NEW STABILITY CONDITIONS FOR TIME-SHARING SERVICING PROCESS IN RANDOM ENVIRONMENT

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We present updated results concerning stability conditions for time-sharing queueing systems with group arrivals and branching secondary flows in random environment.

Keywords: time-sharing queuing system, random environment, stability conditions, iterative-magorant approach.

1. INTRODUCTION

When we investigate control processes in different queuing systems over conflict flows modulated by a random environment we finally construct random sequences of the form $\{(\Gamma_i, \kappa_i, \chi_i); i = 0, 1, \ldots\}$. Such sequence describes in a discrete time scale $\{\tau_i; i = 0, 1, \ldots\}$ server state change, queues length fluctuations and random environment state change correspondingly. As a rule the vectors κ_i, $i = 0, 1, \ldots$ take values from an integer lattice of dimension higher than 1. Besides, the sequence can be periodic (as a periodic Markov chain) if the server performs readjustments and during these doesn't process customers. We use iterative-magorant approach to obtain conditions for the stationary distribution existence. We also demonstrate that averaging of random environment influence should be taken into consideration while applying this approach.

Consider a queueing system with $m < \infty$ servicing nodes, $\Gamma^{(1)}$, $\Gamma^{(2)}$, ..., $\Gamma^{(m)}$, and one node for readjustments, $\Gamma^{(m+1)}$. There also m input flows of customers, Π_1, Π_2, ..., Π_m, one for each servicing node. Customers from each flow enter the appropriate buffer of infinite capacity. A customer from the flow Π_j, $j = \overline{1, m}$, is served at the node $\Gamma^{(j)}$. Input flows are conflict, i.e. a customer from no more that one flow can get service at each time instant. Service duration for a customer from the jth queue has a distribution function $B_j(t)$, $B_j(0+) = 0$. A server readjustment is executed at the node $\Gamma^{(m+1)}$ after each service. The readjustment duration after a service at the node $\Gamma^{(j)}$ has a distribution function $\bar{B}_j(t)$, $\bar{B}_j(+0) = 0$. If all buffers are empty at the end of an readjustment, then the service of the first customer arrived begins. If not all the buffers are empty then the following rule is run. Suppose there are x_j customers in the jth buffer at the end of an readjustment. Put $x = (x_1, x_2, \ldots, x_m)$, assume $x \neq (0, 0, \ldots, 0)$. Then the service starts at the node $\Gamma^{(j)}$ with $j = h(x)$, where $h(\cdot)$ is a mapping of the non-negative integer lattice $X = \{0, 1, \ldots\}^m$ onto $\{1, 2, \ldots, m+1\}$. Here $h(x) = j$ implies $x_j > 0$. Only the zero vector $\overline{0} = (0, 0, \ldots, 0) \in X$ is mapped to $m + 1$. The input flows and branching secondary flows are modulated by the random environment with the finite state set $\{e^{(1)}, e^{(2)}, \ldots, e^{(d)}\}$. The environment may change its state only when a service or an adjustment has just terminated. By $a_{l,k}$ denote the probability to change environment state from $e^{(l)}$ to $e^{(k)}$, $k = \overline{1,d}$. Assuming environment state $e^{(k)}$ on, customer groups arrive to the jth buffer according to the Poisson law with parameter $\lambda_j^{(k)}$. By $p_b^{(j,k)}$ denote the probability of groups size b at the jth flow, $b = 1, 2, \ldots$. We shall assume that the series $f_j^{(k)}(z) = \sum_{b=1}^{\infty} p_b^{(j,k)} z^b$ converges in a circle $|z| < 1 + \varepsilon$ for some $\varepsilon > 0$. This assumption holds in fact for a variety of important group size models, such as bounded group sizes [1] and Bartlett-type group sizes [2]. By $\{p_j^{(k)}(y); y = (y_1, y_2, \ldots, y_m), y_r = 0, 1, \ldots, r = \overline{1,m}\}$ denote the joint distribution of newly born secondary customers when the environment is in the state $e^{(k)}$. For $y = (y_1, y_2, \ldots, y_m)$ here $p_j^{(k)}(y)$ is the probability of y_1 secondary customers arriving into the first queue, y_2 secondary customers arriving into the second queue, etc., and finally y_m secondary customers arriving into the mth queue when the environment is in the state $e^{(k)}$.

The control of the queues in this system follows the *time-sharing algorithm*. The aim of the study is to obtain stability conditions for the fluctuations of queues sizes.

2. MAIN RESULTS

The mathematical model of the queueing system was given in [3]. We recall notation from the cited work. Assume a probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ where all random variables and random elements are defined. Both types of time intervals are called working tacts of the system. Put $\tau_0 = 0$. Let τ_i be the moment of the ith tact's termination, $i = 1, 2, \ldots$. A random element $\Gamma_i \in \Gamma = \{\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(m+1)}\}$ denotes the server's state at the time interval $(\tau_{i-1}, \tau_i]$. A random element $\Gamma_0 \in \Gamma$ denotes the server's state at the epoch τ_0. Further, a random vector $\kappa_i = (\kappa_{1,i}, \kappa_{2,i}, \ldots, \kappa_{m,i})$ describes the sizes of the queues at the time instant τ_i taking into account secondary customers, and a random element $\chi_i \in \{e^{(1)}, e^{(2)}, \ldots, e^{(d)}\}$ is the random environment's state during the time interval $(\tau_i, \tau_{i+1}]$. Lemma 1 from [3] states that given an initial vector $(\Gamma_0, \kappa_0, \chi_0)$ the sequence

$$\{(\Gamma_i, \kappa_i, \chi_i); i = 0, 1, \ldots\} \tag{1}$$

is a Markov chain. We shall write $\tilde{d} = 1$ when the environment states form a single aperiodic class, otherwise by $\tilde{d}$ denote the number of cyclic subclasses. Let $C_1, C_2, \ldots, C_{\tilde{d}}$ be the sets of environment state indices for each subclass. Put for convenience $C_0 = C_{\tilde{d}}$, $C_{\tilde{d}+1} = C_1$. We assume that when $\tilde{d} > 1$ the subclasses are numbered in a way the following transitions $\{e^{(k)}; k \in C_1\} \to \{e^{(k)}; k \in C_2\} \to \ldots \to \{e^{(k)}; k \in C_{\tilde{d}}\}$ take place. Put

$$X_j = \{x\colon x \in X, h(x) = j\}$$

for $j = \overline{1,m}$. For $s = \overline{1,m+1}$, $w \in X$, $k = \overline{1,d}$ and $i = 0, 1, \ldots$ define

$$Q_i^{(s,k)}(w) = \mathsf{P}(\{\omega\colon \Gamma_i = \Gamma^{(s)}, \kappa_i = w, \chi_i = e^{(k)}\}),$$

$$\lambda_+^{(k)} = \lambda^{(1)} + \lambda_2^{(k)} + \ldots + \lambda_m^{(k)},$$

$$F(t, z; j, k) = \exp\{\lambda_j^{(k)}(f_j^{(k)}(z) - 1)t\}.$$

Let functions $P_\zeta(t; j, k)$, $\zeta = 0, 1, \ldots$ be defined with means of series expansion

$$F(t, z; j, k) = \sum_{\zeta=0}^{\infty} z^\zeta P_\zeta(t; j, k).$$

Here $P_\zeta(t; j, k)$ equals the probability of ζ arrivals during time t at flow Π_j when the environment is in the state $e^{(k)}$. Finally, let $y' = (y'_1, y'_2, \ldots, y'_m) \in X$.

Theorem 1 in [3] gives recurrent equations for probabilities $Q_i^{(s,k)}(w)$ with respect to $i = 0, 1, \ldots$. As a corollary of the cited theorem we obtain that the states space of the Markov chain (1) forms at least one communicating class,

$$\{(\Gamma^{(s)}, x, e^{(k)}) : s = 1, 2, \ldots, m+1;\ x \in X;\ k = 1, 2, \ldots, d\},$$

if $\widetilde{d}$ is odd, and for even $\widetilde{d}$ splits into two communicating classes,

$$\begin{gathered}\{(\Gamma^{(j)}, x, e^{(k)}) : j = 1, 2, \ldots, m;\ x \in X;\ k \in C_1 \cup C_3 \cup \ldots \cup C_{\widetilde{d}-1}\} \cup \\ \cup \{(\Gamma^{(m+1)}, x, e^{(k)}) : x \in X;\ k \in C_2 \cup C_4 \cup \ldots \cup C_{\widetilde{d}}\}\end{gathered} \tag{2}$$

and

$$\begin{gathered}\{(\Gamma^{(j)}, x, e^{(k)}) : j = 1, 2, \ldots, m;\ x \in X;\ k \in C_2 \cup C_4 \cup \ldots \cup C_{\widetilde{d}}\} \cup \\ \cup \{(\Gamma^{(m+1)}, x, e^{(k)}) : x \in X;\ k \in C_1 \cup C_3 \cup \ldots \cup C_{\widetilde{d}-1}\}.\end{gathered} \tag{3}$$

In case of even $\widetilde{d}$ two Markov chains with state spaces (2), (3) have to be considered separately. In this paper we present results concerning state space (2). Space (3) can be investigated in a similar way.

For a real or complex vector $v = (v_1, v_2, \ldots, v_m)$ and a vector $w = (w_1, w_2, \ldots, w_m) \in X$ with non-negative integer elements denote the product $v_1^{w_1} v_2^{w_2} \cdots v_m^{w_m}$ by v^w (assuming here $0^0 = 1$). Introduce probability generating functions

$$\mathfrak{M}_i(v, s, k) = \sum_{w \in X} v^w Q_i^{(s,k)}(w),$$

$$\widetilde{\mathfrak{M}}_i(v, j, k) = \sum_{w \in X_j} v^w Q_i^{(n,k)}(w) + Q_i^{(n,k)}(\bar{0}) \frac{\lambda_j^{(k)}}{\lambda_+^{(k)}},$$

$$R_j^{(k)}(v) = v_j^{-1} \sum_{w \in X} v^w p_j^{(k)}(w)$$

and Laplace-Stieltjes transforms

$$q_j^{(k)}(v) = \int_0^\infty \prod_{r=1}^m F(t, v_r; r, k)\, dB_j(t),$$

$$\bar{q}_j^{(k)}(v) = \int_0^\infty \prod_{r=1}^m F(t, v_r; r, k)\, d\tilde{B}_j(t).$$

Here $|v_j| \le 1$ for every $j = \overline{1, m}$. Theorem 3 in [3] gives recurrent equations for probability generating functions defined above. For $\nu = \overline{1, \tilde{d}}$, $k \in C_\nu$, $j = \overline{1, m}$, and $i = 0, 1, \ldots$

$$\mathfrak{M}_{i+1}(v, j, k) = \sum_{l \in C_{\nu-1}} a_{l,k} \Big(q_j^{(l)}(v) R_j^{(l)}(v) \widetilde{\mathfrak{M}}_i(v, j, l) + Q_i^{(n,l)}(\bar{0}) \times \\ \times \frac{\lambda_j^{(l)}}{\lambda_+^{(l)}} (f_j^{(l)}(v_j) - 1) q_j^{(l)}(v) R_j^{(l)}(v) \Big), \tag{4}$$

$$\mathfrak{M}_{i+1}(v, n, k) = \sum_{l \in C_{\nu-1}} a_{l,k} \sum_{r=1}^m \bar{q}_r^{(l)}(v) \mathfrak{M}_i(v, r, l). \tag{5}$$

Theorem 1. *Assume $\tilde{d}$ is odd. For positive $v_1, v_2, \ldots, v_m$ and positive integer g the following inequality holds:*

$$\sum_{k=1}^d \mathfrak{M}_{i+2g}(v, n, k) \le \sum_{l_1, l_2, \ldots, l_{2g}=1}^d \sum_{j_1, j_2, \ldots, j_g=1}^m a_{l_{2g}, l_{2g-1}} a_{l_{2g-1}, l_{2g-2}} \cdots a_{l_2, l_1} \bar{q}_{j_1}^{(l_1)}(v) q_{j_1}^{(l_2)}(v) \times \\ \times R_{j_1}^{(l_2)}(v) \bar{q}_{j_2}^{(l_3)}(v) q_{j_2}^{(l_4)}(v) R_{j_2}^{(l_4)}(v) \cdots \bar{q}_{j_g}^{(l_{2g-1})}(v) q_{j_g}^{(l_{2g})}(v) R_{j_g}^{(l_{2g})}(v) \mathfrak{M}_i(v, n, l_{2g}) + \\ + \sum_{l_1, l_2=1}^d \sum_{j_1=1}^m a_{l_2, l_1} \bar{q}_{j_1}^{(l_1)}(v) q_{j_1}^{(l_2)}(v) R_{j_1}^{(l_2)}(v) Q_{i+2g-2}^{(n,l_2)}(\bar{0}) \frac{\lambda_{j_1}^{(l_2)}}{\lambda_+^{(l_2)}} (f_{j_1}^{(l_2)}(v_{j_1}) - 1) + \\ + \sum_{l_1, l_2, l_3, l_4=1}^d \sum_{j_1, j_2=1}^m a_{l_4, l_3} a_{l_3, l_2} a_{l_2, l_1} \bar{q}_{j_1}^{(l_1)}(v) q_{j_1}^{(l_2)}(v) R_{j_1}^{(l_2)}(v) \bar{q}_{j_2}^{(l_3)}(v) \times \\ \times q_{j_2}^{(l_4)}(v) R_{j_2}^{(l_4)}(v) Q_{i+2g-4}^{(n,l_4)}(\bar{0}) \frac{\lambda_{j_2}^{(l_4)}}{\lambda_+^{(l_4)}} (f_{j_2}^{(l_4)}(v_{j_2}) - 1) + \ldots + \\ + \sum_{l_1, l_2, \ldots, l_{2g}=1}^d \sum_{j_1, j_2, \ldots, j_g=1}^m a_{l_{2g}, l_{2g-1}} a_{l_{2g-1}, l_{2g-2}} \cdots a_{l_2, l_1} \bar{q}_{j_1}^{(l_1)}(v) q_{j_1}^{(l_2)}(v) R_{j_1}^{(l_2)}(v) \bar{q}_{j_2}^{(l_3)}(v) \times \\ \times q_{j_2}^{(l_4)}(v) R_{j_2}^{(l_4)}(v) \cdots \bar{q}_{j_g}^{(l_{2g-1})}(v) q_{j_g}^{(l_{2g})}(v) R_{j_g}^{(l_{2g})}(v) Q_i^{(n,l_{2g})}(\bar{0}) \frac{\lambda_{j_g}^{(l_{2g})}}{\lambda_+^{(l_{2g})}} (f_{j_g}^{(l_{2g})}(v_{j_g}) - 1). \tag{6}$$

The proof follows directly from equations (4), (5) and lemma 1 below.

Lemma 1. *For positive $v_1, v_2, \ldots, v_m$, $j = 1, 2, \ldots, m$ и $l = 1, 2, \ldots, d$ we have*

$$\widetilde{\mathfrak{M}}_i(v, j, l) \le \mathfrak{M}_i(v, n, l). \tag{7}$$

Denote by $\{a_k; k = \overline{1, d}\}$ the stationary distribution of Markov chain $\{\chi_i; i = 0, 1, \ldots\}$, denote by $\pi_{j,r}^{(k)} = \sum_{x \in X} y_r p_j^{(k)}(y)$ the expected number of secondary customers in the rth queue generated by a customer from the jth queue if the environment is in the state $e^{(k)}$, $\pi^{(k)} = (\pi_{j,r}^{(k)})_{j,r=\overline{1,m}}$. Define

$$\pi = (\pi_{j,r})_{j,r=\overline{1,m}} = \sum_{k=1}^{d} \mathfrak{a}_k \pi^{(k)},$$

$$\tilde{\pi} = (\tilde{\pi}_{j,r})_{j,r=\overline{1,m}} = \sum_{k\in C_2\cup C_4\cup\ldots\cup C_{\tilde{d}}} 2\mathfrak{a}_k \pi^{(k)} \qquad \text{for even } \tilde{d},$$

$$\mu_j^{(k)} = \sum_{b=1}^{\infty} b p_b^{(k,j)}, \quad \bar{\lambda}_j^{(k)} = \lambda_j^{(k)} \mu_j^{(k)}, \quad \bar{\lambda}^{(k)} = (\bar{\lambda}_1^{(k)}, \bar{\lambda}_2^{(k)}, \ldots, \bar{\lambda}_m^{(k)})^T,$$

$$\beta_j = \int_0^{\infty} t\, dB_j(t), \quad \bar{\beta}_j = \int_0^{\infty} t\, d\bar{B}_j(t),$$

$$\beta = (\beta_1, \beta_2, \ldots, \beta_m), \quad \bar{\beta} = (\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m).$$

Theorems 2, 3 below give necessary conditions for the stationary distribution existence for Markov chain (1).

Theorem 2. *Assume $\tilde{d}$ is odd and one of the alternatives hold:*

1) the greatest in magnitude eigenvalue R of the matrix π is greater or equal 1, or

2) $R < 1$ and $(\beta + \bar{\beta})(I_m - \pi^T)^{-1}\left(\sum_{l=1}^{d} \mathfrak{a}_l \bar{\lambda}^{(l)}\right) > 1$.

Then for every $(\Gamma^{(s)}, w, e^{(k)}) \in \Gamma \times X \times \{e^{(1)}, e^{(2)}, \ldots, e^{(d)}\}$ and independently of the distribution of $(\Gamma_0, \kappa_0, \chi_0)$ one has

$$\lim_{i\to\infty} Q_i^{(s,k)}(\bar{0}) = 0. \tag{8}$$

Theorem 3. *Assume $\tilde{d}$ is even and one of the alternatives hold:*

1) the greatest in magnitude eigenvalue $\tilde{R}$ of the matrix $\tilde{\pi}$ is greater or equal 1, or

2) $\tilde{R} < 1$ and

$$(\beta + \bar{\beta})(I_m - \tilde{\pi}^T)^{-1}\left(\sum_{l\in C_1\cup C_3\cup\ldots C_{\tilde{d}-1}} 2\mathfrak{a}_l \bar{\lambda}^{(l)}\right)\left(\sum_{j=1}^{m} \bar{\beta}_j\right)\left(\sum_{j=1}^{m}(\beta_j + \bar{\beta}_j)\right)^{-1} +$$

$$+ (\beta + \bar{\beta})(I_m - \tilde{\pi}^T)^{-1}\left(\sum_{l\in C_2\cup C_4\cup\ldots C_{\tilde{d}}} 2\mathfrak{a}_l \bar{\lambda}^{(l)}\right)\left(\sum_{j=1}^{m} \beta_j\right)\left(\sum_{j=1}^{m}(\beta_j + \bar{\beta}_j)\right)^{-1} > 1.$$

Then for every $(\Gamma^{(s)}, w, e^{(k)})$ from (2) and independently of the distribution of $(\Gamma_0, \kappa_0, \chi_0)$ one has (8).

Stability conditions for the fluctuations of queues sizes are given in the next two theorems.

Theorem 4. *Assume $\tilde{d}$ is odd, the greatest in magnitude real eigenvalue R of the non-negative matrix π is less then unity and*

$$(\beta + \bar{\beta})(I_m - \pi^T)^{-1}\left(\sum_{l=1}^{d} \mathfrak{a}_l \bar{\lambda}^{(l)}\right) < 1.$$

Then the sequence $\{\mathsf{E}(\sum_{j=1}^{m}\kappa_{j,i}); i = 0, 1, \ldots\}$ *is bounded.*

Theorem 5. *Assume* $\tilde{d}$ *is even, the states space* (2), *the greatest in magnitude real eigenvalue* $\bar{R}$ *of the non-negative matrix* $\bar{\pi}$ *is less then unity and*

$$(\beta+\bar{\beta})(I_m-\bar{\pi}^T)^{-1}\left(\sum_{l\in C_1\cup C_3\cup\ldots C_{\tilde{d}-1}} 2a_l\bar{\lambda}^{(l)}\right)\left(\sum_{j=1}^{m}\bar{\beta}_j\right)\left(\sum_{j=1}^{m}(\beta_j+\bar{\beta}_j)\right)^{-1}+$$

$$+(\beta+\bar{\beta})(I_m-\bar{\pi}^T)^{-1}\left(\sum_{l\in C_2\cup C_4\cup\ldots C_{\tilde{d}}} 2a_l\bar{\lambda}^{(l)}\right)\left(\sum_{j=1}^{m}\beta_j\right)\left(\sum_{j=1}^{m}(\beta_j+\bar{\beta}_j)\right)^{-1}<1.$$

Then the sequence $\{\mathsf{E}(\sum_{j=1}^{m}\kappa_{j,i}); i = 0, 1, \ldots\}$ *is bounded.*

Inequality (6) helps to prove Theorems 2, 4 using iterative-magoriant approach (see, for example, [3]). Together with Theorem 2 from the cited work the last two theorems give conditions for the stationary distribution existence for the Markov chain under study. Proof of Theorems 3, 5 is done using inequality similar to (6). It is interesting to note that averaging of random environment influence onto input flows and secondary customers is done independently.

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