{j}_l}^-(z)/\phi_{\tilde{j}_{l-1}}^-(z)$ can be written in the form $\phi_{\tilde{j}_l}^-(z)/\phi_{\tilde{j}_{l-1}}^-(z) = q^{\tilde{\gamma}_{l-1} - \tilde{\gamma}_l}(z) + r_{l-1}^1(z)$, where the order at the infinity of the function $r_{l-1}^1(z)$ is equal $\lambda_{l-1}^1 \ge 1$.

We multiply the column $(\tilde{\gamma}_{l-1}, \tilde{\kappa}_{l-1})$ on $q^{\tilde{\gamma}_{l-1} - \tilde{\gamma}_l}(z)$ and subtract it from the column $(\tilde{\gamma}_l, \tilde{\kappa}_l)$. Non-zero elements in the transformed column are $-x_{\tilde{j}_{l-1}}^-(z)q^{\tilde{\gamma}_{l-1} - \tilde{\gamma}_l}(z)$, $x_{\tilde{j}_l}^-(z)$ and $\phi_{\tilde{j}_{l-1}}^-(z)r_{l-1}^1(z)$. The order of the transformed column is equal either $\kappa_{\tilde{j}_{l-1}} - \tilde{\gamma}_{l-1} + \tilde{\gamma}_l$, or $\tilde{\gamma}_{l-1} + \lambda_{l-1}^1$.

We continue this process up to the last pair of columns. For them we will use the following representation $\phi_{\tilde{j}_2}^-(z)/\phi_{\tilde{j}_1}^-(z) = q^{\tilde{\gamma}_1 - \tilde{\gamma}_2}(z) + r_1^1(z)$, where the order at the infinity of the function $r_1^1(z)$ is equal $\lambda_1^1 \ge 1$. By using this representation we get in the column $\tilde{j}_2$ the following non-zero elements $-x_{\tilde{j}_1}^-(z)q^{\tilde{\gamma}_1 - \tilde{\gamma}_2}(z)$, $x_{\tilde{j}_2}^-(z)$ and $\phi_{\tilde{j}_1}^-(z)r_1^1(z)$. The order of this column is equal either $\kappa_{\tilde{j}_1} - \tilde{\gamma}_1 + \tilde{\gamma}_2$, or $\tilde{\gamma}_1 + \lambda_1^1$. The column $(\tilde{\gamma}_1, \tilde{\kappa}_1)$ is not transformed. Therefore, after all these transformations we obtain the matrix-function $X_{n,1}^-(z)$ which is characterized by one of the following two conditions:

1) the order of the only one column is determined by the element of the n-th row of the matrix $X_{n,1}^-(z)$ (this is the column $(\tilde{\gamma}_1, \tilde{\kappa}_1)$);

2) the orders of several (possibly all) columns are determined by the elements of the n-th row of the matrix.

In the first case the matrix-function $X^-_{n,1}(z)$ has a normal form at infinity and thus the partial indices of the matrix-function $D_n(t)$ coincide with the orders of columns of $X^-_{n,1}(z)$.

In the second case, two situations are possible:

a) if the orders at the infinity of less than $n-1$ columns are determined by the last row, then the result follow from the inductive assumption;

b) if the orders at infinity of all $n-1$ column are determined by the last row, then we further proceed with transformation of the matrix-function $X^-_{n,1}(z)$ to the normal form at the infinity by using the above described scheme.

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