# ALGORITHM FOR CALCULATING THE STATIONARY DISTRIBUTION OF THE SYSTEM OPERATING IN A RANDOM ENVIRONMENT

**V. Khramova**
*Belarusian State University*
*Minsk, Belarus*
valentina-khramova@yandex.ru

The *BMAP*/*PH*/*N* queueing system operating in a finite state space Markovian random environment is investigated. Efficient algorithm for calculating the stationary distribution of the system states is proposed.

*Keywords:* Batch Markovian Arrival Process, Phase-type Service, Random Environment, Stationary Distribution

## 1. THE MATHEMATICAL MODEL

We consider the queueing system having $N$ identical servers. The system behavior depends on the state of the stochastic process (random environment) $r_t$, $t \geq 0$, which is assumed to be an irreducible continuous time Markov chain with the state space $\{1, \ldots, R\}$, $R \geq 2$, and the infinitesimal generator $Q$.

The input flow into the system is the following modification of the *BMAP*. In this input flow, the batch arrivals are directed by the process $\nu_t$, $t \geq 0$, (the directing process) with the state space $\{0, 1, \ldots, W\}$. Under the fixed state $r$ of the random environment, this process behaves as an irreducible continuous time Markov chain. Transitions of the chain $\nu_t$, $t \geq 0$, which are accompanied by arrival of $k$-size batch, are described by the matrices $D_k^{(r)}$, $k \geq 0$, $r = \overline{1, R}$, with the generating function $D^{(r)}(z) = \sum\limits_{k=0}^{\infty} D_k{}^{(r)} z^k$, $|z| \leq 1$. The matrix $D^{(r)}(1)$ is an irreducible generator for all $r = \overline{1, R}$. Under the fixed state $r$ of the random environment, the average intensity $\lambda^{(r)}$ (fundamental rate) of the *BMAP* and the intensity $\lambda_b^{(r)}$ of batch arrivals are defined as

$$\lambda^{(r)} = \boldsymbol{\theta}^{(r)}(D^{(r)}(z))'|_{z=1}\boldsymbol{e}, \quad \lambda_b^{(r)} = \boldsymbol{\theta}^{(r)}(-D_0^{(r)})\boldsymbol{e}.$$

Here the row vector $\boldsymbol{\theta}^{(r)}$ is the solution to the equations $\boldsymbol{\theta}^{(r)} D^{(r)}(1) = 0$, $\boldsymbol{\theta}^{(r)}\boldsymbol{e} = 1$, $\boldsymbol{e}$ is a column vector of appropriate size consisting of 1's. The variation coefficient $c_{var}^{(r)}$ of intervals between batch arrivals is given by

$$(c_{var}^{(r)})^2 = 2\lambda_b^{(r)}\boldsymbol{\theta}^{(r)}(-D_0^{(r)})^{-1}\boldsymbol{e} - 1,$$

while the correlation coefficient $c_{cor}^{(r)}$ of intervals between successive batch arrivals is calculated as

$$c_{cor}^{(r)} = (\lambda_b^{(r)}\boldsymbol{\theta}^{(r)}(-D_0^{(r)})^{-1}(D^{(r)}(1) - D_0^{(r)})(-D_0^{(r)})^{-1}\boldsymbol{e} - 1)/(c_{var}^{(r)})^2.$$

The service process is defined by the modification of the phase ($PH$)-type service time distribution. Service time is interpreted as the time until the irreducible continuous time Markov chain $m_t,\ \ t \geq 0$, with the state space $\{1, \ldots, M+1\}$ reaches the absorbing state $M+1$. Under the fixed value $r$ of the random environment, transitions of the chain $m_t,\ \ t \geq 0$, within the state space $\{1, \ldots, M\}$ are defined by an irreducible sub-generator $S^{(r)}$ while the intensities of transition into the absorbing state are defined by the vector $\boldsymbol{S}_0^{(r)} = -S^{(r)}\boldsymbol{e}$. At the service beginning epoch, the state of the process $m_t,\ \ t \geq 0$, is chosen according to the probabilistic row vector $\beta^{(r)},\ \ r = \overline{1, R}$. It is assumed that the state of the process $m_t,\ \ t \geq 0$, is not changed at the epoch of the process $r_t,\ \ t \geq 0$, transitions. Just the exponentially distributed sojourn time of the process $m_t,\ \ t \geq 0$, in the current state is re-started with a new intensity defined by the sub-generator corresponding to the new state of the random environment $r_t,\ \ t \geq 0$.

The system under consideration has an infinite waiting space. If an arriving group of customers sees idle servers a part of the group corresponding to the number of free servers occupy these servers while the rest of the group joins the queue. If the system has all servers being busy at a batch arrival epoch, all customer of the group go to the queue.

## 2. STATIONARY STATE DISTRIBUTION

Let

- $i_t,\ t \geq 0$, be the number of customers in the system, $i_t \geq 0$;
- $r_t,\ t \geq 0$, be the state of random environment, $r_t = \overline{1, R}$;
- $\nu_t,\ t \geq 0$, be the state of the $BMAP$ directing process, $\nu_t = \overline{0, W}$;
- $h_t^{(m)},\ t \geq 0$, be the number of servers in phase $m$ at epoch $t$, $h_t^{(m)} \in \{0 \ldots N\}$, $m = \overline{1, M}$, $h_t^{(1)} + \ldots + h_t^{(M)} = min\{i_t, N\}, t \geq 0$.

Then the behaviour of the system under consideration can be described in terms of the multi-dimensional irreducible continuous-time Markov chain

$$\xi_t = \{i_t, r_t, \nu_t, h_t^{(1)}, \ldots, h_t^{(M)}\},\ t \geq 0. \qquad (1)$$

Denote $p(i, r, \nu, h_1, \ldots, h_M) = \lim_{t \to \infty} P\{i_t = i, r_t = r, \nu_t = \nu,\ h_t^{(1)} = h_1,\ \ldots,\ h_t^{(M)} = h_M\}, i \geq 0, r_t = \overline{1, R}, \nu = \overline{0, W}, h^{(m)} \in \{0 \ldots N\}, m = \overline{1, M}, h^{(1)} + \ldots + h^{(M)} = min\{i, N\}$.

Enumerate the states of the chain $\xi_t,\ t \geq 0$, in direct lexicographic order of components $i, r, \nu$ and then in reverse lexicographic order of components $h^{(1)} \ldots h^{(M)}$.

Compose the row vectors $\boldsymbol{p}_i$ of stationary-state probabilities $p(i, r, \nu, h_1, \ldots, h_M)$, $i \geq 0$, according to the defined order and the row vector $\boldsymbol{p} = (\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots)$. The vector $\boldsymbol{p}$ satisfies the following system of linear algebraic equations:

$$\begin{cases} \boldsymbol{p}A = 0, \\ \boldsymbol{p}\boldsymbol{e} = 1, \end{cases} \qquad (2)$$

where $A$ is the infinitesimal generator of the process $\xi_t$, $t \geq 0$.

To determine the transition rates between the states of the process $\xi_t$, $t \geq 0$, we have to analyze its transition behaviour. Suppose that at the moment $t$ the process $\xi_t$, $t \geq 0$, is in the state $z_t = (i, r, \nu, h_1, \dots, h_M)$. Consider all possible events which can occur in an interval $[t, t+\delta[$ of infinitesimally small length $\delta$ with probability greater than $o(\delta)$. They are:

1) Transition of the *BMAP* directing process $\nu_t, t \geq 0$, from state $\nu$ to state $\nu'$ with generating the $k$-size batch of customers.
2) Transition of the *BMAP* directing process from state $\nu$ to state $\nu'$ without generating customers.
3) Transition of the RE directing process $r_t$, $r_t = \overline{1,R}$, from state $r$ to state $r'$.
4) A phase shift of one server from $m$ to $m'$ without the service completion.
5) A service completion by a server being in phase $m$.

The intensities of the corresponding transitions and the resulting system states are given in Table 1. By the symbol $\boldsymbol{e}_l$ we define the vector of proper size filled with zeroes except the $l$th element which is equal to one, vector $\boldsymbol{h} = \left(h^{(1)}, \dots, h^{(M)}\right)$.

*Table 1* **possible system transitions**

| Event | State | Transition rate | Condition |
|---|---|---|---|
| 1 | $\left(i+k, r, \nu', \boldsymbol{h} + \boldsymbol{e}_{l_1} + \dots + \boldsymbol{e}_{l_k}\right)$ | $(D_k^{(r)})_{\nu,\nu'}\beta_{l_1}^{(r)} \dots \beta_{l_k}^{(r)}$ | $i < N,\ 1 \leq k < N - i$ |
| | $(i+k, r, \nu', \boldsymbol{h} + \boldsymbol{e}_{l_1} + \dots + \boldsymbol{e}_{l_{N-i}})$ | $(D_k^{(r)})_{\nu,\nu'}\beta_{l_1}^{(r)} \dots \beta_{l_{N-i}}^{(r)}$ | $i < N,\ k > N - i$ |
| | $(i+k, r, \nu', \boldsymbol{h})$ | $(D_k^{(r)})_{\nu,\nu'}$ | $i \geq N,\ k \geq 1$ |
| 2 | $(i, r, \nu', \boldsymbol{h})$ | $(D_0^{(r)})_{\nu,\nu'}$ | $i \geq 1,\ k \geq 1$ |
| 3 | $(i, r', \nu, \boldsymbol{h})$ | $(Q)_{r,r'}$ | $i \geq 1,\ k \geq 1$ |
| 4 | $(i, r, \nu', \boldsymbol{h} + \boldsymbol{e}_m + \boldsymbol{e}_{m'})$ | $h^{(m)} S_{mm'}^{(r)}$ | $m \neq m',\ \boldsymbol{h}\boldsymbol{e}_m \geq 1$ |
| 5 | $(i-1, r, \nu, \boldsymbol{h} - \boldsymbol{e}_m)$ | $h^{(m)}(S_0^{(r)})_m$ | $i \leq N,\ \boldsymbol{h}\boldsymbol{e}_m \geq 1$ |
| | $(i-1, r, \nu, \boldsymbol{h} - \boldsymbol{e}_m + \boldsymbol{e}_{m'})$ | $h^{(m)}(S_0^{(r)})_m \beta_{m'}^{(r)}$ | $i > N,\ \boldsymbol{h}\boldsymbol{e}_m \geq 1,\ m \neq m'$ |
| | $(i-1, r, \nu, \boldsymbol{h})$ | $\sum_{m=1}^{M} h^{(m)}(S_0^{(r)})_m \beta_m^{(r)}$ | $i > N$ |

Our aim is to calculate the stationary state distribution of the described queueing model. For the use in the sequel, let us introduce the following notation:

- $\boldsymbol{e}_n$ $(0_n)$ is a column (row) vector of size $n$, consisting of 1's (0's). Suffix may be omitted if the dimension of the vector is clear from context;
- $I$ $(O)$ is an identity (zero) matrix of appropriate dimension (when needed the dimension of this matrix is identified with a suffix);
- $diag\,\{a_l,\ l = \overline{1,L}\}$ is a diagonal matrix with diagonal entries or blocks $a_l$;
- $\otimes$ and $\oplus$ are symbols of the Kronecker product and sum of matrices;
- $\tilde{S}^{(r)} = \begin{pmatrix} 0 & O \\ \mathbf{S}_0^{(r)} & S^{(r)} \end{pmatrix}$, $r = \overline{1,R}$;
- $L(n) = \binom{n+M-1}{M-1}$, $n = \overline{0,N}$;

- $\bar{W} = W + 1$;
- $\mathcal{D}_k^{(n)} = diag\{D_k^{(r)} \otimes I_{L(n)},\ r = \overline{1,R}\},\ n = \overline{0,N},\ k \geq 0$;
- $\mathcal{P}_{k,n} = diag\{I_{\bar{W}} \otimes P_{k,n}(\beta^{(r)}),\ r = \overline{1,R}\},\ k,n = \overline{1,N}$,
- $\mathcal{A}_n = diag\{I_{\bar{W}} \otimes A(n, S^{(r)}),\ r = \overline{1,R}\},\ n = \overline{1,N}$,
- $\mathcal{L}_n = diag\{I_{\bar{W}} \otimes L_{N-n}(N, \tilde{S}^{(r)}),\ r = \overline{1,R}\},\ n = \overline{1,N}$;
- $\mathcal{Q}_N = diag\{I_{\bar{W}} \otimes Q(N, \boldsymbol{S}_0^{(r)}\beta^{(r)}),\ r = \overline{1,R}\}$;
- $\mathcal{C}^{(n)} = Q \otimes I_{\bar{W}} \otimes I_{L(n)} + \mathcal{D}_0^{(n)} + \mathcal{A}_n,\ n = \overline{0,N}$.

**Lemma.** *Infinitesimal generator $A$ of the Markov chain $\xi_t,\ t \geq 0$, has the following block structure:*

$$A = \begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & \cdots \\ A_{1,0} & A_{1,1} & A_{1,2} & A_{1,3} & \cdots \\ O & A_{2,1} & A_{2,2} & A_{2,3} & \cdots \\ O & O & A_{3,2} & A_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3}$$

*where the non-zero blocks $A_{i,j}$ are computed by*

$$A_{i,i-1} = \begin{cases} \mathcal{L}_i, & i = \overline{1,N}, \\ \mathcal{Q}_N, & i \geq N, \end{cases}$$

$$A_{i,i} = \begin{cases} \mathcal{C}^{(i)} + \triangle_i, & i = \overline{0,N-1}, \\ \mathcal{C}^{(N)} + \triangle_{min\{i,N+1\}}, & i \geq N, \end{cases}$$

$$A_{i,j} = \begin{cases} \mathcal{D}_{j-i}^{(i)} \mathcal{P}_{i,min\{j,N\}}, & i = \overline{0,N-1},\ j \geq 1, \\ \mathcal{D}_{j-i}^{(N)}, & i \geq N,\ j \geq N+1. \end{cases}$$

Here $\triangle_i$, $i = \overline{0,N}$, are the diagonal matrices that guarantee $A\boldsymbol{e} = 0$, $P_{i,j}(\beta^{(r)}) = P_i(\beta^{(r)})P_{i+1}(\beta^{(r)}) \ldots P_{j-1}(\beta^{(r)})$, $0 \leq i < j \leq N$, $r = \overline{1,R}$. The detailed description of the matrices $P_j(\beta^{(r)})$, $A(i, S^{(r)})$, $L_{N-1}(N, \tilde{S}^{(r)})$, $r = \overline{1,R}$, and the algorithms for their calculation can be found in [1, 2].

To solve system (2) with the matrix $A$ defined by (3), we use the effective stable procedure [3] based on the special structure of the matrix $A$ (it is upper block Hessenberg) and probabilistic meaning of the unknown vector $\boldsymbol{p}$.

This procedure is given by the following statement.

**Theorem 1.** *The stationary probability vectors $\boldsymbol{p}_i$, $i \geq 0$, are calculated as follows:*

$$\boldsymbol{p}_l = \boldsymbol{p}_0 F_l,\ l \geq 1,$$

*where the matrices $F_l$ are calculated recurrently:*

$$F_l = (\bar{A}_{0,l} + \sum_{i=1}^{l-1} F_i \bar{A}_{i,l})(-\bar{A}_{l,l})^{-1},\ l \geq 1,$$

$$\bar{A}_{i,l} = A_{i,l} + \bar{A}_{i,l+1} G^{max\{0,l+N\}} \cdot G_{min\{N,l\}-1},\ i \geq 0,$$

*the matrix G is calculated from the equation*

$$G = (-\sum_{l=1}^{\infty} A_{N+1,N+l} G^{l-1})^{-1} A_{N+1,N}$$

*the matrices* $G_i$, $i = \overline{0, N-1}$, *вare calculated from the backward recursion:*

$$G_i = (-A_{i+1,i+1} - \sum_{l=i+2}^{\infty} A_{i+1,l} G^{max\{0,l-N\}} \cdot G_{min\{N,l\}-1} \cdot$$

$$\cdot G_{min\{N,l\}-2} \cdot ... \cdot G_{i+1})^{-1} A_{i+1,i}, i = N-1, N-2, ..., 0,$$

*the vector* $\boldsymbol{p}_0$ *is calculated as the unique solution to the following system of linear algebraic equations:*

$$\boldsymbol{p}_0 \tilde{A}_{0,0} = 0, \boldsymbol{p}_0(\sum_{l=0}^{\infty} F_l \boldsymbol{e} + \boldsymbol{e}) = 1.$$

## REFERENCES


1. *Ramaswami V., Lucantoni D.* Algorithm for the multi-server queue with phase-type service // Communications in Statistics-Stochastic Models. 1985. V. 1(3). P. 393–417.
2. *Ramaswami V.* Independent Markov process in parallel // Communications in Statistics-Stochastic Models. 1985. V. 1(3). P. 419–432.
3. *Klimenok V. I., Dudin A. N.* Multi-dimensional asymptotically quasi-Toeplitz Markov chains and their application in queueing theory // Queueing Systems. 2006. P. 245–259.
4. *Klimenok V. I., Khramova V. V., Babitsky A., Dudin A. N.* Steady state analysis of the BMAP|PH|N queue in a random environment // Proceedings of 15th International Conference on Telecommunications - ICT-2008. 2008. P. 91–99.