Nonlinear Phenomena in Complex Systems, vol. **19**, no. 4 (2016), pp. 378 - 386

On Separation of Variables into Relative and Center of Mass Motion for Two-Body System in Three–Dimensional Spaces of Constant Curvature

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(Received 30 March, 2016)

Expressions for variables of the center of mass and relative motions for two-body system with different and equal masses in three–dimensional spaces of constant curvature are introduced in the terms of biquaternions. The problem of separation of the center mass and relative motion variables for action of two particles into biquaternionic form is formulated. We showed that the algebraic nature of these nonseparable variables follows from the fact that the algebra of biquaternions is noncommutative. Some special cases of separation of the center mass and relative motion variables are considered.

PACS numbers: 03.65.-w
Keywords: three–dimensional spaces of constant curvature, biquaternions, two-body system, separation of variables

1. Introduction

Quantum-mechanical problem of the particle motion in the field of force defined by central potential on three-dimensional spherical space S_3 (positive constant curvature) has been first considered by E. Schrödinger[1] and A. Stivenson [2]. The similar problem in the three-dimensional Lobachevsky space ${}^1S_3$ (negative constant curvature) has been solved by Infeld and Shild for thr first time [3]. Since spherical and hyperbolic spaces have different geometrical and physical properties, most authors investigate physical systems on spherical and hyperbolic spaces in a separate ways. However, we believe that for certain problems a generalized approach is more effective for both spaces.

The quantum-mechanical models based on the geometry of spaces of constant curvature have attracted considerable attention due to opportunity of their applications to physical problems as well as their interesting mathematical features (see[4–16]). For example, the model based on the Coulomb interaction on the sphere has been used for description of excited states of excitons in quantum dots (see [11, 12]). The existence of Landau levels for the moving charged particle in curved space were studied in [17, 18]. Among other applications of non-Euclidean geometry in theoretical physics we can highlight the usage of hyperbolic geometry to solve problems of relativistic kinematics [19] and the theory of relativistic nuclear collisions [20]. Thus, the technique of non-Euclidean geometry is useful and effective for a wide range of kinematic and dynamic problems.

Also, we note that studying of quantum mechanics on spaces of constant curvature is important, not only for a development of our

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knowledge of certain fundamental features of quantum mechanics, but also for convenient construction of generalized relativistic theory.

The problem of two interacting particles in spaces of constant curvature is much more complicated because of the fact that variables of the relative motion and center mass in Schrödinger equation can't be separated even for a central potential of interaction [21–23]. In the initial stage there is a problem with the definition of the center mass in spaces of constant curvature (see [21]). However, the use of vectors of spaces constant curvature defined in [24] provides the opportunity to introduce the mentioned variables by analogy with four-velocity (four-momentum) space in the relativistic kinematics. Such approach opens obvious methodical advantages and facilitates a search for some special cases in which complete separation of variables is possible that is important for practical sense.

2. Coordinates of particles on three-dimensional sphere and Lobachevsky space

Three-dimensional spaces of constant curvature of Riemann and Lobachevsky can be embedded into the four-dimensional Euclidean and pseudo-Euclidean space described by biquaternions correspondingly as

$$X = iX_0 + \underline{X}, \text{ where } i^2 = \pm 1. \tag{2.1}$$

The case $i^2 = 1$ i.e. biquaternions over double numbers corresponds to Euclidean space, and $i^2 = -1$ (biquaternions over complex numbers) corresponds to pseudo-Euclidean space.

It should be noted that below in all formulas the upper sign will refer to S_3 space while the lower sign to ${}^1S_3$ space.

Well known that the method of the unified description of the geometry of three-dimensional spaces of constant curvature belongs to Clifford. This method is provided by choice of special form of the biquaternions $X = -\bar{X}^*$, known as biquaternions of Minkovsky type in Synge's terminology in case of the pseudo-Euclidean four-space. Here $\bar{X} = iX_0 - \underline{X}$ is a biquaternion conjugate of a biquaternion X and the symbol * denotes conjugate in system of double and complex numbers correspondingly.

These biquaternions obey the standard multiplicative rule

$$\begin{aligned} Y &= (iX_0' + \underline{X}')(iX_0 + \underline{X}) \\ &= \pm X_0' X_0 - (\underline{X}\ \underline{X}') + iX_0'\underline{X} + iX_0\underline{X}' + [\underline{X}'\ \underline{X}], \end{aligned} \tag{2.2}$$

where underline denotes a three-dimensional vector in right part of (2.2), parentheses means scalar product of three-dimensional vectors and square brackets are cross vector of these vectors.

The uniform equation of three-dimensional surfaces on which realized Riemannian space with constant curvature $1/R^2$ or Lobachevsky space with constant curvature $-1/R^2$ can be written into biquaternionic form as

$$X\ \bar{X} = X_1^2 + X_2^2 + X_3^2 \pm X_0^2 = \pm R^2. \tag{2.3}$$

We require the condition $X_0 > 0$ for Lobachevsky space.

The expression (2.3) is invariant under transformation

$$X' = AX\bar{A}^*, \tag{2.4}$$

where $A\bar{A} = 1$, $A^*\bar{A}^* = 1$. The set of biquaternions (2.1) is invariant under the transformation groups of motions (2.4) in considered spaces. Transformations (2.4) generate the $SO(4,R)$, $SO(3,1)$ groups correspondingly.

Below we take $R = 1$ for the practical reasons.

Let us consider the motion of two noninteracting particles in both spaces (S_3 and ${}^1S_3$). The particles coordinates in embedding four-dimensional space are represented as components of biquaternions

$$X^{(1)} = iX_0^{(1)} + \underline{X}^{(1)}, \quad X^{(2)} = iX_0^{(2)} + \underline{X}^{(2)}. \tag{2.5}$$

Using condition (2.3) we have

$$X^{(1)}\ \bar{X}^{(1)} = \pm 1, \quad X^{(2)}\ \bar{X}^{(2)} = \pm 1. \tag{2.6}$$

These coordinates of particles are not independent.

We will use the Beltrami coordinates which are components of vectors on sphere as independent coordinates

$$\underline{q}^{(1)} = \pm i \frac{\underline{X}^{(1)}}{X_0^{(1)}}, \ \underline{q}^{(2)} = \pm i \frac{\underline{X}^{(2)}}{X_0^{(2)}}. \tag{2.7}$$

For a demonstration of the certain analogy between three-dimensional description of the considered curvature spaces and three-dimensional Euclidean space we give the formula for coordinates of particles $X^{(1)}, X^{(2)}$, which are expressed in terms of the Beltrami coordinates

$$X^{(1)} = \frac{1 + q^{(1)}}{\sqrt{1 + \underline{q}^{(1)}\underline{\bar{q}}^{(1)}}}, \ X^{(2)} = \frac{1 + q^{(2)}}{\sqrt{1 + \underline{q}^{(2)}\underline{\bar{q}}^{(2)}}}. \tag{2.8}$$

We note that definition of vectors (2.7) automatically leads to an identification of opposite points on a sphere and thereby these vectors belong to elliptic space. Therefore the usage of vectors (2.7) for description of the particles motion on the sphere demands to take into account this property. However we can use biquaternions (2.5) to avoid these difficulties. Nevertheless, in parallel with quaternionic variables we will use vectors (2.7) since the problem for elliptic space has independent value.

3. Variables of the center of mass and relative motions for two particle system in three-dimensional spaces of constant curvature

The coordinates of the center of mass for two particles with masses m_1 and m_2 in a biquaternionic form can be written as

$$X_c = \frac{m_1 X^{(1)} + m_2 X^{(2)}}{\sqrt{\pm\left(m_1 X^{(1)} + m_2 X^{(2)}\right)\left(m_1 \bar{X}^{(1)} + m_2 \bar{X}^{(2)}\right)}}. \tag{3.1}$$

We note that in this case three-dimensional coordinates (2.7) of the center of mass are components of vector

$$\underline{q}_c = \pm i \frac{\underline{X}_c}{X_{0c}} = \pm i \frac{m_1 \underline{X}^{(1)} + m_2 \underline{X}^{(2)}}{m_1 X_0^{(1)} + m_2 X_0^{(2)}}, \ X_c = \frac{1 + \underline{q}_c}{\sqrt{1 + \underline{q}_{\,c}\, \underline{\bar{q}}_{\,c}}}. \tag{3.2}$$

This expression in variables (2.7) has the form

$$\underline{q}_c = \frac{m_1 \underline{q}^{(1)}/\sqrt{1 + \underline{q}^{(1)}\underline{\bar{q}}^{(1)}} + m_2 \underline{q}^{(2)}/\sqrt{1 + \underline{q}^{(2)}\underline{\bar{q}}^{(2)}}}{m_1/\sqrt{1 + \underline{q}^{(1)}\underline{\bar{q}}^{(1)}} + m_2/\sqrt{1 + \underline{q}^{(2)}\underline{\bar{q}}^{(2)}}}. \tag{3.3}$$

It is easy to see that expression (3.3) for coordinates of the center of mass coincides with correspondence expression for three-dimensional flat space where masses m_1, m_2 replaced by $m_1 \to m_1/\sqrt{1 + \underline{q}^{(1)}\underline{\bar{q}}^{(1)}}$, $m_2 \to m_2/\sqrt{1 + \underline{q}^{(2)}\underline{\bar{q}}^{(2)}}$.

We note that the feature of the definition of the center mass in our formalism in contrast with other authors ([21]) is manifestly covariant under group of transformations (2.4).

In the paper [25] it was first shown that the model based on the space with conformally flat geometry with the similar coordinate dependence of the mass can be used for an explanation of quark confinement. At the same time quantum-mechanical model based on the geometry of the three-dimensional sphere S_3 has been used for description of excitations in dimensional quantum dot in the work [11] This model provides confinement of quasiparticles and it is interpreted in terms of geometry of three-dimensional Euclidian spaces.

Biquaternionic analog of the variables of relative motion for two-body system is

$$Y_{12} = \pm X^{(2)}\ \bar{X}^{(1)}, \tag{3.4}$$

defined from

$$X^{(2)} = Y_{12}\ X^{(1)}. \tag{3.5}$$

The independent three-dimensional coordinates of the relative motion defined as components of relative motion vector:

$$q_y = \frac{Y_{12} - \bar{Y}_{12}}{Y_{12} + \bar{Y}_{12}} = \left\langle \pm i\frac{\underline{X}^{(2)}}{X_0^{(2)}},\ \mp i\frac{\underline{X}^{(1)}}{X_0^{(1)}} \right\rangle = \left\langle \underline{q}^{(2)},\ -\underline{q}^{(1)} \right\rangle, \tag{3.6}$$

where angle brackets mean addition rule for three-dimensional vectors in Riemannian and Lobachevsky spaces

$$\underline{q}'' = \langle \underline{q}, \underline{q}' \rangle = \frac{\underline{q} + \underline{q}' + [\underline{q}\ \underline{q}']}{1 - (\underline{q}\ \underline{q}')}. \tag{3.7}$$

The formula (3.7) is an algebraic expression for addition of vectors in the elliptic and Lobachevsky three-dimensional spaces or a triangle rule (see Appendix A or more detailed in [24, 26, 27]). Three vectors over double numbers correspond to ordered pairs of points (or direct lines) in the elliptic three-space. Three vectors over complex numbers correspond to ordered pairs of points (or direct lines) in the extended three-dimensional Lobachevsky space. The geometry of spaces of Riemann and Lobachevsky can be derived from properties of vectors (3.6) and formula (3.7).

Now let us introduce four-dimensional Y_1, Y_2 and three-dimensional $q_y^{(1)}$, $q_y^{(2)}$ coordinates with respect to center of mass that are defined identically (3.4) and (3.5) as

$$X^{(1)} = Y_1\ X_c,\ \bar{X}^{(1)} = \bar{X}_c\ \bar{Y}_1 \tag{3.8}$$

meanwhile

$$Y_1 = \pm X^{(1)} \bar{X}_c, \tag{3.9}$$

and respectively

$$X^{(2)} = Y_2\ X_c,\ \bar{X}^{(2)} = \bar{X}_c\ \bar{Y}_2, \tag{3.10}$$

$$Y_2 = \pm X^{(2)} \bar{X}_c. \tag{3.11}$$

It is clear that

$$Y_{12} = Y_2\ \bar{Y}_1. \tag{3.12}$$

Then for the first particle we have

$$\underline{q}^{(1)} = \pm i\frac{\underline{X}^{(1)}}{X_0^{(1)}} = \left\langle \frac{\underline{q}_y}{1 + \frac{m_1}{m_2}\sqrt{1 + \underline{q}_y\ \bar{\underline{q}}_y}},\ \underline{q}_c \right\rangle = \left\langle \underline{q}_y^{(1)}, \underline{q}_c \right\rangle, \tag{3.13}$$

and for the second particle

$$\underline{q}^{(2)} = \pm i\frac{\underline{X}^{(2)}}{X_0^{(2)}} = \left\langle \frac{\underline{q}_y}{1 + \frac{m_2}{m_1}\sqrt{1 + \underline{q}_y\ \bar{\underline{q}}_y}},\ \underline{q}_c \right\rangle = \left\langle \underline{q}_y^{(2)}, \underline{q}_c \right\rangle. \tag{3.14}$$

From formula (3.12) follows

$$\underline{q}_y =< q_2,\ -q_1 >= \left\langle \underline{q}_y^{(2)},\ -\underline{q}_y^{(1)} \right\rangle. \tag{3.15}$$

The introduced variables satisfy the following conditions

$$X_c\ \bar{X}_c = \pm 1,\ Y_{12}\ \bar{Y}_{12} = 1,\ Y_1\ \bar{Y}_1 = 1,\ Y_2\ \bar{Y}_2 = 1. \tag{3.16}$$

4. Classical non-relativistic problem. The separation of variables in an action.

An action of the two-body problem, which forces of interaction depend on the relative variable, in both spaces can be written as

$$W_{12} = \int L dt = \int \left[\frac{1}{2} \left(m_1 \dot{X}^{(1)} \dot{\bar{X}}^{(1)} + m_2 \dot{X}^{(2)} \dot{\bar{X}}^{(2)} \right) - V(Y_{12}) \right] dt. \tag{4.1}$$

Here L is Lagrange function and the dot over symbols indicate the time derivative.

The expression (4.1) will have a standard form if we substitute independent variables $\underline{q}^{(1)}$ and $\underline{q}^{(2)}$. In this case we obtain

$$W = \int \left[\frac{1}{2} \left(m_1 g_{ab}(\underline{q}^{(1)}) \dot{q}_a^{(1)} \dot{q}_b^{(1)} + m_2 g_{ab}(\underline{q}^{(2)}) \dot{q}_a^{(2)} \dot{q}_b^{(2)} \right) - \phi(q_{12}) \right] dt, \tag{4.2}$$

where

$$g_{ab} = \frac{1}{(1+q^2)} \left(\delta_{ab} + \frac{q_a q_b}{1+q^2} \right) \tag{4.3}$$

is a metric tensor of a sphere written in the variables which are components of vectors on sphere. However, feature of this work is that we use action (4.1) written in terms of four-dimensional biquaternionic variables taking into account the additional conditions (2.6) and (3.16).

Taking into account expressions (3.8), (3.10) for $X^{(1)}$, $X^{(2)}$ and condition (3.16) the expression (4.1) can be transformed into

$$W = \frac{1}{2} \int \left[m_1 \left(\pm \dot{Y}_1 \dot{\bar{Y}}_1 \pm \dot{X}_c \dot{\bar{X}}_c \right) + m_2 \left(\pm \dot{Y}_2 \dot{\bar{Y}}_2 \pm \dot{X}_c \dot{\bar{X}}_c \right) + m_1 \left(Y_1 \dot{X}_c \bar{X}_c \dot{\bar{Y}}_1 + \dot{Y}_1 X_c \dot{\bar{X}}_c \bar{Y}_1 \right) \right.$$
$$\left. + m_2 \left(Y_2 \dot{X}_c \bar{X}_c \dot{\bar{Y}}_2 + \dot{Y}_2 X_c \dot{\bar{X}}_c \bar{Y}_2 \right) + 2V(Y_{12}) \right] dt, \tag{4.4}$$

Variables of the center mass and relative motions in the first two terms of expression of a kinetic part (4.4) are separated. The following terms have variables both relative motion and center of mass. The separation of variables in these terms is impossible because of noncommutative biquaternions. Thus, the nature of inseparable variables of the center mass and relative motions can be explained by non-commutative biquaternions in algebraic sense.

5. Some special cases

Nevertheless, formula (4.4) is very convenient for searching of some special cases where complete separation of variables of relative motion and the motion of the center of mass is possible. For example, it is clear that we can separate variables when $\dot{X}_c = 0$. In this case two massive points move along two-dimensional spheres and lie on the same straight line (geodesic line) round the general center of mass. This case known as the dumbbell-shaped figure [21]. The distance from general center to points defined by masses ratio.

Let us rewrite a kinetic part of subintegral function (4.4) using operator Y_{12} (3.4) as follows

$$
\begin{aligned}
L' = &\pm \frac{m_1 m_2 \dot{Y}_{12} \dot{\bar{Y}}_{12} (m_1+m_2)}{m_1^2+m_2^2+2m_1m_2Y_{12(0)}} \mp \frac{m_1^2 m_2^2 \dot{Y}_{12(0)}^2 (m_1+m_2)}{\left(m_1^2+m_2^2+2m_1m_2Y_{12(0)}\right)^2} \\
&\pm (m_1+m_2)\dot{X}_c \dot{\bar{X}}_c + \frac{m_1m_2}{m_1^2+m_2^2+2m_1m_2Y_{12(0)}} \Big[m_1 \Big(Y_{12}\dot{X}_c\bar{X}_c\dot{\bar{Y}}_{12} \\
&+\dot{Y}_{12}X_c\dot{\bar{X}}_c\bar{Y}_{12} \Big) + m_2 \Big(\bar{Y}_{12}\dot{X}_c\bar{X}_c\dot{Y}_{12} + \dot{\bar{Y}}_{12}X_c\dot{\bar{X}}_cY_{12} \Big) \Big] \\
&+ \frac{m_1m_2}{m_1^2+m_2^2+2m_1m_2Y_{12(0)}} \Big[m_1 \Big(\dot{X}_c\bar{X}_c\dot{Y}_{12} + \dot{\bar{Y}}_{12}X_c\dot{\bar{X}}_c \Big) + m_2 \Big(\dot{X}_c\bar{X}_c\dot{\bar{Y}}_{12} + \dot{Y}_{12}X_c\dot{\bar{X}}_c \Big) \Big]. \qquad (5.1)
\end{aligned}
$$

Setting $Y_{12} = \cos r + \underline{n} \sin r$, where r is a distances between points and $n^2 = n^{2*} = 1$ for Riemannian space we find

$$
\begin{aligned}
L' = &\frac{m_1m_2(m_1+m_2)}{F} \Big(\dot{r}^2 + \underline{\dot{n}}^2 \sin^2 r + (\underline{\dot{n}}\dot{X}_c\bar{X}_c\underline{n} - \underline{n}\dot{X}_c\bar{X}_c\underline{\dot{n}}) \sin^2 r \Big) \\
&- \frac{m_1^2m_2^2(m_1+m_2)\sin^2 r\dot{r}^2}{F^2} + (m_1+m_2)\dot{X}_c\dot{\bar{X}}_c \\
&+ \frac{m_1m_2(m_1-m_2)}{F} \Big[(\underline{n}\dot{X}_c\bar{X}_c + \dot{X}_c\bar{X}_c\underline{n})\dot{r} + (\underline{\dot{n}}\dot{X}_c\bar{X}_c - \dot{X}_c\bar{X}_c\underline{\dot{n}}) \sin r \cos r \\
&- (\underline{n}\dot{X}_c\bar{X}_c + \dot{X}_c\bar{X}_c\underline{n})\dot{r}\cos r - (\underline{\dot{n}}\dot{X}_c\bar{X}_c + \dot{X}_c\bar{X}_c\underline{\dot{n}}) \sin r \Big] \qquad (5.2)
\end{aligned}
$$

where $F = m_1^2 + m_2^2 + 2m_1m_2\cos r$.

In the case of equal masses $m_1 = m_2 = m$ we get

$$
L' = m \left[2\sin^2\frac{r}{2} \left(\underline{\dot{n}}^2 + \underline{\dot{n}}\dot{X}_c\bar{X}_c\underline{n} - \underline{n}\dot{X}_c\bar{X}_c\underline{\dot{n}} \right) + \frac{\dot{r}^2}{2} + 2\dot{X}_c\dot{\bar{X}}_c \right]. \qquad (5.3)
$$

By analogy with the Riemannian space we use the substitution $Y_{12} = \cosh r + \underline{n}\sinh r$, where $n^2 = n^{2*} = -1$ for Lobachevsky space as result we get the kinetic part of subintegral function L' as

$$
\begin{aligned}
L' = &\frac{m_1m_2(m_1+m_2)}{F_L} \Big(\dot{r}^2 - \underline{\dot{n}}^2 \sinh^2 r + (\underline{\dot{n}}\dot{X}_c\bar{X}_c\underline{n} - \underline{n}\dot{X}_c\bar{X}_c\underline{\dot{n}}) \sinh^2 r \Big) \\
&- (m_1+m_2)\dot{X}_c\dot{\bar{X}}_c + \frac{m_1^2m_2^2(m_1+m_2)\sinh^2 r\dot{r}^2}{F_L^2} \\
&+ \frac{m_1m_2(m_1-m_2)}{F} \Big[(\underline{\dot{n}}\dot{X}_c\bar{X}_c + \dot{X}_c\bar{X}_c\underline{\dot{n}}) \sinh r \cosh r \\
&+ (\underline{n}\dot{X}_c\bar{X}_c + \dot{X}_c\bar{X}_c\underline{n})\dot{r} - (\underline{n}\dot{X}_c\bar{X}_c + \dot{X}_c\bar{X}_c\underline{n})\dot{r}\cosh r - (\underline{\dot{n}}\dot{X}_c\bar{X}_c + \dot{X}_c\bar{X}_c\underline{\dot{n}}) \sinh r \Big], \qquad (5.4)
\end{aligned}
$$

where $F_L = m_1^2+m_2^2+2m_1m_2\cosh r$. The formula (5.4) may be rewritten for equal masses $m_1 = m_2 = m$ as follows

$$L' = m\left[2\sinh^2\frac{r}{2}\left(-\dot{\underline{n}}^2 + \dot{\underline{n}}\dot{X}_c\bar{X}_c\underline{n} - \underline{n}\dot{X}_c\bar{X}_c\dot{\underline{n}}\right) + \frac{\dot{r}^2}{2} - 2\dot{X}_c\dot{\bar{X}}_c\right]. \quad (5.5)$$

We note that taking into account the homogeneity of three-dimensional spaces of constant curvature S_3 and ${}^1S_3$ we can choose the center of mass as a reference point. Then we get

$$X_c = (i, \underline{0}),\ \dot{X}_c = (0, \dot{\underline{X}}), \quad \underline{n} = i\hat{\underline{n}}, \quad (5.6)$$

where $(i = \pm 1)$, $\hat{\underline{n}}^* = \hat{\underline{n}}$, $\hat{\underline{n}}\ \bar{\hat{\underline{n}}} = \hat{\underline{n}}^2 = 1$, and structure of the expressions (5.1–5.5) is more understandable.

We also should remark that as far as astrophysical data don't exclude the existence of a small curvature of the spatial part of the Universe, the expressions (28)-(31) might be taken into account under calculations of the motion of interacting Galaxies pair.

Let us consider the case when relative distance is much smaller than the radius of curvature, i.e. $r \ll 1$. In this case we can neglect terms that contain r^2 and so, hence $\cos r \approx 1$, $\sin r \approx r$ for Riemannian space and $\cosh r \approx 1$, $\sinh r \approx r$ for Lobachevsky geometry. As a result, the kinetic part of subintegral function for both spaces takes the form

$$L' = \frac{m_1m_2}{m_1+m_2}\dot{r}^2 \pm (m_1+m_2)\dot{X}_c\dot{\bar{X}}_c. \quad (5.7)$$

Thus, separation of variables of the center of mass and the relative motion for two-body system in spaces of constant curvature is possible when relative distance is much smaller than the radius of curvature i.e. we ignore the rotational degree of freedom.

Nevertheless, expression (5.7) is the very rough approximation to formulas (5.2), (5.4). It corresponds to the symmetry of the s-states in quantum mechanical Coulomb problems on three-dimensional sphere and Lobachevsky space and some configurations which are investigated in cosmology.

6. Conclusion

The problem of separation of the center of mass and relative motion variables is formulated for the case of two particles in terms of biquaternions. We showed that algebraic nature of these nonseparable variables follows from the fact that the algebra of biquaternions is noncommutative. It is shown that complete separation of variables of the relative motions and center of mass for two-body problem in spaces of constant curvature is possible only for zero approximation. In this approximation, the rotation degrees of freedom of the system are excluded.

Appendix:

Let us three arbitrary points (vertexes of the triangle) on the three dimensional sphere (or hyperboloid) defined by coordinates which are the components of biquaternios $X^{(1)}, X^{(2)}, X^{(3)}$ correspondingly. We take the all notations of the articles.

We introduce biquaternions as transformations $X^{(1)} \to X^{(2)}, X^{(2)} \to X^{(3)}, X^{(3)} \to X^{(1)}$ as follows $Q_{12} = X^{(2)}\bar{X}^{(1)}, Q_{23} = X^{(3)}\bar{X}^{(2)}, Q_{31} = X^{(1)}\bar{X}^{(3)}$.

We have $Q_{12}\bar{Q}_{12} = 1, Q_{23}\bar{Q}_{23} = 1, Q_{31}\bar{Q}_{31} = 1$ in corresponds with formulas (2.3) and (2.6). Let us write Q_{12}, Q_{23} and Q_{31} in the form which automatic satisfy by these conditions. That is

$$Q_{12} = \frac{1+\underline{q}_{12}}{\sqrt{1+\underline{q}_{12}\bar{\underline{q}}_{12}}}, Q_{23} = \frac{1+\underline{q}_{23}}{\sqrt{1+\underline{q}_{23}\bar{\underline{q}}_{23}}},$$

$$Q_{31} = \frac{1+\underline{q}_{31}}{\sqrt{1+\underline{q}_{31}\bar{\underline{q}}_{31}}}.$$

You can easily check that $Q_{31} = Q_{23}Q_{12}$ then using the standard multiplicative rule (2.2) we obtain

$$Q_{31} = \frac{1+\underline{q}_{31}}{\sqrt{1+\underline{q}_{31}\bar{\underline{q}}_{31}}}$$

$$= \frac{1-\left(\underline{q}_{12}\underline{q}_{23}\right)+\underline{q}_{12}+\underline{q}_{23}+\left[\underline{q}_{12}\underline{q}_{23}\right]}{\sqrt{1+\underline{q}_{12}\bar{\underline{q}}_{12}}\,\sqrt{1+\underline{q}_{23}\bar{\underline{q}}_{23}}}.$$

Thus, for the scalar part we have

$$\frac{1}{\sqrt{1+\underline{q}_{31}\bar{\underline{q}}_{31}}} = \frac{1-\left(\underline{q}_{12}\underline{q}_{23}\right)}{\sqrt{1+\underline{q}_{12}\bar{\underline{q}}_{12}}\,\sqrt{1+\underline{q}_{23}\bar{\underline{q}}_{23}}},$$

and for vector part we get

$$\frac{\underline{q}_{31}}{\sqrt{1+\underline{q}_{31}\bar{\underline{q}}_{31}}} = \frac{\underline{q}_{12}+\underline{q}_{23}+\left[\underline{q}_{12}\underline{q}_{23}\right]}{\sqrt{1+\underline{q}_{12}\bar{\underline{q}}_{12}}\,\sqrt{1+\underline{q}_{23}\bar{\underline{q}}_{23}}}.$$

Divide the vector part by scalar one to get the formula (3.7) of the article.

The vectors in formula (7) corresponds to pairs points, or transformations $X^0 \to X^1, X^0 \to X^2, X^0 \to X^3$, where $X^0 = i$. Thus $Q_{01} = X^{(1)}\bar{X}^{(0)}$, $Q_{02} = X^{(2)}\bar{X}^{(0)}$, $Q_{03} = X^{(3)}\bar{X}^{(0)}$ and

$$Q_{01} = \frac{1+q_{01}}{\sqrt{1+\underline{q}_{01}\bar{\underline{q}}_{01}}}, Q_{02} = \frac{1+q_{02}}{\sqrt{1+\underline{q}_{02}\bar{\underline{q}}_{02}}},$$

$$Q_{03} = \frac{1+q_{03}}{\sqrt{1+\underline{q}_{03}\bar{\underline{q}}_{03}}},$$

where

$$q_{0s} = \pm i\frac{\underline{X}^{(s)}}{X_0^{(s)}},\ s = 1,2,3.$$

Because $Q_{12} = Q_{02}\overline{Q}_{01} = \pm X^{(2)}\bar{X}^{(1)}, Q_{23} = Q_{03}\overline{Q}_{02} = \pm X^{(3)}\bar{X}^{(2)}, Q_{31} = Q_{01}\overline{Q}_{03} = \pm X^{(1)}\bar{X}^{(3)}$ we have formula (3.6) by analog with above derivation for transformations of the arbitrary points.

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