PROCESSOR SHARING SYSTEM WITH NON-HOMOGENEOUS CUSTOMERS AND LIMITATIONS

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We investigate a processor sharing system in which each customer has three interdependent random characteristics: the required number of homogeneous discrete resourse units, capacity (volume) and service time (length). The total capacity of customers and the total number of discrete resourse units are limited. The type of a customer is defined by the number of resourse units required for his servise. We determine a stationary distribution of the number of customers present in the system, as well as the loss probability for a customer of each type.

Keywords: processor sharing, customer capacity, total (customers) capacity, length of a customer.

1. INTRODUCTION

Processor sharing models have been used to solve various problems occuring in computer and communicating systems designing. Presently, they are applicable to situations where a common resourse is shared by a varying number of concurrent users [1] (for example, to WEB-servers modelling [2]).

Let ξ be the length of a customer, i.e. ξ is a customer service time, under condition that there are no other customers in the system during the customer service. We shall also use a notion of residual length of a customer. This is a rest of service time of a customer after some time moment t, under condition that there are no other customers in the system during the time of service termination of the customer.

Results obtained in the present paper is a generalization of ones obtained in [3].

Consider the system that differs from the classical $M/G/1-EPS$ system [2] in the following properties.

1. The system contains N units of some homogeneous discrete resourse. Each customer, independently of his arrival time and characteristics of other customers, requires m ($m \le N$) units of the resourse for his service with probability q_m, $\sum_{m=1}^{N} q_m = 1$. In the sequel we refer to a customer required m units of the resourse as an m-customer or a customer of type m.

2. Independently of other customers and his arrival time, each m-customer is characterized by the random capacity ζ_m (the random variable ζ_m is not necessarily discrete) and the

length ξ_m. The distribution functions

$$F_m(x,t) = \boldsymbol{P}\{\zeta_m < x, \xi_m < t\}, \ m = \overline{1,n},$$

are given. Denote by $\eta(t)$ the number of customers in the system at time t, and denote by $\sigma(t)$ the total capacity, i.e., the total sum of capacities of customers that are in the system at this instant.

3. The total capacity $\sigma(t)$ in the considered system is bounded by a quantity $V > 0$, which is called the memory capacity.

Denote by $L_m(x) = F_m(x, \infty)$ the distribution function of the capacity of m-customer, and denote by $B_m(t) = F_m(\infty, t)$ the distribution function of his length. If at the arrival time τ of an m-customer there are less than m free units of the resourse in the system, the customer is lost and has no effect on further system behavior. If the required amount of free resourse units is available, the customer is nevertheless lost if his capacity x is such that $\sigma(\tau - 0) + x > V$. If there are sufficiently many free resourse units at the arrival time and the condition $\sigma(\tau - 0) + x \le V$ is satisfied, then immediately after the arrival of a customer his service starts; here $\eta(\tau) = \eta(\tau - 0) + 1$ and $\sigma(\tau) = \sigma(\tau - 0) + x$. If τ is the service termination epoch for a customer of capacity x, then $\eta(\tau) = \eta(\tau - 0) - 1$ and $\sigma(\tau) = \sigma(\tau - 0) - x$.

Clearly, for the system in question a stationary mode exists, if the inpute flow parameter a and the first moments $\beta_{11}, \dots, \beta_{N1}$ of lengths of customers of all types are finite.

For the considered system, we find the distribution of the number of customers in the system at an arbitrary time instant in stationary mode, and also stationary loss probabilities for customers of each type.

2. PROCESS AND CHARACTERISTICS

Assume that customers in the considered system at an arbitrary time instant t are enumerated at random; i.e., if the number of customers is k, then there are $k!$ ways to enumerate them, and each enumeration can be chosen with the same probability $1/k!$. Denote by $\nu_j(t)$ the number of resourse units that are used by jth customer at time t, and denote by $\sigma_j(t)$ the capacity of this customer. We denote by $\xi_j^*(t)$ the residual length of the jth customer in the system from the time instant t.

One can easily show that the system under consideration is described by the Markov process

$$\left(\eta(t), \nu_j(t), \sigma_j(t), \xi_j^*(t), j = \overline{1, \eta(t)}\right). \tag{1}$$

Note that in our notations we have $\sigma(t) = \sum_{j=1}^{\eta(t)} \sigma_j(t)$. In what follows, to simplify the notation, we denote $R_k = (r_1, \dots, r_k)$, $Y_k = (y_1, \dots, y_k)$, $R_k^j = (r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_k)$ and similarly $Y_k^j = (y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_k)$. We also assume $r_{(k)} = r_1 + \cdots + r_k$.

Sometimes, in the case $k = 1$, instead of R_1 and Y_1 we write, respectively, r_1 and y_1 or the values that these components take, and in the case $k = 2$, instead of R_2 and Y_2 we write (r_1, r_2) and (y_1, y_2) or their values respectively. In other words, we sometimes specify vectors of small dimensions by indicating their components. We also use the notations $(R_k, m) = (r_1, \dots, r_k, m)$ and $(Y_k, z) = (y_1, \dots, y_k, z)$. We denote the all-one vector of dimension k by 1_k.

We characterize the process (1) by functions with the following probabilistic sense:

$$G_k(x, R_k, Y_k, t) = \boldsymbol{P}\{\eta(t) = k, \sigma(t) < x, \nu_i(t) = r_i, \xi_i^*(t) < y_i, i = \overline{1,k}\}, k = \overline{1,N}, r_{(k)} \le N; \tag{2}$$

$$\Theta_k(R_k, Y_k, t) = \boldsymbol{P}\{\eta(t) = k, \nu_i(t) = r_i, \xi_i^*(t) < y_i, i = \overline{1,k}\} = G_k(V, R_k, Y_k, t), \ k = \overline{1,N}, \tag{3}$$

$r_{(k)} \le N$. We also introduce the functions

$$\Pi_k(R_k, t) = \boldsymbol{P}\{\eta(t) = k, \nu_i(t) = r_i, i = \overline{1,k}\} = \lim_{y_1,\dots,y_k \to \infty} \Theta_k(R_k, Y_k, t), \ k = \overline{1,N}, r_{(k)} \le N; \tag{4}$$

$$P_0(t) = \boldsymbol{P}\{\eta(t) = 0\}; \tag{5}$$

$$P_k(t) = \boldsymbol{P}\{\eta(t) = k\} = \sum_{r_{(k)} \le N} \Pi_k(R_k, t), \ k = \overline{1,N}; \tag{6}$$

Asume that $N < \infty$ or (and) $V < \infty$. Then, a stationary mode exists if $\rho = a\beta_1 < \infty$, where $\beta_1 = q_1\beta_{11} + \dots + q_N\beta_{1N}$ is the first moment of the total service time in the system, i.e. $\eta(t) \Rightarrow \eta$, $\sigma(t) \Rightarrow \sigma$, $\nu_j(t) \Rightarrow \nu_j$ and $\xi_j^*(t) \Rightarrow \xi_j^*$ in the weak convergence sense. So, the following limits exist:

$$g_k(x, R_k, Y_k) = \lim_{t\to\infty} G_k(x, R_k, Y_k, t), \ k = \overline{1,N}, \ r_{(k)} \le N; \tag{7}$$

$$\theta_k(R_k, Y_k) = \lim_{t\to\infty} \Theta_k(R_k, Y_k, t) = g_k(V, R_k, Y_k); \tag{8}$$

$$\pi_k(R_k) = \lim_{t\to\infty} \Pi_k(R_k, t) = \lim_{y_1,\dots,y_k\to\infty} \theta_k(R_k, Y_k), \ k = \overline{1,N}, \ r_{(k)} \le N; \tag{9}$$

$$p_0 = \lim_{t\to\infty} P_0(t); \tag{10}$$

$$p_k = \lim_{t\to\infty} P_k(t) = \sum_{r_{(k)} \le N} \pi_k(R_k), \ k = \overline{1,N}. \tag{11}$$

Note that the functions $G_k(x, R_k, Y_k, t)$, $g_k(x, R_k, Y_k)$ and $\Theta_k(R_k, Y_k, t)$, $\theta_k(R_k, Y_k)$ are symmetric with respect to simultaneous permutations of components with the same indices of the vectors R_k and Y_k due to our random enumeration of customers in the system.

Denote by

$$H_m(x, y) = \boldsymbol{P}\{\zeta_m < x, \xi_m \ge y\} = \int_{v=0}^{x} \int_{u=y}^{\infty} dF_m(v, u) =$$

$$= \boldsymbol{P}\{\zeta_m < x\} - \boldsymbol{P}\{\zeta_m < x, \xi_m < y\} = L_m(x) - F_m(x, y). \tag{12}$$

3. STATIONARY DISTRIBUTION OF THE NUMBER OF CUSTOMERS

Using the method of auxiliary variables [4], we can write out partial differential equations for the functions defined by (2)–(6). Then, passing to the limit as $t \to \infty$, we obtain the following stationary equations for functions (7)–(11):

$$0 = -ap_0 \sum_{m=1}^{N} q_m L_m(V) + \sum_{m=1}^{N} \left. \frac{\partial \theta_1(m,y)}{\partial y} \right|_{y=0}; \tag{13}$$

$$-\frac{\partial \theta_1(m,y)}{\partial y} + \left. \frac{\partial \theta_1(m,y)}{\partial y} \right|_{y=0} = aq_m p_0 F_m(V,y) - a \sum_{j=1}^{N-m} q_j \int_0^V g_1(V-x,m,y) dL_j(x) +$$

$$+ \sum_{j=1}^{N-m} \left. \frac{\partial \theta_2((m,j),(y,z))}{\partial z} \right|_{z=0}, \quad m = \overline{1, N-1}, \; N \geq 2; \tag{14}$$

$$-\frac{\partial \theta_1(N,y)}{\partial y} + \left. \frac{\partial \theta_1(N,y)}{\partial y} \right|_{y=0} = aq_N p_0 F_N(V,y); \tag{15}$$

$$-\frac{1}{k} \sum_{j=1}^{k} \left[\frac{\partial \theta_k(R_k, Y_k)}{\partial y_j} - \left. \frac{\partial \theta_k(R_k, Y_k)}{\partial y_j} \right|_{y_j=0} \right] = \frac{a}{k} \sum_{j=1}^{k} q_{r_j} \int_0^V g_{k-1}(V-x, R_k^j, Y_k^j) d_x F_{r_j}(x, y_j) -$$

$$-a \sum_{m=1}^{N-r_{(k)}} q_m \int_0^V g_k(V-x, R_k, Y_k) dL_m(x) + \sum_{m=1}^{N-r_{(k)}} \left. \frac{\partial \theta_{k+1}((R_k, m),(Y_k, z))}{\partial z} \right|_{z=0},$$

$$k = \overline{2, N-1}, \; r_{(k)} < N, \; N \geq 2; \tag{16}$$

$$-\sum_{j=1}^{N} \left[\frac{\partial \theta_N(1_N, Y_N)}{\partial y_j} - \left. \frac{\partial \theta_N(1_N, Y_N)}{\partial y_j} \right|_{y_j=0} \right] = aq_1 \sum_{j=1}^{N} \int_0^V g_{N-1}(V-x, 1_{N-1}, Y_N^j) d_x F_1(x, y_j),$$

$$N \geq 2. \tag{17}$$

In the stationary mode, we have boundary conditions described by the following equilibrium equations:

$$\left. \frac{\partial \theta_1(m,y)}{\partial y} \right|_{y=0} = aq_m p_0 L_m(V), \; m = \overline{1, N}; \tag{18}$$

$$\left. \frac{\partial \theta_k((R_{k-1}, m),(Y_{k-1}, z))}{\partial z} \right|_{z=0} = aq_m \int_0^V g_{k-1}(V-x, R_{k-1}, Y_{k-1}) dL_m(x),$$

$$k = \overline{2, N}, \; r_{(k-1)} + m \leq N, \; N \geq 2. \tag{19}$$

To these relations, we should add the normalization condition, which can be represented as follows:

$$p_0 + \sum_{k=1}^{N} \sum_{r(k) \leq N} \pi_k(R_k) = 1. \tag{20}$$

Introduce the function $\Phi_j^y(x) = \int_0^y H_j(x,u)du$. Its meaning becomes obvious if we use the representation $F_j(x,u) = L_j(x)B_j(u \mid \zeta_j < x)$, where $B_j(u \mid \zeta_j < x) = \boldsymbol{P}\{\xi_j < u \mid \zeta_j < x\}$ is the conditional distribution function of the length of a j-customer given that his capacity is less than x. Then (12) implies $\Phi_j^y(x) = L_j(x)\int_0^y [1 - B_j(u \mid \zeta_j < x)]du$.

Let us also introduce the following notation for the Stieltjes convolution:

$$F_1 * \cdots * F_n(x) = \mathop{*}_{j=1}^{n} F_j(x).$$

Using the above-mentioned symmetry property of functions (7) and (8) and taking into account boundary conditions (18) and (19), one can show by direct substitution that the solution of equations (13)–(17) can be represented as

$$g_k(x, R_k, Y_k) = Ca^k \mathop{*}_{j=1}^{k} \Phi_{r_j}^{y_j}(x) \prod_{j=1}^{k} q_{r_j}, \ k = \overline{1,N}, \ r_{(k)} \le N, \tag{21}$$

where C is a constant to be specified later from the normalization condition (20). It follows from (8) that

$$\theta_k(R_k, Y_k) = Ca^k \mathop{*}_{j=1}^{k} \Phi_{r_j}^{y_j}(V) \prod_{j=1}^{k} q_{r_j}, \ k = \overline{1,N}, \ r_{(k)} \le N. \tag{22}$$

Introduce the notation $D_j(x) = \int_{u=0}^{x}\int_{y=0}^{\infty} dF_j(u,y)$, $m = \overline{1,N}$. The function $D_j(x)$ has the meaning of "partial" mathematical expectation [5] of the random variable ξ_j with respect to the event $\{\zeta_j < x\}$, i.e., $D_j(x) = \boldsymbol{E}(\xi_j, \zeta_j < x) = \boldsymbol{E}(\xi_j \mid \zeta_j < x)L_j(x)$, where $\boldsymbol{E}(\xi_j \mid \zeta_j < x)$ is the conditional mathematical expectation of the length of a customer of type j given that his capacity is less than x. It is easily seen that

$$D_j(x) = \lim_{y\to\infty} \Phi_j^y(x) = L_j(x) \int_0^\infty [1 - B_j(u \mid \zeta_j < x)]du.$$

Using relation (9), we obtain

$$\pi_k(R_k) = Ca^k \mathop{*}_{j=1}^{k} D_{r_j}(V) \prod_{j=1}^{k} q_{r_j}, \ k = \overline{1,N}, \ r_{(k)} \le N. \tag{23}$$

It follows from (11) that $p_k = Ca^k \sum_{r_{(k)} \le N} \mathop{*}_{j=1}^{k} D_{r_j}(V) \prod_{j=1}^{k} q_{r_j}$, $k = \overline{1,N}$. The latter relation and normalization condition (20) finally yield

$$p_k = p_0 a^k \sum_{r_{(k)} \le N} \mathop{*}_{j=1}^{k} D_{r_j}(V) \prod_{j=1}^{k} q_{r_j}, \ k = \overline{1,N}, \tag{24}$$

where

$$p_0 = C = \left[1 + \sum_{k=1}^{N} a^k \sum_{r_{(k)} \le N} \mathop{*}_{j=1}^{k} D_{r_j}(V) \prod_{j=1}^{k} q_{r_j}\right]^{-1}. \tag{25}$$

4. LOSS PROBABILITY

Finding the stationary loss probability P_m, $m = \overline{1, N}$ for a customer of type m is based on the fact that in stationary mode the average number of customers admited to the system within a time unit (i.e., customers who entered the system during this time period and were not lost) is equal to the average number of customers whose service was terminated within this time period. Thus, taking into account the symmetry of $\theta_k(R_k, Y_k)$ with respect to the above-mentioned permutations of components of vectors (R_k, Y_k), we obtain the following equilibrium equation:

$$aq_m(1 - P_m) = \sum_{j=1}^{N-m+1} \sum_{r_{(j-1)} \le N-m} \left. \frac{\partial \theta_j((R_{j-1}, m), (\infty_{j-1}, z))}{\partial z} \right|_{z=0}, \quad m = \overline{1, N},$$

where $\infty_{j-1} = (\infty, \ldots, \infty)$ is a vector with $j - 1$ components. The latter relation, taking into account (22) and (25), yields

$$P_m = 1 - p_0 \left[L_m(V) + \sum_{j=1}^{N-m} a^j \sum_{r_{(j)} \le N-m} L_m * \left(\mathop{*}_{l=1}^{j} D_{r_l} \right) (V) \prod_{l=1}^{j} q_{r_l} \right]. \qquad (26)$$

Then the total loss probability is defined by the relation

$$P = \sum_{m=1}^{N} q_m P_m = 1 - p_0 \sum_{m=1}^{N} q_m \left[L_m(V) + \sum_{j=1}^{N-m} a^j \sum_{r_{(j)} \le N-m} L_m * \left(\mathop{*}_{l=1}^{j} D_{r_l} \right) (V) \prod_{l=1}^{j} q_{r_l} \right].$$

Note that direct application of obtained relations happens to be inconvenient for computation in the general case. Therefore, "direct" practical application is possible in certain particular cases (see ex. [3]).

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