ON EXISTENCE OF THE MARCHAUD FRACTIONAL DERIVATIVE

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The Marchaud fractional derivative is defined for arbitrary values of $\alpha, \operatorname{Re}\alpha > 0$, by the following formula (see [1, Sec. 5.6]):

$$(\mathbb{D}^{\alpha}_{\pm} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)A_l(\alpha)} \int\limits_0^{\infty} \frac{(\Delta^l_{\pm t} f)(x)}{t^{1+\alpha}} dt, \quad l > \operatorname{Re}\alpha > 0, \tag{1}$$

where

$$A_l(\alpha) = \sum_{k=0}^{l} (-1)^{k-1} \binom{l}{k} k^{\alpha}, \quad (\Delta^l_{\pm t} f)(x) = \sum_{k=0}^{l} (-1)^{k} \binom{l}{k} f(x \mp kt).$$

In particular, this formula has the form

$$(\mathbb{D}^{\alpha}_{\pm} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^{\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt, \quad \text{for} \quad 0 < \operatorname{Re}\alpha < 1, \tag{2}$$

$$(\mathbb{D}^{\alpha}_{\pm} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)(2-2^{\alpha})} \int\limits_0^{\infty} \frac{f(x) - 2f(x-t) + f(x-2t)}{t^{1+\alpha}} dt, \quad \text{for} \quad 1 < \operatorname{Re}\alpha < 2. \tag{3}$$

Definition (1) is a kind of the regularization of the construction appeared due to formal replacement of the positive parameter α in the Liouville fractional integral

$$(I^{\alpha}_{-} f)(x) := \frac{1}{\Gamma(\alpha)} \int\limits_{-\infty}^{x} \frac{f(t)dt}{(x-t)^{1-\alpha}}$$

by the negative one. The approach by Marchaud was to introduce such a regularization which generalizes the Liouville fractional derivative (see [2], [3]). Since the obtained integral $(I^{-\alpha}_{-} f)(x)$ is in general divergent, it is necessary to give a proper sence to this integral.

Sufficient condition for existence of the integral in (1) are presented, e.g., in [1]. In particular, it was shown ([1, Thm. 5.9]) that the Marchaud derivative (1) is defined on bounded function whose derivative of order $[\alpha]$ locally satisfies the Hölder condition with the exponent $\lambda > \alpha - [\alpha]$.

In our report we discuss another class of functions provided an existence of the Marchaud derivative.

Theorem 1. *Let the function $f(x)$ be uniformly approximated on the semi-axes $(-\infty, a] \subset \mathbb{R}$ by the sum of exponents*

$$f(x) \approx \sum_{j=1}^{N} C_j e^{\lambda_j x}, \ 0 \le \lambda_1 < \ldots < \lambda_N, \ C_j \in \mathbb{C}, \tag{4}$$

then there exists the Marchaud derivative (1) of this function for each $x \in (-\infty, a]$.

Sketch of the proof. To describe the idea of the proof we calculate the Marchaud derivative of the function $f(x) = e^{\lambda x}, \ \lambda > 0$, in the case $0 < \alpha < 1$:

$$(\mathbb{D}^{\alpha}_{\pm} e^{\lambda t})(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^{\infty} \frac{e^{\lambda x} - e^{\lambda(x-t)}}{t^{1+\alpha}} dt = \frac{\alpha e^{\lambda x}}{\Gamma(1-\alpha)} \int\limits_0^{\infty} \frac{1 - e^{-\lambda t}}{t^{1+\alpha}} dt.$$

The last integral is surely convergent. Calculating it by parts

$$(\mathbb{D}^{\alpha}_{\pm}e^{\lambda t})(x) = \frac{\alpha}{\Gamma(1-\alpha)}\left\{\frac{e^{\lambda x}}{(-\alpha)}(1-e^{-\lambda t})t^{-\alpha}\Big|_{t=0}^{t=\infty} - \frac{e^{\lambda x}}{(-\alpha)}\int\limits_{0}^{\infty}\lambda e^{-\lambda t}t^{1-\alpha-1}dt\right\}$$

we obtain

$$(\mathbb{D}^{\alpha}_{\pm}e^{\lambda t})(x) = \lambda^{\alpha}e^{\lambda x}.$$

Calculations for the large values of the parameter α can be performed similarly.

Remark 1. *The idea of considering the class of functions (4) goes back to the initial work by Liouville (see [3]).*

Remark 2. *The Marchaud is useful at the study of certain problems for fractional oder operators, see e.g., [4].*

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References

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