A NOTE ON EIGENVALUE DECOMPOSITION ON JACKET TRANSFORM

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Jacket transforms are defined to be $n \times n$ matrices $A = (a_{jk})$ over a field F with the property $AA^\dagger = nI_n$, where $A^\dagger$ is the transpose matrix of elements inverse of A, i.e., $A^\dagger = (a_{kj}^{-1})$, which generalized Hadamard transforms and Center Weighted Hadamard transforms. It has been found that the Jacket transforms are applied to signal and image representation and compression. This paper propose a new eigenvalue decomposition method with Jacket transform. The eigenvalue decomposition methods discussed here may be applied to doubly stochastic processing and the information-theoretic analysis of multiple input multiple output (MIMO) channels.

Keywords: Jacket transform, Eigenvalue decomposition, doubly stochastic processing, MIMO.

1. INTRODUCTION

Hadamard matrices and Hadamard transforms has been of a great deal of interest and has been applied to communications signaling, image processing, signal representation, error correction coding theory, etc. Investigations of Hadamard matrices were connected initially with linear algebra problems, such as finding maximum of determinant. It is well known that for an Hadamard matrix, $detH_n = n2^{n/2}$. Lee [1] proposed the idea of center weighted Hadamard matrices and center weighted Hadamard transform. Further, Lee etc. [2] proposed the following new class of matrices: *Jacket transforms* which generalize real Hadamard transforms, Turyn-type Hadamard transform, Butson-type Hadamard transforms, complex Hadamard transforms and center weighted Hadamard transforms.

Definition 1. *Let $A = (a_{jk})$ be an $n \times n$ matrix whose elements are in a field F (including real fields, complex fields and finite fields, etc). Denoted by $A^\dagger$ the transpose matrix of elements inverse of A, i.e., $A^\dagger = (a_{kj}^{-1})$). A is called a Jacket matrix if $AA^\dagger = A^\dagger A = nI_n$, where I_n is the identity matrix over a field F.* For example,

$$A = \begin{bmatrix} a & \sqrt{ac} \\ \sqrt{ac} & -c \end{bmatrix}, \quad A^{-1} = \frac{1}{2}\begin{bmatrix} \frac{1}{a} & \frac{1}{\sqrt{ac}} \\ \frac{1}{\sqrt{ac}} & -\frac{1}{c} \end{bmatrix}, \tag{1}$$

so A is a 2×2 Jacket matrix. when $a = c = 1$, it is a 2×2 Hadamard matrix.

On one hand, from the definition of Jacket matrices, it is easy to see that the class of Jacket matrices contains complex Hadamard matrices. On the other hands, Lee in [1] defined the *center Weighted Hadamard matrices* W as following:

$$W_4 \equiv \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -w & w & -1 \\ 1 & w & -w & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad W_4^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\frac{1}{w} & \frac{1}{w} & -1 \\ 1 & \frac{1}{w} & -\frac{1}{w} & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \qquad (2)$$

Clearly, by a simply calculation, $W_4 W_4^{\dagger} = 4I_4$. Hence W_4 is a Jacket matrix. In particular, if $w = 1$, it is an Hadamard matrix and if $w = 2$ it is a special center Weighted Hadamard matrix. Hence Center Weighted matrices are also Jacket matrices.

The MIMO communication system in the presence of multiple MIMO co-channel interferers has been extensively studied as a method of combating detrimental effects in wireless fading channels due to its relative simplicity of implementation and feasibility of having multiple antennas (see [3] and [4]). Alamouti (see [5]) proposed a remarkable scheme for transmission by using two transmit antennas. Later, Tarokh etc. [3] [4] created space-time block coding, a new paradigm for communication over Rayleigh fading channels using multiple transmit antennas.

Space-time block codes combine ideas of trellis-coded modulation with a space -time diversity approach. It performs very well in slow fading environments. The codes is based on the orthogonal matrix $A(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}$, where x_1^* is conjugate of x_1. From [5], it is shown that the code is optimal for transmitting signals with full rate. The theory of space-time block codes was further developed by Tarokh, Jafarkhani and Calderbank[3]. They proposed space-time code in terms of orthogonal code matrices. The properties of these marices ensure full diversity equal to the number of antennas and a linear maximal likelihood detection. Based on the central limit theorem and the Monte Carlo simulation, the mean and variance of channel capacity has been studied in [6] for large numbers of antennas. If the number of antennas equals two or three, the exact density function of the capacity is obtained in [7] by aid of the Jacobian transformation on the joint density of the eigenvalues of a Wishart matrix. Another new expression for the joint distribution of the eigenvalues of a Wishart matrix is presented [8]. The distribution of the determinant and trace of a Wishart matrix have been studied [9].

The Hadamard transform and its generalizations in various way have been proposed and used for audio and video coding since these transforms are highly practical value for representing signals and images. Due to the ease and efficiency of these transforms, they are widely used for signal and image representation and compression. The most advantage of these transforms are lied to their inverse transforms that are easily obtained. In order to offer quality of representation over the central region of the image and to retain the simplicity of Hadamard transform, the center weighted Hadamard transform was proposed and studied. With the rapidly development of communication systems that require more transmission and storage capacities of multilevel cases in co-channels for numerous clients, recently, the

Jacket transform and reverse Jacket transform based on Jacket matrix have extensively been proposed and investigated For more applications the readers may be referred to [1] and [2]. With more and more increasing applications of Jacket matrices and Jacket transforms, it will be more interesting and important to study further properties of Jacket matrices and their construction.

It is well known that every invertible matrix is a change-of-basis matrix in linear space. It is often useful to step back from a question about a matrix to a question about some intrinsic property of the linear transform of which it is only one of many possible representations. In other words, which matrices are similar to diagonal matrix by invertible matrix. In many applications, we may ask the invertible matrix whose inverse are easy to obtain. It leads us to the following definition:

Definition 2. *Let A be an $n \times n$ matrix. If there exists a jacket matrix J such that $A = JDJ^{-1}$, where D is a diagonal matrix. We say that A is Jacket similar to the diagonal matrix. Moreover, we say that A is Jacket diagonalizable.*

If A is jacket similar to the diagonal matrix D, then the main diagonal entries of D are all eigenvalues. Further since $A = JDJ^{-1}$, we have $D = J^{-1}AJ$. Note that J is a jacket matrix, its inverse is easy to obtain $J^{-1} = J^{\dagger}$. Hence we can directly calculate the eigenvalues of A by $D = J^{\dagger}AJ$. In this paper, we investigate which matrices have eigenvalue decomposition through jacket matrices.

2. EIGENVALUE DECOMPOSITION OF MATRIX OF ORDER 2

In section, we which matrices of order 2 are jacket diagonalizable, in other words, the eigenvalues of which matrices can be easily to computer. It is known that any jacket matrix of order can be written as follows, $J = PDHEQ$, where P and Q are permutation matrices, D and E are diagonal matrices, H is the standard Hadamard matrix $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Let $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ and $E = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$. Then general jacket of matrix of order 2 is the following form $J = \begin{bmatrix} d_1e_1 & d_1e_2 \\ d_2e_1 & -d_2e_2 \end{bmatrix}$. Now we assume that a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is able to jacket eigenvalue decomposition. In other words, A can be rewritten as follows: $A = J\Lambda J^{-1}$, where $\Lambda = \begin{bmatrix} \lambda_1, & 0 \\ 0 & \lambda_2 \end{bmatrix}$. By matrix multiplication and the property of jacket matrix, we have

$$A = \begin{bmatrix} d_1e_1 & d_1e_2 \\ d_2e_1 & -d_2e_2 \end{bmatrix} \begin{bmatrix} \lambda_1, & 0 \\ 0 & \lambda_2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} \frac{1}{d_1e_1} & \frac{1}{d_2e_1} \\ \frac{1}{d_1e_2} & -\frac{1}{d_2e_2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \lambda_1 + \lambda_2 & \frac{d_1}{d_2}(\lambda_1 - \lambda_2) \\ \frac{d_2}{d_1}(\lambda_1 - \lambda_2) & \lambda_1 + \lambda_2 \end{bmatrix}. \tag{3}$$

Hence we have $a = \frac{1}{2}(\lambda_1 + \lambda_2), b = \frac{d_1}{2d_2}(\lambda_1 - \lambda_2)$, $c = \frac{d_2}{2d_1}(\lambda_1 - \lambda_2), d = \frac{1}{2}(\lambda_1 + \lambda_2)$. In this way, we can obtain the following result:

Theorem 1. *An 2×2 matrix A is Jacket similar to the diagonal matrix if and only if A has the following form $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$, ie., the entries of the main diagonal of a matrix are equal.*

From Theorem, for two by two symmetric matrix $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$,it can be eigenvalue decomposition as the follows: $\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a & a \\ a & -a \end{bmatrix} \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix} \begin{bmatrix} a & a \\ a & -a \end{bmatrix}^{-1}$.

3. EIGENVALUE DECOMPOSITION OF MATRIX OF ORDER 3 AND 4

In this section , we discuss the matrices of order 3 which cab be eigenvalue decomposition though jacket matrices. We just recall the three by three jacket patterns, $J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$, where $\omega^3 = 1$ and $\omega \neq 1$. It is known that any jacket matrix B of order 3 can be expressed in the following form $B = PDJEQ$, where D and E are diagonal matrices and P and Q are permutation matrices. We present the following theorem (the proof is omitted)

Theorem 2. *Let $A = (a_{ij})$ is of order 3. The A is jacket similar to a diagonal matrix Λ if and only if A must be $A = \begin{bmatrix} a & a_{12} & a_{13} \\ \frac{k}{a_{12}} & a & a_{23} \\ \frac{k}{a_{13}} & \frac{k}{a_{23}} & a \end{bmatrix}$. In other words, such the matrix must be eigenvalue decomposition.* For example, let $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$. Hence the matrix A can be decomposed as the following form:

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} a+b+c & 0 & 0 \\ 0 & a+b\omega+c\omega^2 & 0 \\ 0 & 0 & a+b\omega^2+c\omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}^{-1}. \tag{4}$$

Theorem 3. *Any Jacket matrix of order 4 is equivalent to the center weight Jacket matrix.*

Let A be an matrix of order 4. Then A is Jacket similar to diagonal matrix if and only if A has the form $A = \begin{bmatrix} a & (bd_1)/d_2 & (cd_1)/d_3 & (dd_1)/d_4 \\ (fd_2)/d_1 & a & (dd_2)/d_3 & (ed_2)/d_4 \\ (ed_3)/d_1 & (dd_3)/d_2 & a & (fd_3)/d_4 \\ (dd_4)/d_1 & (cd_4)/d_2 & (bd_4)/d_3 & a \end{bmatrix}$.

4. SINGULAR VALUE DECOMPOSITION AND MIMO

with aid of the linear dispersion code framework, Heath and Paulraj presented a space-time coding design for MIMO Rayleigh fading channels, which provided code that had the same ergodic capacity performance as spatial multiplexing but allowed for improved diversity advantage . Jafarhani proposed a quasi-orthogonal space-time block code with full rate, which sacrifice the orthogonality and have relatively simple decoding method. The most important coding matrix $C = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}$ Now we discuss how to factor this matrix by using Jacket matrix. We assume that C can be factored into $J_1 \Lambda J_2$, where J_1 and J_2 are jacket matrices. We assume that $J_1 = \begin{bmatrix} a & a \\ b & -b \end{bmatrix}$ and $J_2 = \begin{bmatrix} c & d \\ c & -d \end{bmatrix}$, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Then $J_1 \Lambda J_2 = \begin{bmatrix} ac(\lambda_1 + \lambda_2 & ad(\lambda_1 - \lambda_2) \\ bc(\lambda_1 - \lambda_2) & bd(\lambda_1 + \lambda_2) \end{bmatrix}$.

For any four by four coding matrix $S = \begin{bmatrix} x_1 & x_2 & 0 & 0 \\ -x_2^* & x_1^* & 0 & 0 \\ 0 & 0 & x_3 & x_4 \\ 0 & 0 & -x_4^* & x_3^* \end{bmatrix} = J_1 \Lambda J_2$, there are similar to singular value decomposition.

5. INTERLEAVED ORTHOGONAL DESIGNS

Zehavi showed that with aid of bitwise interleaving at the encoder output and using an appropriate soft decision metric, the diversity and performance of coded modulation over fading channels are able to be improved. Another related approach is coordinate interleaving. There are several ways to construct space-time block codes, for example, Orthogonal Designs (OD), Quasi-Orthogonal Designs (QOD) and Co-ordinate Interleaved Orthogonal Designs (CIOD). Because their amenability for fast ML decoding, and rate-one with full-rank over quasi-static fading channels [5]. While rate-one, full-rank, square ODs for arbitrary complex constellations exist only for 2 transmit antennas, A Co-ordinate Interleaved Orthogonal Designs exist for exists for 2,3 and 4 transmit antennas with a slight restriction on the complex constellations.

It is known that a rate 1, co-ordinate interleaved orthogonal design of size N exists if and only if $N = 2, 3$ or 4. The interleaved orthogonal design of 4 can be described as (see [5]) S. By the result in last section, it is easy to see that S can be decomposed with jacket matrices. With aid of singular value decomposition, we can easily analysis the conditions for full-diversity and an expression for coding gain obtained.

6. CONCLUSION

This paper has presented a class of 2-by-2 matrices and 3 by 3 matrices which have an eigenvalue decomposition by Jacket matrices. With the aid of the recursive relation, we are able to extend the Eigenvalue decomposition to the high order matrices. These properties may be used for Jacket matrices to be applied to signal processing, coding theory and orthogonal code design. Eigenvalue decomposition can be used in the information-theoretic analysis of multiple input multiple output (MIMO) channels and in analysis of coding gain of Space-Time block codes. The readers may be refereed to [1],[2] and [5].

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