A TANDEM QUEUEING SYSTEM WITH RENEWAL INPUT

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We consider a tandem queueing system consisting of two stations in serious. The first station is represented by a single-server queue with renewal input, PH (Phase-type) service time distribution and waiting room. After service at the first station a customer proceeds to the second station that is described by a single-server queue without a buffer. In case a customer completes the service at the first server and meets the second server being busy, it waits until the second server becomes free and then occupies this server immediately. The waiting period is accompanied by blocking the first stage server operation. The service time of a customer by the second server has the PH-type distribution. We derive the ergodicity condition and calculate the stationary distribution of the system states. The Laplace-Stieltjes transforms of the distribution of the sojourn time at both the stations as well as the whole system are derived. The mean values of these sojourn times are calculated.

Keywords: Tandem queue, renewal input, phase-type service time distribution, $GI/M/1$-type Markov chain.

1. INTRODUCTION

Tandem queueing system can be used for modeling real-life two-node networks as well as for the validation of general decomposition algorithms in networks. Thus, tandem queueing systems have found much interest in the literature. An extensive survey of early papers on tandem queues can be seen in [1]. Most of these papers are devoted to exponential queueing models. Over the last two decades significant results are reached in investigation of tandem queues with a batch Markovian arrival process that can be considered as generalization of stationary Poisson model to the case of correlated bursty traffic. Tandem queues with Markovian input were considered in [2, 3, 4, 5, 6].

In the present paper we consider the dual tandem queue with blocking, non-Markovian input and PH type service time distribution at both the single-server stations.

2. MATHEMATICAL MODEL

We consider $GI/PH/1 \to \bullet/PH/1/0$ tandem queue. The first station is represented by the $GI/PH/1$ queue. The inter-arrival times at the first station are independent random variables with general distribution $A(t)$ and the finite first moment $a_1 = \int\limits_0^\infty t dA(t)$.

After service at the first station a customer proceeds to the second station that is represented by a single-server queue without a buffer. In case the customer completes the service at the first server and meets the second server being busy, it waits until the second server becomes free and then occupies this server immediately. The waiting period is accompanied by blocking the first station server operation.

The service times of a customer at the first and at the second station have PH distributions.

Service time having PH distribution with an irreducible representation (β, S) can be interpreted as a time until the underlying Markov process $m_t, t \geq 0$, with a finite state space $\{1, \ldots, M, *\}$ reaches the single absorbing state $*$ conditional the initial state of this process is selected among the states $\{1, \ldots, M\}$ according to probabilistic vector β. Transition rates of the process m_t within the set $\{1, \ldots, M\}$ are defined by the sub-generator S and transition rates into the absorbing state are given by the entries of the column vector $\mathbf{S}_0 = -S\mathbf{e}$. For more information about PH distributions see [7].

We assume that service process at the rth, $r = 1, 2$, server has PH_r distribution with an irreducible representation $(\beta_r, S^{(r)})$ and is governed by the Markov chain $m_t^{(r)}$, $t \geq 0$, with the state space $\{1, \ldots, M_r, *^{(r)}\}$ where the state $*^{(r)}$ is an absorbing one.

3. STATIONARY DISTRIBUTION OF EMBEDDED MARKOV CHAIN

Let t_n denote the time of the nth arrival at the first station and i_n is the number of customers at this station (including the blocked customer, if any) at the epoch $t_n - 0$, $n \geq 1$. It is easy to see that the process i_n, $n \geq 1$, is non-Markovian. In case of $GI/PH/1$ queue we construct the embedded Markov chain $\{i_n, m_n\}$ by introducing the additional component m_n that denotes the phase of service process at the epoch $t_n + 0$, $n \geq 1$. In the queue under consideration such a technique meets with little success due to the blocking phenomena. Thus, to construct the embedded Markov chain for the considered tandem queue we introduce into consideration the "generalized" service time at the first station.

The generalized service time of a tagged customer is just the service time at the first station if the server of the second station is free at the service completion epoch of the tagged customer at the first server. In opposite case the blocking occurs and the generalized service time consists of the service time of the tagged customer at the first server and the time during which the first server is blocked (waits until the second server becomes free).

Analyzing of generalized service time we observe that the duration of this time can be described by the PH distribution with the state space Ω consisting of pares, $\Omega = \{(m^{(1)}, m^{(2)}),\ (m^{(1)}, *^{(2)}),\ (*^{(1)}, m^{(2)}),\ m^{(1)} = \overline{1, M_1},\ m^{(2)} = \overline{1, M_2},\ (*^{(1)}, *^{(2)})\}$, and an infinitesimal generator

$$\begin{pmatrix} S & \boldsymbol{S}_0 \\ 0 & 0 \end{pmatrix},$$

where

$$S = \begin{pmatrix} S^{(1)} \oplus S^{(2)} & I_1 \otimes S_0^{(2)} & S_0^{(1)} \otimes I_2 \\ 0 & S^{(1)} & 0 \\ 0 & 0 & S^{(2)} \end{pmatrix}, \boldsymbol{S}_0 = -S\boldsymbol{e}.$$

The vector $\boldsymbol{\beta}$ for this PH has form $\boldsymbol{\beta} = (\boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2, 0_{M_1+M_2})$.

Remark 1. The phases (states) of generalized service process given by pares from Ω are related to the phases of PH_1, PH_2 service processes as follows:

(i) the phase $(m^{(1)}, m^{(2)})$, $m^{(1)} = \overline{1, M_1}$, $m^{(2)} = \overline{1, M_2}$, corresponds to busy servers at both the stations with PH_1, PH_2 in the phases $m^{(1)}$, $m^{(2)}$ respectively;

(ii) the phase $(m^{(1)}, *^{(2)})$, $m^{(1)} = \overline{1, M_1}$, corresponds to busy first sever with PH_1 in the phase $m^{(1)}$ and idle second server. Analogously, the phase $(*^{(1)}, m^{(2)})$, $m^{(2)} = \overline{1, M_2}$, takes place when the first server is idle and the second sever is in the phase $m^{(2)}$;

(iii) the absorbing phase $(*^{(1)}, *^{(2)})$ corresponds to the case when both the stations are empty.

Remark 2. The dimension of the state space Ω is $M_1M_2 + M_1 + M_2$. The pairs from the set Ω are listed in the lexicographic order. In the following we assume that the pair that is placed on the mth position in the list will be considered as mth state of the underlying process of the generalized service time.

Now we are able to construct the embedded Markov chain describing the queue under consideration. Let m_n be the phase of generalized service time at the epoch $t_n + 0$, $n \geq 1$. It is easy to see that the process $\xi_n = \{i_n, m_n\}$, $n \geq 1$, is an irreducible Markov chain with the state space $\{(0, m), m = 1, \dots, K_0; (i, m), i > 0, m = 1, \dots, K\}$ where $K = M_1M_2 + M_1 + M_2$ and $K_0 = M_1M_2 + M_1$. In the following we will assume that the states of the chain ξ_n are numerated in the lexicographic order.

Lemma 1. *The transition probability matrix of the chain ξ_n, $n \geq 1$, has the following block structure*

$$P = \begin{pmatrix} \tilde{B}_0 & C_0 & 0 & 0 & \dots \\ B_1 & A_1 & A_0 & 0 & \dots \\ B_2 & A_2 & A_1 & A_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$A_n = \int_0^\infty P(n,t)dA(t),\ B_n = \int_0^\infty \int_0^t P(n,x)\boldsymbol{S}_0 dx \boldsymbol{\beta}^*(t-x)dA(t),\ n \geq 0,$$

$$\tilde{B}_0 = \tilde{I}B_0,\ C_0 = \tilde{I}A_0,\ \boldsymbol{\beta}^*(y) = (\boldsymbol{\beta}_1 \otimes \boldsymbol{\beta}_2 e^{S^{(2)}y}, \boldsymbol{\beta}_1(1 - \boldsymbol{\beta}_2 e^{S^{(2)}y}\boldsymbol{e})),$$

$$\tilde{I} = \begin{pmatrix} I_{M_1M_2} & 0 & 0 \\ 0 & I_{M_1} & 0_{M_1 \times M_2} \end{pmatrix},$$

and $P(n,t)$, $n \geq 0$, *are are defined by* $\sum_{n=0}^{\infty} P(n,t)z^n = e^{(S+\mathbf{S}_0\beta z)t}$.

Corollary 1. *The process* $\xi_n, n \geq 1$, *is of the* $GI/M/1$ *type Markov chain (see [7]).*

Theorem 1. *Stationary distribution of the Markov chain* ξ_n, $n \geq 1$, *exists if and only if the inequality*

$$\rho = a_1^{-1} b_1^{(g)} < 1$$

is fulfilled. Here $b_1^{(g)} = -\beta S^{-1}\mathbf{e}$ *is the mean value of generalized service time.*

Denote the stationary state probabilities of the chain ξ_n by $\pi(0, m,)$, $i \geq 0$, $m = \overline{1, K_0}$, $\pi(i, m,)$, $i > 0$, $m = \overline{1, K}$. Introduce the notation for row vectors of these probabilities

$$\boldsymbol{\pi}_0 = (\pi(0,1), \pi(0,2), \ldots, \pi(0,K_0)), \ \boldsymbol{\pi}_i = (\pi(i,1), \pi(i,2), \ldots, \pi(i,K)), \ i > 0.$$

Theorem 2. *The stationary probability vectors* $\boldsymbol{\pi}_i$, $i \geq 0$, *are calculated as follows:*

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_1 \mathcal{R}^{i-1}, \ i \geq 2,$$

where the matrix $\mathcal{R}$ *is the minimal non-negative solution of the matrix equation*

$$\mathcal{R} = \sum_{j=0}^{\infty} \mathcal{R}^j A_j,$$

and the vector $(\boldsymbol{\pi}_0, \ \boldsymbol{\pi}_1)$ *is the unique solution of the system*

$$(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1) = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1)T, \ \boldsymbol{\pi}_0 \mathbf{e} + \boldsymbol{\pi}_1 (I - \mathcal{R})^{-1} \mathbf{e} = 1,$$

where

$$T = \begin{pmatrix} \tilde{B}_0 & C_0 \\ \sum_{j=1}^{\infty} \mathcal{R}^{j-1} B_j & \sum_{j=1}^{\infty} \mathcal{R}^{j-1} A_j \end{pmatrix}.$$

4. STATIONARY DISTRIBUTION AT AN ARBITRARY TIME. SOJOURN TIME DISTRIBUTION

Define the state of the system at an arbitrary time t as (i_t, m_t) where i_t is the number of customers at the first station (including the blocked customer, if any); m_t is the state of the PH_2 service process at the moment t if $i_t = 0$ and m_t is the state of the generalized service time at the moment t if $i_t > 0$.

The process of the system states at an arbitrary time $\zeta_t = \{i_t, m_t\}$, $t \geq 0$, has state space $\{(0, m), m = 1, \ldots, M_2, *^{(2)}; \ (i, m), i \geq 1, m \subset \Omega\}$.

Enumerate the states of the process ζ_t in the lexicographic order.

Let $\mathbf{p}_i$ be the vector of steady-state probabilities of this process corresponding the value i of the first component, $i \geq 0$.

Theorem 3. *The vectors $\boldsymbol{p}_i$, $i \geq 0$, are calculated as follows:*

$$\boldsymbol{p}_0 = a_1{}^{-1}[\boldsymbol{\pi}_0 \tilde{I}\Phi_0 + \boldsymbol{\pi}_1 \sum_{n=1}^{\infty} \mathcal{R}^{n-1}\Phi_n],$$

$$\boldsymbol{p}_1 = a_1{}^{-1}[\boldsymbol{\pi}_0 \tilde{I}\tilde{A}_0 + \boldsymbol{\pi}_1 \sum_{n=1}^{\infty} \mathcal{R}^{n-1}\tilde{A}_n],\ \boldsymbol{p}_i = a_1{}^{-1}\boldsymbol{\pi}_1 \mathcal{R}^{i-2} \sum_{n=0}^{\infty} \mathcal{R}^n \tilde{A}_n,\ i > 1,$$

where $\Phi_n = \int\limits_0^{\infty}\int\limits_0^{t} P(n,x)\boldsymbol{S}_0 dx(\boldsymbol{\beta}_2, 0)e^{\bar{S}^{(2)}(t-x)}(1-A(t))dt,\ \bar{S}^{(2)} = \begin{pmatrix} S^{(2)} & \boldsymbol{S}_0^{(2)} \\ 0 & 0 \end{pmatrix},\ \tilde{A}_n = \int\limits_0^{\infty} P(n,t)(1-A(t))dt,\ n \geq 0.$

Having the stationary distributions $\boldsymbol{\pi}_i$, $i \geq 0$, $\boldsymbol{p}_i$, $i \geq 0$, been calculated we can find a number of stationary performance measures of the system under consideration. Some of them are calculated as follows:

- Mean number of customers at the first station at the arrival epoch

$$L = \boldsymbol{\pi}_1(I - \mathcal{R})^{-2}\boldsymbol{e};$$

- Probability that both servers are busy at the arrival epoch

$$P_{busy}^{(1,2)} = \boldsymbol{\pi}_1(I - \mathcal{R})^{-1} diag\,\{I_{M_1M_2},\, 0_{M_1+M_2}\}\boldsymbol{e}.$$

- Probability that the server of the first station is busy and the server of the second station is idle at the arrival epoch

$$P_{busy}^{(1)} = \boldsymbol{\pi}_1(I - \mathcal{R})^{-1} diag\,\{0_{M_1M_2},\, I_{M_1}, 0_{M_2}\}\boldsymbol{e}.$$

- Probability that the first server is blocked at the arrival epoch

$$P_{block} = \boldsymbol{\pi}_1(I - \mathcal{R})^{-1} diag\,\{0_{M_1M_2},\, 0_{M_1}, I_{M_2}\}\boldsymbol{e}.$$

- Probability that the server of the first station is idle and the server of the second station is busy at an arbitrary time

$$\tilde{P}_{idle}^{(1)} = \boldsymbol{p}_0 diag\,\{I_{M_2},\, 0\}\boldsymbol{e}.$$

- Probability that both the stations are empty at an arbitrary time

$$\tilde{P}_{idle}^{(1,2)} = \boldsymbol{p}_0 diag\,\{0_{M_2},\, 1\}\boldsymbol{e}.$$

Theorem 4. *The Laplace-Stieltjes transform of the stationary distribution of the actual sojourn time is calculated as follows:*

(i) at the first station

$$v_1(\theta) = [\boldsymbol{\pi}_0 \tilde{I} + \boldsymbol{\pi}_1(I - \mathcal{R}\varphi_g(\theta))^{-1}\varphi_g(\theta)](\theta I - S)^{-1}\boldsymbol{S}_0;$$

(ii) in the whole system

$$v(\theta) = v_1(\theta)\varphi_2(\theta)).$$

Here $\varphi_g(\theta)) = \boldsymbol{\beta}(\theta I - S)^{-1}\boldsymbol{S}_0$ *and* $\varphi_2(\theta)) = \boldsymbol{\beta}_2(\theta I - S^{(2)})^{-1}\boldsymbol{S}_0^{(2)}$ *are the Laplace-Stieltjes transforms of distribution of generalize service time and service time at the second station respectively.*

Corollary 2. *The mean actual sojourn time is calculated as follows:*
(*i*) *at the first station*

$$\bar{v}_1 = [\pi_0 \tilde{I} + \pi_1 (I - \mathcal{R})^{-1}](-S)^{-1} e + L b_1^{(g)};$$

(*ii*) *in the whole system:* $\bar{v} = \bar{v}_1 + b_1^{(2)}$,
where $b_1^{(g)}$ *and* $b_1^{(2)}$ *are the mean values of generalize service time and service time at the second station respectively.*

5. CONCLUSION

In this paper, the $GI/PH/1 \to \bullet/PH/1/0$ tandem queue with blocking is studied. The condition for the existence of the stationary distribution is derived and the algorithms for calculating the steady state probabilities are presented. The Laplace-Stieltjes transform of the distribution of the actual sojourn time at both stations as well as at the whole system are derived. Formulas for the mean values of these times are presented. The results of this paper can be applied to areas such as capacity planning, performance evaluations, and optimization of real-life tandem queues and two-node networks.

REFERENCES

1. *Gnedenko B. W., Konig D.* Handbuch der Bedienungstheorie // Berlin: Akademie Verlag, 1983.
2. *Breuer L., Dudin A. N., Klimenok V. I., Tsarenkov G. V.* A two-phase $BMAP/G/1/N \to \cdot PH/1/M-1$ system with blocking // Automation and Remote Control. 2004. V. 65. P. 117–130.
3. *Gomez-Corral A.* A tandem queue with blocking and Markovian arrival process // Queueing Systems. 2002. V. 41. P. 343–370.
4. *Gomez-Corral A., Martos M. E.* Performance of two-station tandem queues with blocking: The impact of several flows of signals // Performance Evaluation. 2006. 63. P. 910–938.
5. *Klimenok V. I., Breuer L., Tsarenkov G. V., Dudin A. N.* The $BMAP/G/1/N \to \cdot/PH/1/M-1$ tandem queue with losses // Performance Evaluation. 2005. V. 61. P. 17–60.
6. *Kim C.S., Klimenok V., Tsarenkov G.V., Breuer L., Dudin A.N.* The $BMAP/G/1 \to \cdot/PH/1/M$ tandem queue with feedback and losses // Performance Evaluation. 2007. 64. P. 802–818.
7. *Neuts M.F.* Matrix-Geometric Solutions in Stochastic Models - An Algorithmic Approach // Johns Hopkins University Press, 1981.