IDLE PERIOD IN A CLOSED SINGLE-LINE SYSTEM WITH REPEATED CALLS

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Queue length and server state stationary distributions in the queueing system under consideration are studied in previous papers of the author. Here, on the basis of these distributions, formulas for the distribution function, the mean value and the variance of the idle channel period are obtained. The numerical values computed according to these formulas are compared with the simulated results and some dependences of these values on the system parameters are established.

Keywords: queueing systems, repeated calls, idle channel period, simulation experiment

1. INTRODUCTION

The definitions of idle and busy channel periods in a single-server queueing system with repeated calls differ from those in a system with a queue ([1], [2]). Although in the present paper we consider a system with repetitions, these definitions are treated as in a system with a queue: a busy period is the time from a start of a service till the moment the server for the first time is free again, and the idle period is the time from the end of a service till the beginning of the succeeding service. This means that the busy periods coincide with the service times and only the investigation of the idle period is of interest.

We study a single-line (channel, server) queueing system with N customers (subscribers). These customers are identified as sources of primary orders (calls, demands). Each such source produces a Poisson process of primary (initial) calls with intensity λ. If the server is free (idle) at the instant of a primary call arrival it begins service immediately and after service completion becomes again a source of primary calls. Otherwise, if the channel is busy, it forms a source of repeated calls. Such a source produces a Poisson process of repeated (secondary) calls with intensity μ. If an incoming repeated call finds a free line, it operates in the same way as the primary calls: begins service and after its completion forms a source of primary calls. Otherwise, if the line is engaged at the moment of a repeated call arrival, the system state does not change.

The service time is a random variable ξ with distribution function $G(x)$ both for primary and repeated calls. The intervals between repeated trials, as well as between primary ones and the service times, are assumed to be mutually independent. Let

$$g(x) = G'(x),\ \overline{g}(s) = \int_0^\infty e^{-sx} dG(x) = \int_0^\infty e^{-sx} g(x)dx,\ \nu^{-1} = -\overline{g}'(0),$$

$$\eta(x)\Delta = P\{\xi \in (x, x+\Delta)/\xi \geq x\}, \tag{1}$$

i.e. ν^{-1} is the mean of the service time ξ and $\eta(x)$ is the service rate at instant x after start of a service,

$$\eta(x) = \frac{g(x)}{1 - G(x)}. \tag{2}$$

By $C(t)$ we denote the number of busy channels at moment t, by $z(t)$ - the time from the last service starting moment before t (in case $C(t) = 1$) and by $R(t)$ - the number of repeated calls sources at moment t (a sort of queue). For the sake of simplicity these sources will be called *secondary subscribers*. Formulas for the stationary distributions of the channel state, the queue length and their probability characteristics have been derived in previous papers of the author. Using these distributions, expressions for the distribution function, the mean value and the variance of the idle period are obtained in the present article. The mean values, computed according to these expressions are compared with simulated results. Some results, clarifying the dependence of the idle period mean on the parameters of the system, are numerically established.

2. JOINT DISTRIBUTION OF THE CHANNEL STATE AND THE QUEUE LENGTH

The joint distribution of $C(t)$, $R(t)$ and $z(t)$ in steady state

$$p_{1n}(x)dx = \lim_{t\to\infty} p_{1n}(x,t)dx = \tag{3}$$

$$= \lim_{t\to\infty} P\{C(t) = 1, R(t) = n, x \leq z(t) < x + dx\},$$

as well as the stationary joint distribution of $C(t)$ and $R(t)$

$$p_{in} = \lim_{t\to\infty} p_{in}(t) = \tag{4}$$

$$= \lim_{t\to\infty} P\{C(t) = i, R(t) = n\}, \ i = 0, 1, \ \ n = 0, 1, ..., N$$

are studied in [3]. Considering the possible transitions of the system for a short time interval Δt, we get a system of partial differential equations for the distributions $p_{1n}(x,t)$, $p_{in}(t)$, $i = 0, 1, \ \ n = 0, 1, ..., N$. By means of Laplace - Stieltjes transforms of this system's equations and taking limit as $t \to \infty$ we get a system of ordinary differential equations for the stationary distributions (3)-(4). Solving the system, we obtain formulas for these distributions, thus proving their existence. The following theorem is proved in [3]:

Theorem 1. *The stationary distributions (3)-(4), always exist and are given by the formulas*

$$p_{1N}(x) = p_{1N} = p_{0N} = 0,$$

$$p_{1n}(x) = (-1)^{N-n-1}(1 - G(x)) \sum_{k=0}^{n} \binom{N-k-1}{n-k} e^{-(N-k-1)\lambda x} \frac{\varphi_k}{g_{N-k-1}} C, \tag{5}$$

$$p_{1n} = (-1)^{N-n-1} \sum_{k=0}^{n} \binom{N-k-1}{n-k} \frac{\varphi_k}{g_{N-k-1}} r_k C,$$

$$p_{0n} = (-1)^{N-n-1} a_n^{-1} \sum_{k=0}^{n} \binom{N-k-1}{n-k} \varphi_k C, \quad n = 0, 1, ..., N-1. \tag{6}$$

Furthermore, for the stationary distribution of the channel state

$$P_i = \lim_{t \to \infty} P\{C(t) = i,\},\ i = 0, 1 \tag{7}$$

and the secondary subscribers mean value ER we have

$$P_1 = \nu^{-1} \varphi_{N-1} C, \quad P_0 = 1 - P_1, \tag{8}$$

$$ER = \left[(N-1)\nu^{-1} \varphi_{N-1} + \frac{(1-g_1)(\mu - \lambda)\varphi_{N-2}}{\mu \lambda g_1} \right] C. \tag{9}$$

Here

$$C = N\lambda\mu \left[\mu \left(1 + N\lambda\nu^{-1}\right) \varphi_{N-1} + \frac{1-g_1}{g_1} (\mu - \lambda) \varphi_{N-2} \right]^{-1}, \tag{10}$$

$$\varphi_n = (-1)^n \binom{N-1}{n} g_{N-1} (1 + A_n + B_n + C_n),\ n = 0, 1, ..., N-1, \tag{11}$$

$$a_n = (N-n)\lambda + n\mu, \tag{12}$$

$$g_n = \overline{g}(n\lambda),\ n = 0, 1, ..., N-1 \tag{13}$$

and A_n, B_n and C_n are defined by the recurrent relations

$$A_0 = B_0 = C_0 = 0, \tag{14}$$

$$A_n = \frac{1 - S_{N-n}}{S_{N-n}} \frac{a_n}{(N-n)\mu} (A_{n-1} + B_{n-1}), \tag{15}$$

$$B_n = \frac{a_0}{a_{n-1}} (A_{n-1} + C_{n-1}) + B_{n-1}, \tag{16}$$

$$C_n = \frac{1 - g_{N-n}}{g_{N-n}} \frac{a_n}{(N-n)\mu} (1 + C_{n-1}),\ n = 1, 2, ..., N-1. \tag{17}$$

3. IDLE CHANNEL PERIOD

Suppose at moment t an idle channel period starts and let us denote its length by $\zeta(t)$. For the stationary distribution, the mean value and the variance of $\zeta(t)$ the following theorem holds:

Theorem 2. *The stationary distribution $F_\zeta(x)$, the mean value $E\zeta$ and the variance $D\zeta$ of the idle period $\zeta(t)$ are given by the expressions*

$$F_\zeta(x) = \frac{1}{\nu P_1} \sum_{n=0}^{N-1} (-1)^{N-n-1} \left(1 - e^{-(N-n)\lambda x - n\mu x}\right) \sum_{k=0}^{n} \binom{N-k-1}{n-k} \varphi_k C, \tag{18}$$

$$E\zeta = \frac{P_0}{\nu P_1}, \tag{19}$$

$$D\zeta = \frac{2}{\nu P_1}\sum_{n=0}^{N-1} C(-1)^{N-n-1}\frac{1}{a_n^2}\sum_{k=0}^{n}\binom{N-n+k-1}{k}\varphi_{n-k} - (E\zeta)^2, \tag{20}$$

where the stationary channel state distribution P_0, P_1 (7) as well as the quantities φ_k, g_{N-k-1}, a_n and C are given in Theorem 1.

Proof. The distribution $F_\zeta(x)$ is equal to

$$F_\zeta(x) = \lim_{t\to\infty} P\left\{\zeta(t) < x / C(t) = 0,\ C(t-0) = 1\right\}. \tag{21}$$

We express the probabilities participating here by the stationary distribution 3

$$\lim_{t\to\infty} P\left\{\zeta(t) < x,\ C(t) = 0,\ C(t-0) = 1\right\} = \tag{22}$$

$$= \sum_{n=0}^{N-1}\left(1 - e^{-(N-n)\lambda x - n\mu x}\right)\int_0^\infty p_{1n}(u)\eta(u)du,$$

$$\lim_{t\to\infty} P\left\{C(t) = 0,\ C(t-0) = 1\right\} = \sum_{n=0}^{N-1}\int_0^\infty p_{1n}(u)\eta(u)du. \tag{23}$$

Using formulas (5) and (2) we substitute in (22), (23) and get consecutively:

$$\int_0^\infty p_{1n}(u)\eta(u)du = (-1)^{N-n-1}\sum_{k=0}^{n}\binom{N-k-1}{n-k}\varphi_k C,$$

$$\lim_{t\to\infty} P\left\{\zeta(t) < x,\ C(t) = 0,\ C(t-0) = 1\right\} =$$

$$= \sum_{n=0}^{N-1}(-1)^{N-n-1}\sum_{k=0}^{n}\binom{N-k-1}{n-k}\varphi_k C\left(1 - e^{-(N-n)\lambda x - n\mu x}\right),$$

$$\lim_{t\to\infty} P\left\{C(t) = 0,\ C(t-0) = 1\right\} =$$

$$= \sum_{n=0}^{N-1}(-1)^{N-n-1}\sum_{k=0}^{n}\binom{N-k-1}{n-k}\varphi_k C = \varphi_{N-1}C.$$

From here, taking into account (21) and (8) we obtain formula (18) and using this formula we get

$$E\zeta = \int_0^\infty x dF_\zeta(x) =$$

$$= \frac{1}{\nu P_1}\sum_{n=0}^{N-1}(-1)^{N-n-1}\sum_{k=0}^{n}\binom{N-k-1}{n-k}\varphi_k C\frac{1}{(N-n)\lambda + n\mu}.$$

The comparison of the last equation with the expressions (12) and (6) for the distribution p_{0n} gives 19. The proof of (20) is similar. □

4. SIMULATED RESULTS

Queue length simulations in the case of exponentially distributed service time with mean ν^{-1} are presented in [4]. They are performed using the software system "MATLAB"[5]. As in this case the quantity $\eta(x)(1)$ is equal to ν and does not depend on x, the system state at time t is fully determined by the number $R(t)$ of repeated subscribers, the channel state $C(t)$ and some initial state $(R(t_0), C(t_0))$. The simulation follows the short time Δt changes of the process $(R(t), C(t))$, with the assumption that $(C(t_0) = 0,\ R(t_0) = ER)$, where ER is the stationary mean value of the secondary subscribers, calculated according to formulas (9), (10-17) in the case of exponentially distributed service time with parameter ν.

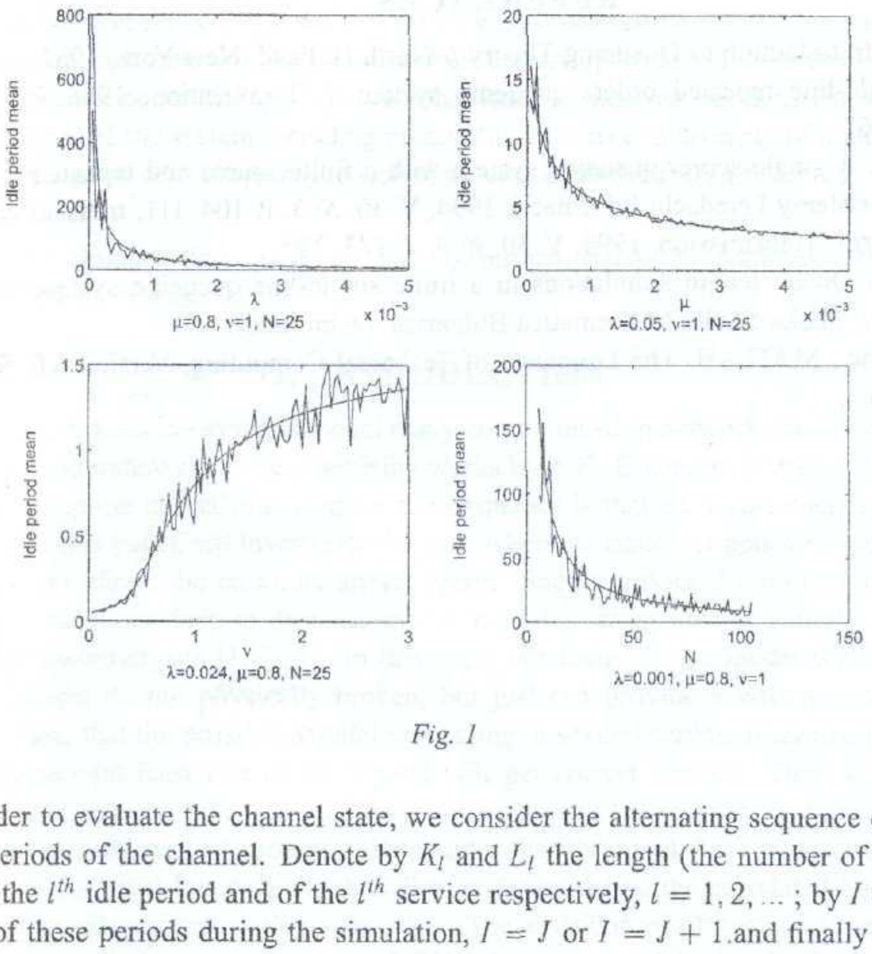

Fig. 1

In order to evaluate the channel state, we consider the alternating sequence of idle and service periods of the channel. Denote by K_l and L_l the length (the number of simulation steps) of the l^{th} idle period and of the l^{th} service respectively, $l = 1, 2, \ldots$; by I and J - the number of these periods during the simulation, $I = J$ or $I = J + 1$.and finally by n - the number of simulation steps,

$$\sum_{l=1}^{I} K_l + \sum_{l=1}^{J} L_l = n.$$

Then the obtained empirical mean values of the idle and service periods, multiplied by the single step length Δt should be close to the real means of these distributions.

Each of the graphs on Figure 1 represents the values of the idle period mean, computed according to formula (19) (in the case of exponentially distributed service time), and the empirical means obtained via simulations. This is done for 100 values of the varying parameter with 10000 steps for each simulation. We can see that the observed values are near to the theoretical ones for a wide range of parameters values. Moreover, the graphs show specific properties of the idle period mean as a function of any particular parameter, which ones should be checked for other distributions of the service time or proved theoretically.

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