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PARTIAL DIFFERENTIAL EQUATIONS

Boundary Value Problems for a Weakly Loaded Operator of a Second-Order Hyperbolic Equation in a Cylindrical Domain

V. I. Korzyuk, M. T. Dzhenaliev, and I. S. Kozlovskaya

Belarusian State University, Minsk, Belarus
Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
e-mail: korzyuk@bsu.by, muvasharkhan@gmail.com, kozlovskaja@bsu.by

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Abstract—By using methods of functional analysis, we prove the existence and uniqueness, in appropriate function spaces, of solutions of boundary value problems for a second-order linear hyperbolic equation weakly loaded by a differential operator. The order of the operator in the loaded term with respect to its derivatives is equal to 2.

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1. INTRODUCTION

The study of boundary value problems for loaded equations is related to numerous applications. Such problems were considered in many papers [1–9].

Note that the above-mentioned papers mainly deal with boundary value problems for a loaded heat equation. In the present paper, we consider equations of the hyperbolic type with the use of the method of energy inequalities and averaging operators with variable step [10]. In the monograph [10, Subsec. 3.6], the notion of generalized classical solution was introduced for mixed problems for hyperbolic equations, and boundary value problems in a cylindrical domain with the Dirichlet conditions, Neumann conditions (with the conormal derivative), and with conditions expressed via tangential derivatives were considered. In the present paper, we study these problems for a similar equation with a weakly loaded operator.

A "weakly loaded operator" is an operator whose coefficients are sufficiently small compared with the coefficients in the main operator of the equation. This will be used in the derivation of an energy inequality and the proof of the existence of solutions of the problems considered.

In many cases, methods of functional analysis permit one to prove existence and uniqueness theorems for generalized solutions by considering an extension of the operator of the original problem in the strong or weak topology. However, as a rule, generalized solutions of problems for hyperbolic equations do not have all derivatives occurring in the equation.

Here we prove the existence and uniqueness of a strong solution of mixed problems in function spaces for the case in which the space of solutions contains all derivatives occurring in the equation. Therefore, in this case, generalized solutions are treated as solutions satisfying the boundary conditions almost everywhere.

2. STATEMENT OF THE PROBLEM

For a function $u:\ \mathbb{R}^{n+1} \supset Q \ni \boldsymbol{x} \to u(\boldsymbol{x}) \in \mathbb{R}$ in a cylindrical domain $Q = (0,T) \times \Omega$ of the independent variables $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$, consider the loaded linear second-order differential equation

$$\mathfrak{L}u(\boldsymbol{x}) = \frac{\partial^2 u(\boldsymbol{x})}{\partial x_0^2} - A^{(0)}u(\boldsymbol{x}) - B^{(0)}u(t,\boldsymbol{x}') + A^{(1)}u(\boldsymbol{x}) + B^{(1)}u(t,\boldsymbol{x}') = f(\boldsymbol{x}). \tag{2.1}$$

Here $\mathbb{R}^{n+1}$ is the $(n+1)$-dimensional Euclidean space,

$$A^{(0)}u(\boldsymbol{x}) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a^{(ij)}(\boldsymbol{x})\frac{\partial u(\boldsymbol{x})}{\partial x_j}\right), \quad A^{(1)}u(\boldsymbol{x}) = \sum_{i=0}^{n} a^{(i)}(\boldsymbol{x})\frac{\partial u(\boldsymbol{x})}{\partial x_i} + \widetilde{a}(\boldsymbol{x})u(\boldsymbol{x}),$$

$$B^{(0)}u = \sum_{i,j=1}^{n} b^{(ij)}(t,\boldsymbol{x}')\frac{\partial^2 u(t,\boldsymbol{x}')}{\partial x_j^2}, \quad B^{(1)}u(t,\boldsymbol{x}') = \sum_{i=0}^{n} b^{(i)}(t,\boldsymbol{x}')\frac{\partial u(t,\boldsymbol{x}')}{\partial x_i} + b^{(0)}(t,\boldsymbol{x}')u(t,\boldsymbol{x}'),$$

and Ω is a bounded domain in the space $\mathbb{R}^n$ of independent variables $\boldsymbol{x}'$ with piecewise smooth boundary $\partial\Omega$.

Assume that Eq. (2.1) is hyperbolic with respect to the direction $\boldsymbol{\eta} = (1, 0, 0, \ldots, 0)$. To this end, we require that the quadratic form consisting of the coefficients $a^{(ij)}$ of the operator $A^{(0)}$ is positive; i.e.,

$$\sum_{i,j=1}^{n} a^{(ij)}(\boldsymbol{x})\xi_i\xi_j \geq \alpha^{(0)} \sum_{i=1}^{n} \xi_i^2 = \alpha^{(0)}|\boldsymbol{\xi}|^2 \tag{2.2}$$

for any vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and for some positive number $\alpha^{(0)}$ uniformly with respect to $\boldsymbol{x} \in \bar{\Omega}$, where $\bar{\Omega}$ is the closure of the domain Ω.

The coefficients of the operator $A^{(0)}$ satisfy the condition of symmetry with respect to indices, $a^{(ij)} = a^{(ji)}$ for arbitrary $i, j = 1, \ldots, n$. In addition, $a^{(ij)} \in C^2(\overline{Q})$ and $a^{(k)}, \tilde{a} \in C^1(\overline{Q})$, $k = 0, \ldots, n$, where $C(\overline{Q})$ is the set of continuous functions and $C^s(\overline{Q})$ $(s = 1, 2)$ is the set of functions that are continuously differentiable up to order s and are defined on the closure $\overline{Q}$ of the domain Q.

Then the condition $a^{(ij)} \in C(\overline{Q})$ implies the inequality

$$\sum_{i,j=1}^{n} (a^{(ij)}(\boldsymbol{x}))\xi_i\xi_j \leq \alpha^{(1)}|\boldsymbol{\xi}|^2 \tag{2.3}$$

for any vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and any point $\boldsymbol{x} \in \overline{Q}$, where $\alpha^{(1)}$ is some positive number. Obviously, the relation $\alpha^{(0)} \leq \alpha^{(1)}$ holds. Since $a^{(k)}, \tilde{a} \in C^1(\overline{Q})$, $k = 0, \ldots, n$, it follows that the coefficients of the operator $A^{(1)}$ satisfy the estimates

$$\sum_{k=0}^{n} a^{(k)}\xi_k \leq \alpha^{(2)}|\boldsymbol{\xi}|^2, \qquad |\tilde{a}(\boldsymbol{x})| \leq \alpha^{(3)} \tag{2.4}$$

for all $\boldsymbol{x} \in \overline{Q}$, where the $\alpha^{(j)}$, $j = 2, 3$, are some positive constants.

Similar conditions hold for the coefficients of the operators $B^{(0)}$ and $B^{(1)}$. They satisfy the symmetry condition $b^{(ij)} = b^{(ji)}$ $(i, j = 1, \ldots, n)$, $b^{(ij)} \in C^2(\overline{\Omega})$, $b^{(i)} \in C^1(\Omega)$ $(i = 0, \ldots, n)$, and the inequality

$$\left(\sum_{i,j=1}^{n} b^{(ij)}(t,\boldsymbol{x}')\xi_{ij}\right)^2 \leq \beta^{(0)}|\boldsymbol{\xi}|^2 = \beta^{(0)} \sum_{i,j=1}^{n} \xi_{ij}^2, \tag{2.5}$$

$\boldsymbol{\xi} = (\xi_{11}, \ldots, \xi_{ij}, \ldots, \xi_{nn}) \in \mathbb{R}^{n^2}$, where $\beta^{(0)}$ is a positive constant.

A similar estimate holds for the operator $B^{(1)}$; namely,

$$(B^{(1)}u(t,\boldsymbol{x}'))^2 \leq \beta^{(1)}\left[\sum_{i=0}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2(t,\boldsymbol{x}') + u^2(t,\boldsymbol{x}')\right] \tag{2.6}$$

for some positive number $\beta^{(1)}$ and for arbitrary functions $u \in C^1(\overline{Q})$.

We supplement Eq. (2.1) with the Cauchy conditions

$$l_0u = u(0,\boldsymbol{x}') = \varphi(\boldsymbol{x}'), \qquad l_1u = \frac{\partial u(0,\boldsymbol{x}')}{\partial x_0} = \psi(\boldsymbol{x}'), \tag{2.7}$$

and one of the boundary conditions

$$u|_{\Gamma} = 0, \tag{2.8}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{N}}\right|_{\Gamma} = 0, \tag{2.9}$$

$$\left.\frac{\partial u}{\partial \boldsymbol{\tau}^{(1)}}\right|_{\Gamma} = \cdots = \left.\frac{\partial u}{\partial \boldsymbol{\tau}^{(n)}}\right|_{\Gamma} = 0, \qquad \int\limits_{\partial\Omega} \sum_{i,j=1}^{n} a^{(ij)}(\boldsymbol{x})\nu_i\nu_j \frac{\partial u}{\partial \boldsymbol{\nu}}\, u\, ds \le 0, \tag{2.10}$$

where $\Gamma = (0,T)\times\partial\Omega$,

$$\left.\frac{\partial u}{\partial \boldsymbol{N}}\right|_{\Gamma} = \sum_{i,j=1}^{n} a^{(ij)}(\boldsymbol{x})\, \left.\frac{\partial u}{\partial x_j}\nu_i\right|_{\Gamma},$$

$\boldsymbol{\nu} = (0,\nu_1,\dots,\nu_n)$ is the unit outward normal vector with respect to the domain Q, and $\boldsymbol{\tau}^{(1)},\dots,\boldsymbol{\tau}^{(n)}$ are n linearly independent tangent vectors defined almost everywhere at points of the hypersurface Γ.

Therefore, we have the following boundary value (mixed) problems for the loaded hyperbolic equation (2.1).

MP1: the first mixed problem (2.1), (2.7), (2.8) with the Dirichlet boundary conditions on the lateral surface of Γ.

MP2: the second mixed problem (2.1), (2.7), (2.9).

MP3: the third mixed problem (2.1), (2.7), (2.10).

Note that one can consider mixed problems with mixed boundary conditions of the form (2.8)–(2.10) on Γ. In this case, Γ consists of finitely many parts, and some of conditions (2.8)–(2.10) is posed on each of them.

3. FUNCTION SPACES AND STATEMENT OF THE PROBLEMS IN OPERATOR FORM

Problems **MP1**–**MP3** differ by the boundary conditions (2.8)–(2.10). Having in mind these conditions, let us introduce some subsets of the set $C^2(\overline{Q})$.

By $C^2(\overline{Q};(2.s))$, $s = 8,9,10$, we denote the subset of functions in $C^2(\overline{Q})$ satisfying conditions $(2.s)$.

For functions $\omega:\ \mathbb{R}^n \supset \Omega \ni \boldsymbol{x}' \to \omega(\boldsymbol{x}') \in \mathbb{R}$ defined on $\Omega^{(0)} = \{\boldsymbol{x}\in\overline{Q}|\ x_0 = 0\}$, consider the restrictions of conditions (2.8)–(2.10), that is,

$$\omega|_{\partial\Omega} = 0, \tag{3.1}$$

$$\left.\frac{\partial \omega}{\partial \boldsymbol{N}}\right|_{\partial\Omega} = 0, \qquad \left.\frac{\partial \omega}{\partial \boldsymbol{N}}\right|_{\partial\Omega} = \sum_{i,j=1}^{n} a^{(ij)}(0,\boldsymbol{x}')\nu_i\frac{\partial w}{\partial x_j}, \tag{3.2}$$

$$\left.\frac{\partial \omega}{\partial \tilde{\boldsymbol{\tau}}^{(1)}}\right|_{\partial\Omega} = \cdots = \left.\frac{\partial \omega}{\partial \tilde{\boldsymbol{\tau}}^{(n-1)}}\right|_{\partial\Omega} = 0, \qquad \int\limits_{\partial\Omega} \sum_{i,j=1}^{n} a^{(ij)}(0,\boldsymbol{x}')\nu_i\nu_j \frac{\partial \omega}{\partial \boldsymbol{\nu}}\, \omega\, ds \le 0, \tag{3.3}$$

where $\tilde{\boldsymbol{\tau}}^{(1)},\dots,\tilde{\boldsymbol{\tau}}^{(n-1)}$ are linearly independent unit vectors tangent to the hypersurface $\partial\Omega$ and $\boldsymbol{\nu} = (\nu_1,\dots,\nu_n)$ is the unit outward (with respect to the domain Ω) normal to the surface $\partial\Omega$ at the points $\boldsymbol{x}'$.

By $C^2(\overline{\Omega};(3.s))$, $s = 1,2,3$, we denote the subset of functions in $C^2(\overline{\Omega})$ satisfying conditions $(3.s)$, and by $H^2(\Omega)$ we denote the Sobolev space of functions that are defined in the domain Ω and, together with their generalized derivatives of order ≤ 2, are Lebesgue square integrable. Let $H^2(\Omega,(3.s))$ $(s = 1,2,3)$ be the subspaces of $H^2(\Omega)$ whose elements satisfy the boundary conditions $(3.s)$.

Condition 3.1. The boundary $\partial\Omega$ is piecewise smooth and satisfies the condition that the spaces $H^2(\Omega,(3.s))$ $(s=1,2,3)$ can be obtained as the closures of the respective spaces $C^2(\overline{\Omega};(3.s))$ in the norm of the space $H^2(\Omega)$.

Let $H^1(\Omega,(3.s))$ $(s=1,2,3)$ be the closures of the respective sets $C^2(\overline{\Omega};(3.s))$ in the norm of the space $H^1(\Omega)$.

For functions $u\in C^2(\overline{Q})$, we consider the norm defined by the formula

$$\|u\|_B=\left\|\frac{\partial^2 u}{\partial x_0^2}\right\|_{L_2(Q)}+\sup_{0\le x_0\le T}\left(\|u\|_{H^2(\Omega)}+\sum_{i=1}^n\left\|\frac{\partial^2 u}{\partial x_0\,\partial x_i}\right\|_{L_2(\Omega)}+\left\|\frac{\partial u}{\partial x_0}\right\|_{L_2(\Omega)}\right)(x_0). \tag{3.4}$$

Let us introduce the Banach spaces $B^{(s)}(Q)$, $s=1,2,3$, which are obtained as the closure of the sets $C^2(\overline{Q};(2.s+7))$ in the norm defined by (3.4).

By $\mathfrak{H}^1(Q)$ we denote the Hilbert space of Lebesgue square integrable functions u for which there exist generalized derivatives $\partial u/\partial x_i\in L_2(Q)$, $i=1,\ldots,n$. The inner product and the norm are defined by the formulas

$$(u,v)_{\mathfrak{H}^1(Q)}=(u,v)_{L_2(Q)}+\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i},\frac{\partial v}{\partial x_i}\right)_{L_2(Q)},\quad \|u\|_{\mathfrak{H}^1(Q)}=(u,u)^{1/2}_{\mathfrak{H}^1(Q)},\quad u,v\in\mathfrak{H}^1(Q). \tag{3.5}$$

Let $\mathring{\mathfrak{H}}^1(Q)$ be the subspace of $\mathfrak{H}^1(Q)$ obtained as the closure of the set $C^2(\overline{Q};(2.8))$ in the norm (3.5).

We introduce the operators $\boldsymbol{L}^s$, $s=1,2,3$, as follows:

$$\boldsymbol{L}^s:\ B^{(s)}(Q)\ni u\to\boldsymbol{L}^{(s)}u=\{\mathfrak{L}u,l_0u,l_1u\}\in\boldsymbol{H}^s=\mathring{\mathfrak{H}}^1(Q)\times H^2(\Omega;(3.s))\times H^1(\Omega;(3.s)).$$

Now each problem **MP**s, $s=1,2,3$, can be treated as the operator equation

$$\boldsymbol{L}^{(s)}u=\boldsymbol{F}^{(s)},\qquad s=1,2,3, \tag{3.6}$$

with domain

$$\mathfrak{D}(\boldsymbol{L}^{(s)})=\{C^2(\overline{Q};(2.s+7))|\mathfrak{L}u|_\Gamma=0\},$$

where $\boldsymbol{F}^{(s)}=\{f(\boldsymbol{x}),\varphi(\boldsymbol{x}'),\psi(\boldsymbol{x}')\}\in\boldsymbol{H}^{(s)}$.

4. ENERGY INEQUALITIES

Let us prove energy inequalities for the operators $\boldsymbol{L}^{(s)}$, $s=1,2,3$, in Eqs. (3.6) for functions in $\mathfrak{D}(\boldsymbol{L}^{(s)})$. By using these inequalities, we will introduce the extension of the operators $\boldsymbol{L}^{(s)}$ to closed operators $\overline{\boldsymbol{L}^{(s)}}$ by closure, and then we prove existence and uniqueness theorems for strong solutions.

Lemma 4.1. *For each $s=1,2,3$ and for arbitrary functions $u,v\in B^{(s)}(Q)$, one has the relation*

$$\begin{aligned}(A^{(0)}u,A^{(0)}v)_{L_2(\Omega)}(x_0)&=\sum_{i,j,k,l=1}^n\left(a^{(ij)}\frac{\partial^2 u}{\partial x_j\,\partial x_k},a^{(kl)}\frac{\partial^2 v}{\partial x_i\,\partial x_l}\right)_{L_2(\Omega)}(x_0)+\mathcal{A}^{(1)}(u,v;x_0)\\&=\mathcal{A}^{(0)}(u,v;x_0)+\mathcal{A}^{(1)}(u,v;x_0)\end{aligned} \tag{4.1}$$

and the estimate

$$\|A^{(0)}u\|^2_{L_2(\Omega)}(x_0)\ge(\alpha^{(0)})^2-\varepsilon^{(0)}\sum_{i,j=1}^n\left\|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right\|^2_{L_2(\Omega)}(x_0)-c^{(1)}(\varepsilon^{(0)})\sum_{i=1}^n\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L_2(\Omega)}(x_0) \tag{4.2}$$

for all $x_0\in[0,T]$, where the positive constants $(\alpha^{(0)})^2\varepsilon^{(0)}$ and $c^{(1)}(\varepsilon^{(0)})$ are independent of the elements $u\in B^{(s)}(Q)$, $s=1,2,3$, and $\varepsilon^{(0)}$ is an arbitrary positive number.

The proof of Lemma 4.1 can be found in the monograph [10, pp. 124–127]. Here $\varepsilon^{(0)} > 0$ can be chosen arbitrarily so as to ensure that the difference $(\alpha^{(0)})^2 - \varepsilon^{(0)}$ is positive. The number $c^{(1)}(\varepsilon^{(0)})$ grows proportionally as $\varepsilon^{(0)}$ decreases.

Theorem 4.1. *Let Condition* 3.1 *be satisfied, and let the coefficients of Eq.* (2.1) *satisfy the smoothness constraints in Section* 2. *Then, for sufficiently small* $\beta^{(0)}$ *and* $\beta^{(1)}$, *one has the energy inequalities*

$$\|u\|_B \le c\|\boldsymbol{L}^{(s)}u\|_{\boldsymbol{H}^{(s)}} \tag{4.3}$$

for any function u *in the corresponding domain* $\mathfrak{D}(\boldsymbol{L}^{(s)})$ *and for each operator* $\boldsymbol{L}^{(1)}$, $\boldsymbol{L}^{(2)}$, *and* $\boldsymbol{L}^{(3)}$, *where* c *is a positive constant independent of* u.

Proof. We represent the expression $\mathfrak{L}u\frac{\partial}{\partial x_0}A^0(u)$ in the form

$$\begin{aligned}2\left(\mathfrak{L}u\frac{\partial}{\partial x_0}A^{(0)}u\right)(\boldsymbol{x}) = 2\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a^{(ij)}\frac{\partial^2 u}{\partial x_0^2}\,\frac{\partial^2 u}{\partial x_0\,\partial x_j}\right)(\boldsymbol{x}) - \frac{\partial}{\partial x_0}(\mathfrak{A}^{(0)}(u,u)(\boldsymbol{x}))\\ + \mathfrak{A}^{(1)}(u,u)(\boldsymbol{x}) + 2\frac{\partial}{\partial x_0}(A^{(0)}uA^{(1)}u)(\boldsymbol{x})\\ - \mathfrak{B}^{(0)}(u,u)(t,\boldsymbol{x}) + \mathfrak{B}^{(1)}(u,u)(t,\boldsymbol{x}),\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}\mathfrak{A}^{(0)}(u,u)(\boldsymbol{x}) &= \sum_{i,j=1}^{n}\left(a^{(ij)}\frac{\partial^2 u}{\partial x_0\partial x_i}\,\frac{\partial^2 u}{\partial x_0\partial x_j}\right)(\boldsymbol{x}) + (A^{(0)}u)^2(\boldsymbol{x}),\\ \mathfrak{A}^{(1)}(u,u)(\boldsymbol{x}) &= \sum_{i,j=1}^{n}\left[\frac{\partial a^{(ij)}}{\partial x_0}\,\frac{\partial^2 u}{\partial x_0\partial x_i}\,\frac{\partial^2 u}{\partial x_0\partial x_j} + 2\frac{\partial^2 u}{\partial x_0^2}\,\frac{\partial}{\partial x_i}\left(\frac{\partial a^{(ij)}}{\partial x_0}\,\frac{\partial u}{\partial x_j}\right)\right](\boldsymbol{x})\\ &\quad - 2\left(A^{(0)}u\frac{\partial}{\partial x_0}A^{(1)}u\right)(\boldsymbol{x}),\\ \mathfrak{B}^{(0)}(u,u) &= 2B^{(0)}u(t,\boldsymbol{x}')A^{(0)}u(\boldsymbol{x}'), \qquad \mathfrak{B}^{(1)}(u,u) = 2B^{(1)}u(t,\boldsymbol{x}')A^{(0)}u(\boldsymbol{x}').\end{aligned}$$

On the other hand, we have

$$\left(\mathfrak{L}u\frac{\partial}{\partial x_0}A^{(0)}u\right)(\boldsymbol{x}) = \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\mathfrak{L}u\frac{\partial}{\partial x_0}\left(a^{(ij)}\frac{\partial u}{\partial x_j}\right)\right)(\boldsymbol{x}) - \sum_{i,j=1}^{n}\frac{\partial \mathfrak{L}u}{\partial x_i}\,\frac{\partial}{\partial x_0}\left(a^{(ij)}\frac{\partial u}{\partial x_j}\right)(\boldsymbol{x}). \tag{4.5}$$

By $Q^{(\tau)}$ we denote the subdomain of Q of height τ; i.e., $Q^{(\tau)} = (0,\tau)\times\Omega$, $0 < \tau < T$. We integrate relations (4.4) and (4.5) over $Q^{(\tau)}$. As a result, by using the boundary condition $(2.s+7)$ depending on the operator $\boldsymbol{L}^{(s)}$, $s = 1, 2, 3$, we obtain the relation

$$\begin{aligned}&\int_\Omega \mathfrak{A}^{(0)}(u,u)(\tau,\boldsymbol{x}')\,d\boldsymbol{x}' + \int_\Omega \mathfrak{B}^{(0)}(u,u)(t,\tau,\boldsymbol{x}')\,d\boldsymbol{x}'\\ &\quad = \int_{Q^{(\tau)}} \mathfrak{A}^{(1)}(u,u)(\boldsymbol{x})\,d\boldsymbol{x} + \int_\Omega [(2A^{(0)}uA^{(1)}u)(\tau,\boldsymbol{x}') + \mathfrak{B}^{(1)}(u,u)(t,\tau,\boldsymbol{x}')]\,d\boldsymbol{x}'\\ &\qquad + \int_\Omega [\mathfrak{A}^{(0)}(u,u)(0,\boldsymbol{x}') - 2(A^{(0)}uA^{(1)}u)(0,\boldsymbol{x}')\\ &\qquad + 2B^{(0)}u(t,\boldsymbol{x}')A^{(0)}u(0,\boldsymbol{x}') - 2B^{(1)}u(t,\boldsymbol{x}')\,A^{(0)}(0,\boldsymbol{x}')]\,d\boldsymbol{x}' - \int_{Q^{(\tau)}} \mathfrak{F}^{(0)}(u,u)(\boldsymbol{x})\,d\boldsymbol{x},\end{aligned} \tag{4.6}$$

where

$$\mathfrak{F}^{(0)}(u,u)(\boldsymbol{x}) = 2\sum_{i,j=1}^{n}\left(\frac{\partial \mathfrak{L}u}{\partial x_i}\frac{\partial}{\partial x_0}\left(a^{(ij)}\frac{\partial u}{\partial x_j}\right)\right)(\boldsymbol{x}).$$

Now we estimate the left-hand side of relation (4.6) from below by some nonnegative expression and the right-hand side from above.

By inequality (2.2), we have

$$\int\limits_{\Omega}\mathfrak{A}^{(0)}(u,u)(\tau,\boldsymbol{x}')\,d\boldsymbol{x}' \geq \alpha^{(0)}\sum_{i=1}^{n}\left\|\frac{\partial^2 u}{\partial x_0 \partial x_i}\right\|^2_{L_2(\Omega)}(\tau) + \|A^{(0)}u\|^2_{L_2(\Omega)}(\tau). \tag{4.7}$$

We estimate the expression $\mathfrak{B}^{(0)}(u,u)$ from above as follows:

$$\left|\int\limits_{\Omega}\mathfrak{B}^{(0)}(u,u)(t,\tau,\boldsymbol{x}')\,d\boldsymbol{x}'\right| \leq \frac{1}{\varepsilon^{(1)}}\|B^{(0)}u\|^2_{L_2(\Omega)}(t) + \varepsilon^{(1)}\|A^{(0)}u\|^2_{L_2(\Omega)}(\tau). \tag{4.8}$$

By virtue of inequality (2.5), we have

$$\|B^{(0)}u\|^2_{L_2(\Omega)}(t) \leq \beta^{(0)}\sum_{i,j=1}^{n}\left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|^2_{L_2(\Omega)}(t). \tag{4.9}$$

By using the Cauchy–Schwarz inequality and the conditions for the coefficients of Eq. (2.1), we estimate the remaining terms in relation (4.6),

$$\begin{aligned}
\left|\int\limits_{Q^{(\tau)}}\mathfrak{A}^{(1)}(u,u)(\boldsymbol{x})\,d\boldsymbol{x}\right| &\leq c^{(2)}\sum_{1\leq|\boldsymbol{\alpha}|\leq 2}\|D^{\boldsymbol{\alpha}}u\|^2_{L_2(Q^{(\tau)})},\\
2\left|\int\limits_{\Omega}(A^{(0)}uA^{(1)}u)(\tau,\boldsymbol{x}')\,d\boldsymbol{x}'\right| &\leq \varepsilon^{(2)}\|A^{(0)}u\|^2_{L_2(\Omega)}(\tau)\\
&\quad + \frac{c^{(2)}}{\varepsilon^{(2)}}\left(\|u\|^2_{H^1(\Omega)} + \left\|\frac{\partial u}{\partial x_0}\right\|^2_{L_2(\Omega)}\right)(\tau),\\
\left|\int\limits_{\Omega}\mathfrak{B}^{(1)}(u,u)(t,\tau,\boldsymbol{x}')\,d\boldsymbol{x}'\right| &\leq \varepsilon^{(3)}\|A^{(0)}u\|^2_{L_2(\Omega)}(\tau) + \frac{1}{\varepsilon^{(3)}}\|B^{(1)}u\|^2_{L_2(\Omega)}(t),\\
2\left|\int\limits_{\Omega}A^{(0)}(0,\boldsymbol{x}')[B^{(0)}u(t,\boldsymbol{x}') - B^{(1)}u(t,\boldsymbol{x}')]\,d\boldsymbol{x}'\right|&\\
&\leq \varepsilon^{(4)}(\|B^{(0)}u\|^2_{L_2(\Omega)} + \|B^{(1)}u\|^2_{L_2(\Omega)})(t) + \frac{c^{(3)}}{\varepsilon^{(4)}}\|l_0u\|^2_{H^2(\Omega)},\\
\int\limits_{\Omega}|(\mathfrak{A}^{(0)}(u,u) - 2A^{(0)}uA^{(1)}u)(0,\boldsymbol{x}')|\,d\boldsymbol{x}' &\leq c^{(4)}(\|l_0u\|^2_{H^2(\Omega)} + \|l_1u\|^2_{H^1(\Omega)}),\\
\left|\int\limits_{Q^{(\tau)}}\mathfrak{F}^{(0)}(u,u)(\boldsymbol{x})\,d\boldsymbol{x}\right| &\leq c^{(5)}\sum_{j=1}^{n}\left(\left\|\frac{\partial}{\partial x_j}\mathfrak{L}u\right\|^2_{L_2(Q)} + \left\|\frac{\partial^2 u}{\partial x_0 \partial x_j}\right\|^2_{L_2(Q^{(\tau)})}\right),
\end{aligned} \tag{4.10}$$

where $D^{\boldsymbol{\alpha}} = \dfrac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_0^{\alpha_0}\ldots\partial x_n^{\alpha_n}}$, $\boldsymbol{\alpha} = (\alpha_0,\ldots,\alpha_n)$, $|\boldsymbol{\alpha}| = \alpha_0 + \cdots + \alpha_n$, and the α_j are nonnegative integers, $j = 0,\ldots,n$.

From relation (4.6) and the estimates (4.7)–(4.10), we obtain the inequality

$$\alpha^{(0)}\sum_{i=1}^{n}\left\|\frac{\partial^2 u}{\partial x_0\partial x_i}\right\|^2_{L_2(\Omega)}(\tau)+(1-\varepsilon^{(1)}-\varepsilon^{(2)}-\varepsilon^{(3)})\|A^{(0)}u\|^2_{L_2(\Omega)}(\tau)$$
$$\le \frac{c^{(2)}}{\varepsilon^{(2)}}\left(\|u\|^2_{H^1(\Omega)}+\left\|\frac{\partial u}{\partial x_0}\right\|^2_{L_2(\Omega)}\right)(\tau)+\left(\frac{1}{\varepsilon^{(1)}}+\varepsilon^{(4)}\right)\|B^{(0)}u\|^2_{L_2(\Omega)}(t)$$
$$+\left(\frac{1}{\varepsilon^{(3)}}+\varepsilon^{(4)}\right)\|B^{(1)}u\|^2_{L_2(\Omega)}(t)+c^{(6)}\|u\|^2_{H^2(Q^{(\tau)})}+c^{(7)}\|\boldsymbol{L}^{(s)}u\|^2_{\boldsymbol{H}^{(s)}}, \qquad (4.11)$$

where $c^{(6)}$ and $c^{(7)}$ are some positive constants defined via $c^{(j)}$, $j=2,\dots,5$, and $\varepsilon^{(4)}$.

On the left-hand side in inequality (4.11), we introduce the terms $\left\|\frac{\partial^2 u}{\partial x_0^2}\right\|^2_{L_2(Q^{(\tau)})}$, $\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L_2(\Omega)}(\tau)$, $i=0,\dots,n$, and $\|u\|_{L_2(\Omega)}(\tau)$. To this end, consider the expressions

$$\mathfrak{L}u\frac{\partial^2 u}{\partial x_0^2}=\left(\frac{\partial^2 u}{\partial x_0^2}\right)^2+(A^{(1)}u+B^{(1)}u-A^{(0)}u-B^{(0)}u)\frac{\partial^2 u}{\partial x_0^2}, \qquad (4.12)$$

$$\frac{\partial}{\partial x_0}\left(\frac{\partial u}{\partial x_i}\right)^2=2\,\frac{\partial^2 u}{\partial x_0\,\partial x_i}\,\frac{\partial u}{\partial x_i},\qquad i=0,\dots,n,\qquad \frac{\partial}{\partial x_0}\,u^2=2u\frac{\partial u}{\partial x_0}. \qquad (4.13)$$

We integrate relation (4.12) over the domain $Q^{(\tau)}$ and obtain the corresponding estimates with the use of the Cauchy–Schwarz inequality. As a result, we obtain

$$\frac{1}{2}\left\|\frac{\partial^2 u}{\partial x_0^2}\right\|^2_{L_2(Q^{(\tau)})}\le\frac{1}{2}\int\limits_{Q^{(\tau)}}|A^{(0)}u+B^{(0)}u-A^{(1)}u-B^{(1)}u|^2\,d\boldsymbol{x}$$
$$\le c^{(8)}\|u\|^2_{H^2(Q^{(\tau)})}+c^{(9)}(\|B^{(0)}u\|^2_{L_2(\Omega)}$$
$$+\|B^{(1)}u\|^2_{L_2(\Omega)})(t)+c^{(10)}\|\boldsymbol{L}^{(s)}u\|^2_{\boldsymbol{H}^{(s)}}. \qquad (4.14)$$

Likewise, by integrating relation (4.13) over the domain $Q^{(\tau)}$, we derive relations that readily imply the inequalities

$$\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L_2(\Omega)}\le\left\|\frac{\partial^2 u}{\partial x_0\partial x_i}\right\|^2_{L_2(Q^{(\tau)})}+\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L_2(Q^{(\tau)})}+\left\|\frac{\partial}{\partial x_i}l_0u\right\|^2_{L_2(\Omega)},\qquad i=0,\dots,n, \qquad (4.15)$$

$$\|u\|^2_{L_2(\Omega)}\le\left\|\frac{\partial u}{\partial x_0}\right\|^2_{L_2(Q^{(\tau)})}+\|u\|^2_{L_2(Q^{(\tau)})}+\|l_0u\|^2_{L_2(\Omega)}. \qquad (4.16)$$

We add inequalities (4.14)–(4.16) multiplied by appropriate constants to inequality (4.11). For example, we multiply inequalities (4.15) and (4.16) by $2c^{(2)}/\varepsilon^{(2)}+c^{(1)}c^{(11)}$; as a result, we obtain the new inequality

$$\frac{1}{2}\left\|\frac{\partial^2 u}{\partial x_0^2}\right\|^2_{L_2(Q^{(\tau)})}+c^{(11)}\|A^{(0)}u\|^2_{L_2(\Omega)}(\tau)+\left(\frac{c^{(2)}}{\varepsilon^{(2)}}+c^{(1)}c^{(11)}\right)\left(\|u\|^2_{H^1(\Omega)}+\left\|\frac{\partial u}{\partial x_0}\right\|^2_{L_2(\Omega)}\right)(\tau)$$
$$\le c^{(12)}\|u\|^2_{H^2(Q^{(\tau)})}+c^{(13)}\|B^{(0)}u\|^2_{L_2(\Omega)}(t)+c^{(14)}\|B^{(1)}u\|^2_{L_2(\Omega)}(t)+c^{(15)}\|\boldsymbol{L}^{(s)}u\|^2_{\boldsymbol{H}^{(s)}}, \qquad (4.17)$$

where $c^{(11)}=1-\varepsilon^{(1)}-\varepsilon^{(2)}-\varepsilon^{(3)}$, $c^{(12)}$ depends on the smoothness of the coefficients of the operators $A^{(0)}$ and $A^{(1)}$ as well as on constants bounding them and their derivatives, $c^{(13)}=1/\varepsilon^{(1)}+\varepsilon^{(4)}$, and

$c^{(14)} = 1/\varepsilon^{(3)} + \varepsilon^{(4)}$. Here the $\varepsilon^{(j)}$, $j = 1, 2, 3$, are numbers such that the constant $c^{(11)}$ is positive. By Lemma 4.1, from inequality (4.16), we obtain the relation

$$\left\| \frac{\partial^2 u}{\partial x_0^2} \right\|^2_{L_2(Q^{(\tau)})} + \left(\|u\|^2_{H^2(\Omega)} + \left\| \frac{\partial u}{\partial x_0} \right\|^2_{L_2(\Omega)} \right)(x_0) \le c^{(10)} c^{(12)} \|u\|^2_{H^2(Q^{(\tau)})} + c^{(16)} (c^{(13)} \|B^{(0)} u\|^2_{L_2(\Omega)}(t) + c^{(14)} \|B^{(1)} u\|^2_{L_2(\Omega)}(t) + c^{(15)} \|\boldsymbol{L}^{(s)} u\|^2_{\boldsymbol{H}^{(s)}}), \tag{4.18}$$

where

$$\frac{1}{c^{(16)}} = \min \left\{ \frac{1}{2}, c^{(11)} [(\alpha^{(0)})^2 - \varepsilon^{(0)}], \frac{c^{(2)}}{\varepsilon^{(2)}} \right\}, \qquad (\alpha^{(0)})^2 - \varepsilon^{(0)} > 0.$$

By applying the Gronwall inequality [10, p. 188] to inequality (4.18), we obtain a relation close to the energy inequality (4.3) to be proved,

$$\|u\|^2_B \le c^{(17)} \|\boldsymbol{L}^{(s)} u\|^2_{\boldsymbol{H}^{(s)}} + c^{(18)} \|B^{(0)} u\|^2_{L_2(\Omega)}(t) + c^{(19)} \|B^{(1)} u\|^2_{L_2(\Omega)}(t), \tag{4.19}$$

where $c^{(17)} = c^{(15)} e^{c^{(12)} c^{(16)} T}$, $c^{(18)} = c^{(13)} c^{(16)} e^{c^{(12)} c^{(16)} T}$, and $c^{(19)} = c^{(14)} c^{(16)} e^{c^{(12)} c^{(16)} T}$. In inequalities (2.5) and (2.6), we assume that the constants $\beta^{(0)}$ and $\beta^{(1)}$ are chosen so as to ensure that

$$\beta^{(0)} c^{(18)} < 1, \qquad \beta^{(1)} c^{(19)} < 1. \tag{4.20}$$

One can readily see that if condition (4.20) is satisfied, then

$$c^{(18)} \|B^{(0)} u\|^2_{L_2(\Omega)}(t), \ c^{(19)} \|B^{(1)} u\|^2_{L_2(\Omega)}(t) \le c^{(20)} \|u\|^2_B, \tag{4.21}$$

where $c^{(20)} = \max\{\beta^{(0)} c^{(18)}, \beta^{(1)} c^{(19)}\}$, and $c^{(20)} < 1$.

Therefore, now relations (4.21) and (4.19) imply the desired energy inequality (4.3).

5. EXISTENCE AND UNIQUENESS OF STRONG SOLUTIONS OF THE MIXED PROBLEMS $\mathbf{MP}s$, $s = 1, 2, 3$

By using a criterion for the closability of an operator and by performing a straightforward verification, we show that all operators $\boldsymbol{L}^{(s)} : B^{(s)}(Q) \to \boldsymbol{H}^{(s)}$, $s = 1, 2, 3$, admit closures $\overline{\boldsymbol{L}^{(s)}}$. In what follows, we consider the operators $\overline{\boldsymbol{L}^{(s)}}$ [10].

Definition 5.1. The solutions of the operator equations

$$\overline{\boldsymbol{L}^{(s)}} u = \boldsymbol{F}^{(s)}, \qquad \boldsymbol{F}^{(s)} \in \boldsymbol{H}^{(s)}, \qquad s = 1, 2, 3, \tag{5.1}$$

are referred to as *strong solutions*, or, in our case of the spaces $B^{(s)}(Q)$, as *generalized classical solutions* of the mixed problems $\mathbf{MP}s$, $s = 1, 2, 3$.

By using a passage to the limit, for the operators $\overline{\boldsymbol{L}^{(s)}}$, $s = 1, 2, 3$, from the energy inequality (4.3), we obtain the energy inequality

$$\|u\|_B \le c \|\overline{\boldsymbol{L}^{(s)}} u\|_{\boldsymbol{H}^{(s)}} \tag{5.2}$$

for arbitrary $u \in \mathfrak{D}(\overline{\boldsymbol{L}^{(s)}})$, where c is the same positive constant as in inequality (4.3).

Theorem 5.1. *Assume that Condition* 3.1 *is satisfied and the coefficients of Eq.* (2.1) *satisfy the smoothness conditions in Section* 2. *Then for arbitrary functions* $f \in \mathring{\mathfrak{H}}^1(Q)$, $\varphi \in H^2(\Omega, (3.s))$, *and* $\psi \in H^1(\Omega, (3.s))$, $s = 1, 2, 3$, *there exists a unique generalized-classical solution* $u \in B^{(s)}(Q)$ *of the corresponding mixed problem* $\mathbf{MP}s$ $(s = 1, 2, 3)$, *and the following estimate holds*:

$$\|u\|_B \le c(\|f\|_{\mathfrak{H}^1(Q)} + \|\varphi\|_{H^2(\Omega)} + \|\psi\|_{H^1(\Omega)}). \tag{5.3}$$

Proof. The uniqueness and the estimate (5.3) for a generalized-classical solution of each mixed problem $\mathbf{MP}s$, if the solution exists at all, follow from the energy inequality (5.2).

It is well known (see the proof of Theorem 3.2.2 in [10]) that, by Definition 5.1, the proof of the existence of a strong solution can be reduced to proving the relation $\mathfrak{R}(\overline{\boldsymbol{L}^{(s)}}) = \overline{\mathfrak{R}(\boldsymbol{L}^{(s)})}$, i.e., to proving that the range $\mathfrak{R}(\boldsymbol{L}^{(s)})$ of the operator $\boldsymbol{L}^{(s)}$ is dense in the space $\boldsymbol{H}^{(s)}$. The proof of this assertion can be reduced to the following statement: for each $s = 1, 2, 3$, the relation

$$(\boldsymbol{L}^{(s)}u, \boldsymbol{v})_{\boldsymbol{H}^{(s)}} = (\mathfrak{L}u, v)_{\mathfrak{H}^1(Q)} + (l_0u, v^{(0)})_{H^2(\Omega)} + (l_1u, v^{(1)})_{H^1(\Omega)} = 0 \qquad (5.4)$$

holds for any function $u \in \mathfrak{D}(\boldsymbol{L}^{(s)})$ if and only if $v = 0$ in $\mathring{\mathfrak{H}}^1(Q)$, $v^{(0)} = 0$ in $H^2(\Omega; (3.s))$, and $v^{(1)} = 0$ in $H^1(\Omega; (3.s))$.

By $\mathfrak{L}_0$ we denote the leading part of the operator $\mathfrak{L}$; i.e., $\mathfrak{L}_0 = \partial^2/\partial x_0^2 - A^{(0)} - B^{(0)}$. In addition, by $\mathring{\mathfrak{D}}(\boldsymbol{L}^{(s)})$ we denote the subset of $\mathfrak{D}(\boldsymbol{L}^{(s)})$ whose elements u satisfy the condition $l_0u = l_1u = 0$. In our notation, we have $\mathring{\mathbf{L}}_0^{(s)} = \{\mathring{\mathfrak{L}}_0, l_0, l_1\}$.

Now, in a special case, Eq. (5.4) can be represented in the form

$$(\mathfrak{L}_0u, v)_{\mathfrak{H}^1(Q)} = (\mathfrak{L}_0u, v)_{L_2(Q)} + \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}\mathfrak{L}_0u, \frac{\partial}{\partial x_j}v\right)_{L_2(Q)} = 0, \qquad (5.5)$$

where u is an arbitrary function in $\mathring{\mathfrak{D}}(\mathring{\mathbf{L}}_0^{(s)})$. The values of $\mathfrak{L}_0u$ and $\dfrac{\partial}{\partial x_j}\mathfrak{L}_0u$ in relation (5.5) are linearly independent. Therefore, if the sets $\{\mathfrak{L}_0u|\ u \in \mathring{\mathfrak{D}}(\mathring{\mathbf{L}}_0^{(s)})\}$ and $\left\{\dfrac{\partial}{\partial x_j}\mathfrak{L}_0u|\ u \in \mathring{\mathfrak{D}}(\mathring{\mathbf{L}}_0^{(s)})\right\}$, $j = 1, \dots, n$, are dense in $L_2(Q)$, then from relation (5.5), we obtain orthogonality conditions for each term,

$$(\mathfrak{L}_0u, v)_{L_2(Q)} = 0, \qquad \left(\frac{\partial}{\partial x_j}\mathfrak{L}_0u, \frac{\partial}{\partial x_j}v\right)_{L_2(Q)} = 0, \qquad j = 1, \dots, n. \qquad (5.6)$$

By Assertion 3.4.5 in [10], instead of relations (5.6), one can consider the equivalent relations

$$(J_{(k)}\mathfrak{L}_0u, v)_{L_2(Q)} = 0, \qquad \left(J_{(k)}\frac{\partial}{\partial x_j}\mathfrak{L}_0u, w^{(j)}\right)_{L_2(Q)} = 0, \qquad j = 1, \dots, n, \qquad (5.7)$$

which hold for any function $u \in \mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)})$ and for some elements v and $w^{(j)}$ of the space $L_2(Q)$. Here the $J_{(k)}$ are the corresponding averaging operators with variable step [10, 11–13], which take into account the boundary conditions of the considered problems $\mathbf{MP}s$, $s = 1, 2, 3$, on the boundary ∂Q of the domain Q.

Therefore, the subsequent proof of Theorem 5.1 is based on proving that the sets $\{J_{(k)}\mathfrak{L}_0u|\ u \in \mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)})\}$ and $\left\{J_{(k)}\dfrac{\partial}{\partial x_j}\mathfrak{L}_0u|\ u \in \mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)})\right\}$, $j = 1, \dots, n$, are dense in the space $L_2(Q)$. We state this result in the form of a lemma.

Lemma 5.1. *Let the assumptions of Theorem* 5.1 *be satisfied. Then relations* (5.7) *hold for the elements v and $w^{(j)}$, $j = 1, \dots, n$, of the space $L_2(Q)$ and for arbitrary functions $u \in \mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)})$ if and only if $v = w^{(j)} = 0$ in $L_2(Q)$.*

We prove Lemma 5.1 after finishing the proof of Theorem 5.1.

Therefore, by using Assertion 3.4.5 from [10], from Lemma 5.1, we find that the sets $\{\mathfrak{L}_0u|\ u \in \mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)})\}$ and $\left\{\dfrac{\partial}{\partial x_j}\mathfrak{L}_0u|\ u \in \mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)})\right\}$, $j = 1, \dots, n$, are dense in $L_2(Q)$. Consequently, as was mentioned above, from relation (5.5), we obtain relation (5.6), and from Lemma 5.1 and relations (5.6), we obtain $v = 0$ and $\dfrac{\partial}{\partial x_j}v = 0$; i.e., $v = 0$ in $\mathfrak{H}^1(Q)$.

Since $v=0$ in $\mathfrak{H}^1(Q)$, it follows that the representation (5.5) holds, and relation (5.4) acquires the form

$$(l_0u, v^{(0)})_{H^2(\Omega)} + (l_1u, v^{(1)})_{H^1(\Omega)} = 0$$

for any function $u \in \mathfrak{D}(\boldsymbol{L}^{(s)})$. Since the operators l_0, l_1, $\dfrac{\partial}{\partial x_j} l_k$, and $\dfrac{\partial^2}{\partial x_j \partial x_k} l_0$, $j,k = 1,\ldots,n$, are linearly independent and their ranges are dense in $L_2(\Omega)$ for $\mathfrak{D}(l_i) = \mathfrak{D}(\boldsymbol{L}^{(3)})$, we have $v^{(0)} = 0$ in $H^2(\Omega;(3.s))$ and $v^{(1)} = 0$ in $H^1(\Omega;(3.s))$, $s=1,2,3$.

In the general case of the operator $\boldsymbol{L}^{(s)}$, one can prove the relation $\overline{\mathfrak{R}(\boldsymbol{L}^{(s)})} = H$ for the formally adjoint problem with the use of averaging operators with variable step or the method of continuation with respect to a parameter (see [10, Th. 3.2.3]).

Proof of Lemma 5.1. Consider the first relation in (5.7). If $v=0$, then, obviously, this relation is true.

Let us prove the converse. Let the first relation in (5.7) be expressed for some $v \in L_2(Q)$, where u is an arbitrary function in $\mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)})$. We represent it in the form

$$(\mathfrak{L}_0 u, J^*_{(k)} v)_{L_2(Q)} = 0, \tag{5.8}$$

where $J^*_{(k)}$ is the operator adjoint to $J_{(k)}$ with variable step [10, p. 45; 12, p. 45]. By integrating by parts on the left-hand side in relation (5.8), we obtain the relation

$$(u, \mathfrak{L}_0 J^*_{(k)} v)_{L_2(Q)} = \mathfrak{M}(u, v; \partial Q), \tag{5.9}$$

where

$$\begin{aligned}
\mathfrak{M}(u,v;\partial Q) = {} & \left(\frac{\partial u}{\partial x_0}(T,\boldsymbol{x}'), J^*_{(k)} v(T,\boldsymbol{x}')\right)_{L_2(\Omega)} - \left(u(T,\boldsymbol{x}'), \frac{\partial}{\partial x_0} J^*_{(k)} v(T,\boldsymbol{x}')\right)_{L_2(\Omega)} \\
& - (l_1 u, J^*_{(k)} v(0,\boldsymbol{x}'))_{L_2(\Omega)} + \left(l_0 u, \frac{\partial}{\partial x_0} J^*_{(k)} v(0,\boldsymbol{x}')\right)_{L_2(\Omega)} - \int_0^T\!\!\int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{N}}\bigg|_P J^*_{(k)} v|_\Gamma \, dx_0\, ds \\
& + \int_0^T\!\!\int_{\partial\Omega} u|_P \frac{\partial}{\partial \boldsymbol{N}} J^*_{(k)} v|_\Gamma \, dx_0\, ds - T\int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{B}}\bigg|_{\partial\Omega}(t) J^*_{(k)} v|_{\partial\Omega}(t)\, ds \\
& + T\int_{\partial\Omega} u|_{\partial\Omega}(t) \frac{\partial}{\partial \boldsymbol{B}} J^*_{(k)} v|_{\partial\Omega}(t)\, ds,
\end{aligned} \tag{5.10}$$

$$\frac{\partial u}{\partial \boldsymbol{B}}\bigg|_{\partial\Omega}(t) = \sum_{i,j=1}^n b^{(ij)}(t,\boldsymbol{x}') \left(\frac{\partial u}{\partial x_j}\nu_i\right)(t)\bigg|_{\partial\Omega},$$

$$\frac{\partial}{\partial \boldsymbol{B}} J^*_{(k)} v|_{\partial\Omega}(t) = \sum_{i,j=1}^n b^{(ij)}(t,\boldsymbol{x}') \left(\frac{\partial}{\partial x_j} J^*_{(k)} v \nu_i\right)(t)\bigg|_{\partial\Omega}.$$

Since Q and ∂Q occurring in relation (5.9) are two distinct sets over which the integration is performed, it follows that, by virtue of the arbitrary choice of the functions $u \in \mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)})$, we obtain two equations

$$(u, \mathfrak{L}_0 J^*_{(k)} v)_{L_2(Q)} = 0, \tag{5.11}$$

$$\mathfrak{M}(u,v;\partial Q) = 0. \tag{5.12}$$

By varying the functions u in the sets $\mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)})$, from (5.12), we obtain adjoint conditions for the function $J^*_{(k)} v$. These conditions depend on conditions (2.5)–(2.7), i.e., on the considered problems **MP**s, $s = 1, 2, 3$. It follows from relations (5.10) and (2.5)–(2.7) that relation (5.12) holds if and only if the function $v \in L_2(Q)$ averaged by the operator $J^*_{(k)}$ satisfies the boundary conditions

$$J^*_{(k)} v|_{x_0=T} = \frac{\partial}{\partial x_0} J^*_{(k)} v|_{x_0=T} = J^*_{(k)} v(t)|_{\partial\Omega} = \frac{\partial}{\partial \boldsymbol{B}} J^*_{(k)} v(t)|_{\partial\Omega} = 0 \tag{5.13}$$

for all considered problems **MP**s, $s = 1, 2, 3$. In addition to these conditions on Γ, depending on conditions (2.5)–(2.7), we additionally consider one of the following conditions $(s = 1, 2, 3)$:

$$J^*_{(k)} v|_\Gamma = 0, \tag{5.14}$$

$$\frac{\partial}{\partial \boldsymbol{N}} J^*_{(k)} v|_\Gamma = 0, \tag{5.15}$$

$$\frac{\partial}{\partial \boldsymbol{N}} J^*_{(k)} v|_\Gamma = J^*_{(k)} v = 0 \tag{5.16}$$

in the case of problems **MP**1–**MP**3, respectively.

By using the passage to the limit, we generalize relation (5.11) to any function $u \in L_2(Q)$, because the set $\mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)})$ is dense in $L_2(Q)$. By $\widetilde{Q}^{(\tau)}$ we denote the complement in the domain Q of the subdomain $Q^{(\tau)}$. In relation (5.11), set

$$u(\boldsymbol{x}) = \begin{cases} \dfrac{\partial}{\partial x_0} J^*_{(k)} v(\boldsymbol{x}) & \text{for} \quad \boldsymbol{x} \in \tilde{Q}^{(\tau)}, \\ 0 & \text{for} \quad \boldsymbol{x} \in Q^{(\tau)}. \end{cases}$$

As a result, we obtain the relation

$$\left(\frac{\partial}{\partial x_0} J^*_{(k)} v, \mathfrak{L}_0, J^*_{(k)} v \right)_{L_2(\tilde{Q}^{(\tau)})} = 0.$$

Next, by following the proof of the energy inequality (4.3) and by using conditions (5.13)–(5.16), we obtain $\|J^*_{(k)} v\|_{L_2(Q)} = 0$. Since $J^*_{(k)} v \to v$ in $L_2(Q)$ as $k \to \infty$, it follows that $v = 0$. We have thereby shown that the set $\{\mathfrak{L}_0 |\ u \in \mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)})\}$ is dense in $L_2(Q)$. Therefore, the sets

$$\left\{ \frac{\partial}{\partial x_j} \mathfrak{L}_0 |\ u \in \mathring{\mathfrak{D}}(\boldsymbol{L}_0^{(s)}) \right\}$$

are dense in $L_2(Q)$ for all $j = 1, \ldots, n$ as well. This follows from the properties of the operator $\partial/\partial x_j$.

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