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MATHEMATICS

Nearly Kähler and Hermitian *f*-Structures on Homogeneous *k*-Symmetric Spaces

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1. INTRODUCTION

Almost Hermitian structures are among the most important differential-geometric structures on smooth manifolds; basic classes of such structures (which include Hermitian, Kähler, nearly Kähler, and other structures) have invariant realizations. In other words, there is a rich source of homogeneous manifolds G/H endowed with almost Hermitian structures of various classes invariant with respect to the Lie group G. For example, it is a classical fact that Hermitian structures on Hermitian symmetric spaces are Kähler. As to invariant nearly Kähler structures, these structures are realized on naturally reductive 3-symmetric spaces, which are always endowed with the canonical almost complex structure J (see [1–3]).

Metric f-structures of classical type (for which $f^3 + f = 0$ [4]), which include almost Hermitian and metric almost contact structures, have become one of the main objects in the extensive theory of generalized Hermitian geometry being developed starting in the mid 1980s years (see, e.g., [5]). In this theory, basic classes of metric f-structures (including Hermitian, Kähler, Killing, nearly Kähler, and other f-structures) also arose; however, they have long lacked proper invariant realizations. The situation changed after the discovery of a rich source of canonical f-structures on regular Φ-spaces, in particular, on homogeneous k-symmetric spaces [6]. It turned out that the canonical f-structure on naturally reductive 4-symmetric spaces is nearly Kähler and Hermitian [7, 8]. The same is true for both canonical f-structures, f_1 and f_2, on homogeneous 5-symmetric spaces [7–9].

In this paper, we show that all of the base canonical f-structures on naturally reductive k-symmetric spaces (where $k \geq 3$) are nearly Kähler f-structures. We also give a criterion for a canonical f-structure being the sum or difference of two base f-structures to be nearly Kähler. We describe those base f-structures which are also Hermitian f-structures and give a criterion for the remaining base structures to be Hermitian. The results obtained in this paper are of general character and imply some of the previous results, including those mentioned above, as corollaries. By way of example, results on canonical f-structures on homogeneous 6-symmetric spaces are refined.

2. NEARLY KÄHLER AND HERMITIAN f-STRUCTURES

Recall that an f-structure on a smooth manifold M generates the two mutually complementary distributions $\mathfrak{L} = \mathrm{Im} f$ and $\mathfrak{M} = \mathrm{Ker} f$, whose dimensions are called, respectively, the rank and the defect of the f-structure. In the special cases of defects 0 and 1, we obtain almost complex and almost contact structures, respectively. The distributions $\mathfrak{L}$ and $\mathfrak{M}$ are called the first and the second fundamental distribution of the f-structure.

Let $(M, g = \langle\cdot, \cdot\rangle)$ be a (pseudo)Riemannian manifold. An f-structure on this manifold is said to be metric if $\langle fX, Y\rangle + \langle X, fY\rangle = 0$ for all $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the module of smooth vector fields on M. For a metric f-structure, the distributions $\mathfrak{L}$ and $\mathfrak{M}$ are orthogonal. In the special case of defect 0, metric f-structures are almost Hermitian.

A fundamental role in the geometry of metric f-structures is played by the composition tensor T of type (2, 1), which makes it possible to introduce the structure of adjoint Q-algebra on $\mathfrak{X}(M)$ by means of the formula $X * Y = T(X, Y)$ [5, 10]. This allows us to define some classes of metric f-structures in terms of this Q-algebra. The form of the tensor T is well known [5, 10]:

$$T(X, Y) = \frac{1}{4} f(\nabla_{fX}(f) fY - \nabla_{f^2X}(f) f^2 Y),$$

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where ∇ is the Levi-Civita connection on the (pseudo)Riemannian manifold (M, g) and $X, Y \in \mathfrak{X}(M)$.

Below, we specify some of the most important classes of metric f-structures. A metric f-structure is called a nearly Kähler f-structure (or, briefly, an NKf-structure) if $\nabla_{fX}(f)fX = 0$ for all $X \in \mathfrak{X}(M)$ [7]. If the adjoint Q-algebra is Abelian (i.e., $T(X, Y) = 0$), then the metric f-structure is said to be an Hermitian f-structure (or, briefly, an Hf-structure) [5]. We denote the classes of NKf-structures and Hf-structures by **NKf** and **Hf**, respectively. Note that, in the special case of f-structures with defect 0, the classes **NKf** and **Hf** coincide with well-known classes of almost Hermitian structures, namely, the class **NK** of nearly Kähler structures and the class **H** of Hermitian structures (see, e.g., [10]).

3. INVARIANT f-STRUCTURES ON NATURALLY REDUCTIVE SPACES

Let $(G/H, g, f)$ be a homogeneous reductive space with Lie group G, invariant (pseudo)Riemannian metric g, and invariant metric f-structure, and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the corresponding reductive decomposition of the Lie algebra $\mathfrak{g}$ of the group G. In what follows, we use the same symbols to denote invariant structures on G/H and their values at the point $o = H$. A metric f-structure generates the orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where the subspaces $\mathfrak{m}_1 = \mathrm{Im} f$ and $\mathfrak{m}_2 = \mathrm{Ker} f$ determine, respectively, the first and the second fundamental distributions for f. Recall that $(G/H, g)$ is said to be naturally reductive if $g([X, Y]_\mathfrak{m}, Z) = g(X, [Y, Z]_\mathfrak{m})$ for all $X, Y, Z \in \mathfrak{m}$, where the subscript $\mathfrak{m}$ denotes the projection on $\mathfrak{m}$ with respect to the reductive decomposition.

The following two theorems are used in the further considerations.

Theorem 1 [11]. *An invariant metric f-structure on a naturally reductive space $(G/H, g)$ is nearly Kähler if and only if $[fX, f^2X] \in \mathfrak{h}$ for all $X \in \mathfrak{m}$.*

Theorem 2 [12]. *An invariant* NKf-*structure on a naturally reductive space $(G/H, g)$ is Hermitian if and only if $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_2 \oplus \mathfrak{h}$.*

4. CANONICAL f-STRUCTURES ON HOMOGENEOUS Φ-SPACES

Suppose that G/H is the homogeneous Φ-space determined by an automorphism Φ of the Lie group G and $\varphi = d\Phi_e$ is the corresponding automorphism of the Lie algebra $\mathfrak{g}$. The space G/H is called a regular Φ-space [13, 6] if $\mathfrak{g} = \mathfrak{h} \oplus A\mathfrak{g}$, where $A = \varphi - \mathrm{id}$. This decomposition of the Lie algebra $\mathfrak{g}$ is invariant with respect to φ and is the canonical reductive decomposition of the Lie algebra [13]. Let θ denote the restriction of φ to $\mathfrak{m} = A\mathfrak{g}$. Recall that all homogeneous Φ-spaces of order k (that is, such that $\Phi^k = \mathrm{id}$) are regular [13]. Such spaces are also called homogeneous k-symmetric spaces [14].

An invariant affinor structure F on a regular Φ-space G/H is said to be canonical if its value at the point o is a polynomial in θ: $F = F(\theta)$ [6]. All canonical structures form a commutative subalgebra $\mathcal{A}(\theta)$ in the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. The algebra $\mathcal{A}(\theta)$ contains structures of classical type (almost product structures, almost complex structures, and f-structures); they were completely described in [6]. In particular, for homogeneous Φ-spaces of order k, exact computational formulas were obtained. The formulas for canonical f-structures are

$$f = \frac{2}{k}\sum_{m=1}^{u}\left(\sum_{j=1}^{u}\zeta_j \sin\frac{2\pi mj}{k}\right)(\theta^m - \theta^{k-m}),$$

where

$$u = \begin{cases} n, & \text{if} \quad k = 2n+1 \\ n-1, & \text{if} \quad k = 2n, \end{cases}$$

and $\zeta_j \in \{-1, 0, 1\}$, $j = 1, 2, \ldots, u$; moreover, not all of the numbers ζ_j are zero. The canonical f-structures f_j determined by the sets $(\zeta_1, \ldots, \zeta_j, \ldots, \zeta_u)$ in which $\zeta_j = 1$ and the remaining components vanish are called base canonical f-structures. Detailed formulas for classical canonical structures were given in [6] for the small orders $k = 3, 4, 5$. For $k = 3$, the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ already mentioned was obtained [1], and for $k = 4$, we have the structure $f = \frac{1}{2}(\theta - \theta^3)$. We do not give formulas for $k = 5$ (see [6]); we only mention that two of the canonical f-structures in this case are base and the two remaining ones are almost complex.

In what follows, we assume that, on a homogeneous Φ-space G/H of order k, an invariant (pseudo)Riemannian metric determined by a symmetric bilinear form $g = \langle\cdot, \cdot\rangle$ on $\mathfrak{m} \times \mathfrak{m}$ invariant with respect to $\mathrm{Ad}(H)$ and θ is given. As is known, all canonical f-structures on $(G/H, g)$ are metric f-structures with respect to such a metric. In the case of a semisimple Lie group G, a classical example of a metric g with above properties is the so-called standard metric induced by the Killing form of the Lie algebra $\mathfrak{g}$. Note also that, on any regular Φ-space, such a metric is naturally reductive with respect to the canonical reductive decomposition [13].

5. CANONICAL NEARLY KÄHLER AND HERMITIAN f-STRUCTURES

Consider a homogeneous Φ-space G/H of any order k ($k \geq 2$). Let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras, and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the canonical reductive decomposition. In what follows, we use the following notation: $s = \left[\frac{k-1}{2}\right]$ (integer part), $u = s$ (for odd k), and $u = s + 1$ (for even k). Let us write the following φ-invariant decomposition of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_u, \quad (1)$$

where the subspaces $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_u$ correspond to the pairs of conjugate kth roots of unity from the spectrum of the operator θ (some of the subspaces may vanish); in particular, the subspace $\mathfrak{m}_{s+1}$ corresponds to the root −1.

Below, we specify commutator relations for the subspaces introduced above.

Theorem 3. *Suppose that G/H is a homogeneous Φ-space of order k ($k \geq 2$); $\mathfrak{m}$ is the corresponding canonical reductive complement with decomposition* (1); *$i, j = 0, 1, \ldots, u$; $i \geq j$; and $\mathfrak{m}_{i+j}$ denotes $\mathfrak{m}_{k-(i+j)}$ if $i + j > u$.*

Then, the following commutator relations are valid:

$$[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}.$$

Note that, for $k = 2$, Theorem 3 gives the well-known commutator relations for symmetric spaces, namely, $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Similar relations in different form were independently obtained in [15].

Consider canonical f-structures on homogeneous Φ-spaces G/H of order k. It follows from the construction of these structures (see [6]) that any of them is trivial on the subspace $\mathfrak{m}_{s+1}$ (which is defined only for even k). For $i = 1, 2, \ldots, s$, let f_i denote the base canonical f-structure whose image is the subspace $\mathfrak{m}_i$.

By using Theorems 1 and 3 and a number of additional relations, we have proved the following theorem.

Theorem 4. *Let $(G/H, g)$ be a homogeneous Φ-space of order k with naturally reductive metric g.*

Then, any canonical base f-structure f_i with $i = 1, 2, \ldots, s$ on G/H is a nearly Kähler f-structure.

Note that this theorem generalizes known results on the inclusion of canonical base f-structures in the class **NKf** for the orders $k = 3, 4, 5$ (see [3, 7, 11]).

For sums and differences of canonical base f-structures, the following theorem is valid.

Theorem 5. *Let $(G/H, g)$ be a naturally reductive homogeneous Φ-space of order k. Suppose that f_i and f_j are canonical base f-structures on G/H for $i, j = 1, 2, \ldots, s$, $i > j$, $\mathfrak{m}_i$, and $\mathfrak{m}_j$ are the corresponding subspaces from decomposition* (1); *and $\mathfrak{m}_{i+j}$ denotes $\mathfrak{m}_{k-(i+j)}$* if *$i + j > u$. Then,*

$$f_i \pm f_j \in \mathbf{NKf} \Leftrightarrow [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i \pm j}.$$

Let us study f-structures in the class **Hf**. Applying Theorems 2–4 for canonical base f-structures, we have proved the following theorem.

Theorem 6. *Let $M = G/H$ be a homogeneous Φ-space of order k with naturally reductive metric.*

Then, for any canonical base f-structure f_i on M, the following assertions hold:

(i) *if $3i \neq k$, then f_i belongs to the class* **Hf**;

(ii) *if $3i = k$, then $f_i \in$* **Hf** *$\Leftrightarrow [\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}$.*

6. CANONICAL f-STRUCTURES ON HOMOGENEOUS 6-SYMMETRIC SPACES

As mentioned, the theorems stated above imply a number of known results for the orders $k = 3, 4, 5$ as corollaries. Let us apply them to $k = 6$.

Using the general formula for canonical f-structures given in [6], we can write formulas for all canonical f-structures (up to sign) in the case of homogeneous Φ-spaces of order 6:

$$f_1 = \frac{\sqrt{3}}{6}(\theta + \theta^2 - \theta^4 - \theta^5), \quad f_2 = \frac{\sqrt{3}}{6}(\theta - \theta^2 + \theta^4 - \theta^5),$$

$$f_3 = f_1 + f_2, \quad f_4 = f_1 - f_2.$$

Note, that the structures f_1 and f_2 are base, and the decomposition (1) of the canonical reductive complement $\mathfrak{m}$ has the form

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

Note also that the image of the structures f_3 and f_4 is the subspace $\mathfrak{m}_1 \oplus \mathfrak{m}_2$.

Detailing the results obtained above for the case under consideration, we arrive at the following theorem.

Theorem 7. *Let $(G/H, g)$ be a naturally reductive homogeneous Φ-space of order* 6. *Then:*

(i) *the canonical structures f_1 and f_2 are* NKf-*structures;*

(ii) *the canonical structure $f_3 = f_1 + f_2$ is an* NKf-*structure if and only if $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_3$;*

(iii) *the canonical structure $f_4 = f_1 - f_2$ is an* NKf-*structure if and only if $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$;*

(iv) *the canonical structure f_1 is a Hermitian f-structure;*

(v) *the canonical structure f_2 belongs to the class* **Hf** *if and only if $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h}$.*

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