OVERLOADING IN QUEUEING NETWORK NODES

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In this paper a problem of a limit distribution calculation in a queueing network with an overloaded node and with a node replaced by another queueing network are considered.

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1. INTRODUCTION

In this paper a problem of a limit distribution calculation in a queueing network with an overloaded node is considered. This problem originates from an analogy between queueing networks and hydrodynamical models. So it is natural to analyze an overloaded regime in queueing networks as an analogy of a turbulence flow in a fluid.

2. QUEUEING NETWORK WITH OVERLOADED NODES

In this section we consider queueing networks in which some nodes are overloaded. In the overloaded node a service intensity does not exceed an input flow intensity and so in this node an infinite queue is created. Then a Markov process which describes numbers of customers in the network nodes does not have a limit distribution. We construct a model which allows to calculate the limit distribution of customers numbers in underloaded nodes of the network.

In this model each overloaded node becomes a source of a Poisson flow of customers with an intensity which coincides with a service intensity. This assumption changes a system of balance equations for input flow intensities of the network nodes and allows to calculate a limit distribution in underloaded nodes by the Jackson product theorem.

3. UNDERLOADED NETWORK

Consider the queueing network G with the Poisson input flow and the intensity λ and the nodes set $I_0 = \{0\} \cup S_0,\ S_0 = \{1, ..., m\}$. Here the node 0 is the external source and the node k consists of a single server with exponentially distributed service times and the parameter $\mu_k,\ k \in S_0$. A customers displacement is defined by the route matrix $\Theta = \left\|\theta_{ij}\right\|_{i,j \in I_0}$, where θ_{ij} is the probability of a customer transition from the node i to the node j. Suppose that all nondiagonal elements of the matrix Θ are positive. Then the route matrix Θ is indivisible and consequently there is the vector $(\gamma_s,\ s \in S_0)$, $\gamma_k > 0,\ k \in S_0$, so that the system

$$\lambda_k - \sum_{j \in S_0} \lambda_j \theta_{jk} = \lambda \theta_{0k},\ k \in S_0, \tag{1}$$

has the single solution [1] $\lambda(\gamma_s,\ s \in S_0)$.

Numbers of customers in the network nodes are described by the Markov process $x(t)$ with the state set $X = \{x = (x_s,\ s \in S_0) : \ x_k \geq 0,\ k \in S_0\}$ and positive transition intensities

$$L(x, x + e_k) = \lambda\theta_{0k},\ L(x + x_k, x) = \mu_k\theta_{k0},$$

$$L(x + e_k, x + e_i) = \mu_k\theta_{ki},\ k, i \in S_0,\ k \neq i.$$

Here $e_k = (e_{ks},\ s \in S_0)$ is $|S_0|$-dimensional vector in which the component $e_{kk} = 1$ and all others – 0, $|M|$ is a number of elements in a finite set M. If

$$\rho_i = \frac{\lambda\gamma_i}{\mu_i} < 1,\ i \in S_0, \tag{2}$$

then the process $x(t)$ is ergodic [2] and its limit distribution $P(x)$ is calculated by the formula [1]

$$P(x) = \prod_{i \in S_0} \pi_i(x_i),\ \pi_i(x_i) = (1 - \rho_i)\rho_i^{x_i},\ x \in X. \tag{3}$$

Consider the queueing network G with the Poisson input flow and the intensity λ and the nodes set $I_0 = \{0\} \cup S_0,\ S_0 = \{1, ..., m\}$. Here the node 0 is the external source and the node k consists of a single server with exponentially distributed service times and the parameter $\mu_k,\ k \in S_0$. A customers displacement is defined by the route matrix $\Theta = \left\|\theta_{ij}\right\|_{i,j \in I_0}$, where θ_{ij} is the probability of a customer transition from the node i to the node j. Suppose that all nondiagonal elements of the matrix Θ are positive. Then the route matrix Θ is indivisible and consequently there is the vector $(\gamma_s,\ s \in S_0),\ \gamma_k > 0,\ k \in S_0$, so that the system

$$\lambda_k - \sum_{j \in S_0} \lambda_j\theta_{jk} = \lambda\theta_{0k},\ k \in S_0, \tag{4}$$

has the single solution [1] $\lambda(\gamma_s,\ s \in S_0)$.

4. OVERLOADED NETWORK

Suppose that the condition (2) which characterizes underloading of all network nodes and guarantees the process $x(t)$ ergodicity (and a calculation of its limit distribution by the formula (3)) is not true. Assume that there is a nonempty set $I_0 \subset S_0$ and a positive number λ_0 satisfying the condition

$$\frac{\mu_k}{\gamma_k} = \lambda_0,\ k \in I_0,\ \lambda_0 < \min_{k \in S_1} \frac{\mu_k}{\gamma_k} = \lambda_0',\ S_1 = S_0 \backslash I_0.$$

Construct a Markov model of an overloaded network by an assumption

$$\lambda_0 \leq \lambda < \lambda_0'. \tag{5}$$

Suppose that overloaded nodes $k \in I_0$ work as sources of Poisson flows with the intensities μ_k and absorb all other arriving customers. To define input intensities Λ_k of underloaded nodes $k \in S_1$ the system of balance equations (1) (in overloaded nodes λ_k is replaced by μ_k, $k \in I_0$) is replaced by the system

$$\Lambda_k - \sum_{j \in S_1}^{m} \Lambda_j \theta_{jk} = \lambda \theta_{0k} + \sum_{j \in I_0} \mu_j \theta_{j\,k}, \; k \in S_1. \tag{6}$$

To find Λ_k, $k \in S_1$, on the nodes set $J_1 = \{0\} \cup S_1$ define the route matrix $\Pi = ||\pi_{ij}||_{i,j \in J_1}$:

$$\pi_{i\,j} = \theta_{i\,j}, \; i, j \in S_1,$$

$$\pi_{0\,j} = \frac{\lambda}{\lambda + \mu}\theta_{0\,j} + \sum_{t \in I_0} \frac{\mu_t}{\lambda + \mu}\theta_{t\,j}, \; j \in S_1, \; \mu = \sum_{t \in I_0} \mu_t,$$

$$\pi_{j\,0} = \theta_{j\,0} + \sum_{t \in I_0} \theta_{j\,t}, \; j \in S_1,$$

$$\pi_{0\,0} = \frac{\lambda}{\lambda + \mu}(\theta_{0\,0} + \sum_{t \in I_0} \theta_{0\,t}) + \sum_{t \in I_0} \frac{\mu_t}{\lambda + \mu}(\theta_{t\,0} + \sum_{s \in I_0} \theta_{t\,s}).$$

Then the system (6) may be rewritten in the form ($\Lambda = \lambda + \mu$)

$$\Lambda_k - \sum_{j \in S_1} \Lambda_j \pi_{j\,k} = \Lambda \pi_{0\,k} = \lambda \theta_{0k} + \sum_{j \in I_0} \mu_j \theta_{j\,k}, \; k \in S_1. \tag{7}$$

By a construction the route matrix $\Pi = \Pi(\lambda)$ is indivisible for any $\lambda > 0$. Then for any $\lambda > 0$ from [1] obtain that there is the vector

$$(\Lambda_s, \; s \in S_1) = (\Lambda_s(\lambda), \; s \in S_1), \; \Lambda_k(\lambda) > 0, \; k \in S_1,$$

which is the single solution of the system (7). Consequently there is the reversible matrix T (with the dimension $|S_1| \times |S_1|$ and with elements which do not depend on λ) satisfying the equalities

$$(\Lambda_s, \; s \in S_1) = \left(\lambda \theta_{0s} + \sum_{j \in I_0} \mu_j \theta_{j\,s}, \; s \in S_1 \right) T.$$

Denote

$$a_s = (\theta_{0s}, \; s \in S_1)T, \; b_s = \left(\sum_{j \in I_0} \mu_j \theta_{j\,s} \right) T, \; s \in S_1,$$

then $\Lambda_s(\lambda) = a_s \lambda + b_s$, $\lambda \geq 0$ and so

$$a_s > 0, \; b_s \geq 0, \; s \in S_1. \tag{8}$$

Numbers of customers in the underloaded nodes of the overloaded network are described by the Markov process $y(t)$ with the state set

$$Y = \{y = (y_s, \; s \in S_1) : \; y_k \geq 0, \; k \in S_1\}$$

and with the positive transition intensities

$$\overline{L}(y, y+E_k) = \Lambda\pi_{0k},\ \overline{L}(y+E_k, y) = \mu_k\pi_{k0},$$

$$\overline{L}(y+E_k, y+E_i) = \mu_k\pi_{ki},\ k, i \in S_1,\ k \neq i.$$

Here $E_k = (E_{ks},\ s \in S_1)$ is $|S_1|$-dimensional vector with the component $E_{kk} = 1$ and with all other components - 0.

The formulas (1), (5), (7) lead to the equalities $\lambda_0\gamma_k = \Lambda_k(\lambda_0),\ k \in S_1$. So from (5), (8) obtain

$$\mu_k > \lambda\gamma_k = \frac{\lambda}{\lambda_0}(a_k\lambda_0 + b_k) \geq (a_k\lambda + b_k) = \Lambda_k(\lambda),\ k \in S_1. \tag{9}$$

The formulas (1), (5) give that

$$\Lambda_t(\lambda) \geq \Lambda_t(\lambda_0) = \lambda_0\theta_{0\,t} + \sum_{j\in S_0}\lambda_0\gamma_j\theta_{j\,t} = \mu_t,\ t \in I_0. \tag{10}$$

So in the overloaded network the underloaded nodes remain underloaded and overloaded nodes - overloaded. Thus in the conditions (5) the process $y(t)$ is ergodic [2] and its limit distribution $\overline{P}(y)$ is calculated by the formula [1]

$$\overline{P}(y) = \prod_{i\in S_1}\overline{\pi}_i(y_i),\ \overline{\pi}_i(y_i) = (1-\overline{\rho}_i)\overline{\rho}_i^{y_i},\ y \in Y, \tag{11}$$

where

$$\overline{\rho}_i = \frac{\Lambda_i}{\mu_i},\ i \in S_1.$$

Denote $\lambda_1 = \min_{k\in S_1}\frac{\mu_k - b_k}{a_k}$. As the formula (9) is true so we obtain $\frac{\mu_k - b_k}{a_k} > \frac{\mu_k}{\gamma_k} = \frac{\mu_k}{a_k + b_k/\lambda_0},\ k \in S_1$, and consequently $\mu_k > \Lambda_k(\lambda_0),\ k \in S_1$,

$$\lambda_0' < \lambda_1.$$

All considerations of this subsection may be generalized to the case when the conditions (5) are widen to

$$\lambda_0 \leq \lambda < \lambda_1. \tag{12}$$

Indeed, in the conditions (12) the inequality $\mu_k > \Lambda_k(\lambda),\ k \in S_1$, is true automatically and the formula $\Lambda_t(\lambda) \geq \mu_t,\ t \in I_0$, follows from the formulas (8), (10). The conditions (5) distinguish the first subset of overloaded nodes I_0. The second subset I_1 is defined by the equality $I_1 = \{i : \frac{\mu_i - b_i}{a_i} = \lambda_1\}$. Then the consideration of this subsection may be repeated.

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