# A DISCRETE-TIME QUEUEING SYSTEM WITH GENERAL RETRIAL TIMES

**I. Atencia, I. Fortes, S. Sánchez**

*University of Málaga*

*Málaga, Spain*

iatencia@ctima.uma.es

The standard models of classical queueing theory are systems operating in continuous time. But in practice there are many systems which shows an inherent generic slotted time scale (for example time–shared computing systems) and demands a study of discrete time systems. One of the advantages of dealing with discrete-time models is that they have been found more appropriate that their continuous-time counterpart for modelling computer and telecommunication systems.

*Keywords:* Discrete-time, general retrial times, Bernoulli feedback.

## 1. INTRODUCTION

The study of discrete-time queues was initiated by Meisling [1], Birdsall et al. [2], and also by Powell et al. [3]. One of the most outstanding works of the queueing theory has been recently carried out by Yang and Li [4], who extended the retrial queues to the discrete time systems.

Retrial queues are queueing systems which arise naturally in many telecommunication and computer systems and are characterized by the fact that a customer who finds the server busy upon arrival must leave the service area and joins a retrial group (which will be called *orbit*) in order to reinitiate his request after some random time.

There is a wide literature on retrial queues and for a good survey of the results and fundamental methods of retrial queues, we refer for example [5] and [6].

Most of the investigations in queueing systems consider the server as reliable, in this paper we adopt the more realistic approach of unreliable server, that is, we admit the possibility that the server is subjected to breakdowns. In many practical situations, the server is unreliable from time to time (hardware breakdowns, preventive maintenance, spare replacement, ...). This work is an attempt to extend the queueing theory on server breakdowns to the discrete–time retrial queues. We study a discrete–time Geo/G/1 retrial queue with starting failures and general retrial times. With this paper we generalize the previous work mentioned in discrete–time retrial queue with unreliable server due to starting failures because we consider general service and general retrial times. Besides, we introduce the feedback phenomenon in our system. Although the feedback phenomenon, introduced by Takacs [7], occurs in many practical situations see [8]. Feedback is present for example in telecommunication systems where the messages with errors are send again. Also, feedback can be introduced as a mechanism to program the service of customers when the its service is divided in an aleatory number of stages.

## 2. THE MATHEMATICAL MODEL

We consider a discrete-time queueing system where customers arrive at the system according to a geometric arrival process with probability $p$. An arriving customer who finds the server busy or down joins a group of blocked customers called *orbit* with a FCFS discipline, that is, only the customer at the head of the orbit is allowed for access to the server. An arriving customer (external o repeated) who finds the server idle must turn on the service station. If the server is started successfully (with probability $\nu$), the customer gets service immediately, otherwise, if the server is started unsuccessfully (with a probability $\bar{\nu} = 1 - \nu$) the repair for the server begins immediately and the customer must join the orbit. Successive interretrial times follow a general distribution $\{a_i\}_{i=0}^{\infty}$ with generating function $A(x) = \sum_{i=0}^{\infty} a_i x^i$. Service and repair times are governed by arbitrary distributions $\{s_{1,i}\}_{i=1}^{\infty}$, $\{s_{2,i}\}_{i=1}^{\infty}$ with generating functions $S_1(x) = \sum_{i=1}^{\infty} s_{1,i} x^i$, $S_2(x) = \sum_{i=1}^{\infty} s_{2,i} x^i$ and the n-th factorial moments $\beta_{1,n}$ and $\beta_{2,n}$ respectively.

After service completion, the customer decides either to join the retrial group again for another service with probability $\theta$ or leaves the system with complementary probability $\bar{\theta}$.

It is assumed that interarrival times, service times and repair times are mutually independent. We will denote $\rho_i = p\,\beta_{i,1}$, $i = 1, 2$. In order to avoid trivial cases, it is also supposed $0 < p < 1$, $0 < \nu \leq 1$, $0 \leq \theta < 1$.

## 3. THE MARKOV CHAIN

At time $m^+$ (the instant immediately after time slot $m$), the system can be described by the process

$$Y_m = (C_m, \xi_{0,m}, \xi_{1,m}, \xi_{2,m}, N_m)$$

where $C_m$ denotes the state of the server 0, 1 or 2 according to whether the server is free, busy or down and $N_m$ the number of unsatisfied customers. If $C_m = 0$ and $N_m > 0$, $\xi_{0,m}$ represents the remaining retrial time. If $C_m = 1$, $\xi_{1,m}$ corresponds to the remaining service time of the customer currently being served and if $C_m = 2$, $\xi_{2,m}$ is the remaining repair time.

It can be shown that $\{Y_m\,,\ m \in \mathbb{N}\}$ is the Markov chain of our queueing system, whose states space is

$$\{(0,0);\ (0,i,k) : i \geq 1, k \geq 1;\ (1,i,k) : i \geq 1, k \geq 0;\ (2,i,k) : i \geq 1, k \geq 1\}.$$

Our goal is to find the stationary distribution

$$\begin{aligned}
\pi_{0,0} &= \lim_{m\to\infty} P[C_m = 0\,,\ N_m = 0] \\
\pi_{0,i,k} &= \lim_{m\to\infty} P[C_m = 0\,,\ \xi_{0,m} = i\,,\ N_m = k]\,;\ i \geq 1\,,\ k \geq 1 \\
\pi_{1,i,k} &= \lim_{m\to\infty} P[C_m = 1\,,\ \xi_{1,m} = i\,,\ N_m = k]\,;\ i \geq 1\,,\ k \geq 0 \\
\pi_{2,i,k} &= \lim_{m\to\infty} P[C_m = 2\,,\ \xi_{2,m} = i\,,\ N_m = k]\,;\ i \geq 1\,,\ k \geq 1
\end{aligned}$$

of the Markov chain $\{Y_m, \ m \in \mathbb{N}\}$.

The Kolmogorov equations for the stationary distribution of the system are

$$\pi_{0,0} = \bar{p}\,\pi_{0,0} + \bar{\theta}\bar{p}\,\pi_{1,1,0} \tag{1}$$

$$\pi_{0,i,k} = \bar{p}\,\pi_{0,i+1,k} + \theta\bar{p}\,a_i\pi_{1,1,k-1} + \bar{\theta}\bar{p}\,a_i\,\pi_{1,1,k} + \bar{p}\,a_i\,\pi_{2,i,k},\ i \geq 1, k \geq 1 \tag{2}$$

$$\begin{aligned}\pi_{1,i,k} &= \delta_{0,k}\,p\,\nu\,s_{1,i}\,\pi_{0,0} + \bar{p}\,\nu\,s_{1,i}\,\pi_{0,1,k+1} + (1-\delta_{0,k})\,p\,\nu\,s_{1,i}\sum_{j=1}^{\infty}\pi_{0,j,k} + \\ &+ (1-\delta_{0,k})\,\theta\,p\,\nu\,s_{1,i}\,\pi_{1,1,k-1} + (\bar{\theta}\,p\,\nu\,s_{1,i} + \theta\,\bar{p}\,a_0\,\nu\,s_{1,i})\,\pi_{1,1,k} + \\ &+ \bar{\theta}\,\bar{p}\,a_0\,\nu\,s_{1,i}\,\pi_{1,1,k+1} + (1-\delta_{0,k})\,p\,\pi_{1,i+1,k-1} + \bar{p}\,\pi_{1,i+1,k} + \\ &+ (1-\delta_{0,k})\,p\,\nu\,s_{1,i}\,\pi_{2,1,k} + \bar{p}\,a_0\,\nu\,s_{1,i}\,\pi_{2,1,k+1},\ i \geq 1,\ k \geq 0\end{aligned} \tag{3}$$

$$\begin{aligned}\pi_{2,i,k} &= \delta_{1,k}\,p\,\bar{\nu}\,s_{2,i}\,\pi_{0,0} + (1-\delta_{1,k})\,p\,\bar{\nu}\,s_{2,i}\sum_{j=1}^{\infty}\pi_{0,j,k-1} + \bar{p}\,\bar{\nu}\,s_{2,i}\,\pi_{0,1,k} + \\ &+ (1-\delta_{1,k})\,\theta\,p\,\bar{\nu}\,s_{2,i}\,\pi_{1,1,k-2} + (\bar{\theta}\,p\,\bar{\nu}\,s_{2,i} + \theta\,\bar{p}\,a_0\,\bar{\nu}\,s_{2,i})\,\pi_{1,1,k-1} + \\ &+ \bar{\theta}\,\bar{p}\,a_0\,\bar{\nu}\,s_{2,i}\,\pi_{1,1,k} + (1-\delta_{1,k})\,p\,\bar{\nu}\,s_{2,i}\,\pi_{2,1,k-1} + \bar{p}\,a_0\,\bar{\nu}\,s_{2,i}\,\pi_{2,1,k} + \\ &+ (1-\delta_{1,k})\,p\,\pi_{2,i+1,k-1} + \bar{p}\,\pi_{2,i+1,k},\ \ i \geq 1,\ \ k \geq 1,\end{aligned} \tag{4}$$

and the normalizing condition is

$$\pi_{0,0} + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{0,i,k} + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i,k} + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{2,i,k} = 1.$$

To solve equations (1) – (4) we introduce the following generating functions and auxiliary generating functions:

$$\begin{aligned}\varphi_0(x,z) &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{0,i,k}\,x^i\,z^k; &\varphi_{0,i}(z) &= \textstyle\sum_{k=1}^{\infty}\pi_{0,i,k}\,z^k;\ i \geq 1\\ \varphi_1(x,z) &= \textstyle\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\pi_{1,i,k}\,x^i\,z^k; &\varphi_{1,i}(z) &= \textstyle\sum_{k=0}^{\infty}\pi_{1,i,k}\,z^k;\ i \geq 1\\ \varphi_2(x,z) &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{2,i,k}\,x^i\,z^k; &\varphi_{2,i}(z) &= \textstyle\sum_{k=0}^{\infty}\pi_{2,i,k}\,z^k;\ i \geq 1.\end{aligned}$$

Multiplying equations (2) – (4) by $z^k$, summing over $k$ and the result by $x^i$ and summing over $i$, nd having into account (1) we obtain:

$$\begin{aligned}\frac{x-\bar{p}}{x}\varphi_0(x,z) &= \bar{p}\,(\bar{\theta}+\theta\,z)[A(x)-a_0]\varphi_{1,1}(z) - \bar{p}\,\varphi_{0,1}(z) + \\ &+ \bar{p}\,[A(x)-a_0]\varphi_{2,1}(z) - p\,[A(x)-a_0]\,\pi_{0,0},\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{x-(\bar{p}+p\,z)}{x}\varphi_1(x,z) &= [\frac{(\bar{p}\,a_0+p\,z)(\bar{\theta}+\theta\,z)}{z}\,\nu\,S_1(x) - (\bar{p}+p\,z)]\,\varphi_{1,1}(z) + \\ &+ p\,\nu\,s_1(x)\,\varphi_0(1,z) + \frac{\bar{p}}{z}\,\nu\,S_1(x)\,\varphi_{0,1}(z) + \\ &+ \frac{\bar{p}\,a_0+p\,z}{z}\nu S_1 x\,\varphi_{2,1}(z) + \frac{p\,(z-a_0)}{z}\,\nu\,S_1(x)\pi_{0,0},\end{aligned} \tag{6}$$

$$\frac{x-(\bar p+p\,z)}{x}\varphi_2(x,z) = [(\bar p\,a_0+p\,z)\,\bar\nu\,S_2(x)-(\bar p+p\,z)]\,\varphi_{2,1}(z)+ \\ + p\,z\,\bar\nu\,S_2(x)\,\varphi_0(1,z)+\bar p\,\bar\nu\,S_2(x)\,\varphi_{0,1}(z)+ \\ + (\bar p\,a_0+p\,z)(\bar\theta+\theta\,z)\,\bar\nu\,S_2(x)\,\varphi_{1,1}(z)+ \\ + p\,(z-a_0)\,\bar\nu\,S_2(x)\,\pi_{0,0}. \quad (7)$$

Setting $x=1$ in (5) we have

$$p\varphi_0(1,z)=\bar p(\bar\theta+\theta z)(1-a_0)\varphi_{1,1}(z)-\bar p\varphi_{0,1}(z)+\bar p(1-a_0)\varphi_{2,1}(z)-p(1-a_0)\pi_{0,0},$$

and substituting the above equation into (6) and (7), we obtain

$$\frac{x-(\bar p+p\,z)}{x}\varphi_1(x,z) = [\frac{z+\bar p\,a_0\,(1-z)}{z}(\bar\theta+\theta\,z)\,\nu\,S_1(x)-(\bar p+p\,z)]\,\varphi_{1,1}(z)+ \\ + \frac{\bar p\,(1-z)}{z}\,\bar\nu\,S_1(x)\varphi_{0,1}(z)+\frac{z+\bar p\,a_0\,(1-z)}{z}\,\nu\,S_1(x)\,\varphi_{2,1}(z)+ \\ + \frac{p\,a_0\,(z-1)}{z}\,\nu\,S_1(x)\,\pi_{0,0}, \quad (8)$$

$$\frac{x-(\bar p+p\,z)}{x}\varphi_2(x,z) = [(z+\bar p\,a_0\,(1-z))\,\bar\nu\,S_2(x)-(\bar p+p\,z)]\varphi_{2,1}(z)+ \\ + (z+\bar p\,a_0\,(1-z))(\bar\theta+\theta\,z)\,\bar\nu\,S_2(x)\,\varphi_{1,1}(z)+ \\ + \bar p\,\bar\nu\,S_2(x)\,(1-z)\,\varphi_{0,1}(z)+a_0\,(z-1)\,\bar\nu\,S_2(x)\,\pi_{0,0}. \quad (9)$$

Choosing $x=\bar p$ in (5) and $x=\bar p+p\,z$ in (6) and (7) and after some algebra we can find the generating functions $\varphi_{0,1}(z)$ $\varphi_{1,1}(z)$ and $\varphi_{2,1}(z)$:

$$\varphi_{0,1}(z)=\frac{p\,z[A(\bar p)-a_0][\bar p+p\,z-\nu S_1(\bar p+p\,z)(\bar\theta+\theta\,z)-\bar\nu\,z\,S_2(\bar p+p\,z)]}{\Omega(z)}\,\frac{\pi_{0,0}}{\bar p}, \quad (10)$$

$$\varphi_{1,1}(z) = \frac{p\,A(\bar p)\,(1-z)\,\nu\,S_1(\bar p+p\,z)}{\Omega(z)}\,\pi_{0,0}, \quad (11)$$

$$\varphi_{2,1}(z) = \frac{p\,A(\bar p)\,z\,(1-z)\,\bar\nu\,S_2(\bar p+p\,z)}{\Omega(z)}\,\pi_{0,0}, \quad (12)$$

where $\Omega(z)=[\nu\,S_1(\bar p+p\,z)\,(\bar\theta+\theta\,z)+\bar\nu\,z\,S_2(\bar p+p\,z)][z+(1-z)\,\bar p\,A(\bar p)]-z\,(\bar p+p\,z)$.

Let us note that the above generating functions are defined for $z\in[0,1)$ and in $z=1$ can be extended by continuity. By substituting (11) − (12) into (5), (8) and (9) we can derive the generating functions obtaining the following theorem:

**Theorem 1.** *The generating functions of the stationary distribution of the chain are given by*

$$\varphi_0(x,z) = \frac{A(x)-A(\bar p)}{x-\bar p}\times \\ \times \frac{p\,x\,z\,[\bar p+p\,z-\nu\,S_1(\bar p+p\,z)(\bar\theta+\theta\,z)-\bar\nu\,z\,S_2(\bar p+p\,z)]}{\Omega(z)}\,\pi_{0,0}$$

$$\varphi_1(x,z) = \frac{S_1(x)-S_1(\bar p+p\,z)}{x-(\bar p+p\,z)}\cdot\frac{p\,x\,A(\bar p)\,(1-z)\,\nu\,(\bar p+p\,z)}{\Omega(z)}\,\pi_{0,0}$$

$$\varphi_2(x,z) = \frac{S_2(x)-S_2(\bar p + p z)}{x-(\bar p + p z)} \cdot \frac{p\,x\,z\,A(\bar p)\,(1-z)\,\bar\nu\,(\bar p + p z)}{\Omega(z)}\,\pi_{0,0},$$

*where* $\pi_{0,0} = \dfrac{p + \bar p A(\bar p) - \nu\theta - \bar\nu - \nu\rho_1 - \bar\nu\rho_2}{A(\bar p)\,\nu\,\bar\theta}$.

Let us present some performance measures for the system at the stationary regime:
The system is occupied with probability

$$\varphi_0(1,1)+\varphi_1(1,1)+\varphi_2(1,1) = \frac{A(\bar p)\,(\nu\bar\theta - \bar p) + \nu\theta + \bar\nu + \nu\rho_1 + \bar\nu\rho_2 - p}{\nu\bar\theta A(\bar p)}.$$

The server is idle with probability $\pi_{0,0}+\varphi_0(1,1) = \dfrac{\nu\bar\theta - \nu\rho_1 - \bar\nu\rho_2}{\nu\bar\theta}$.
The mean number of customers in the orbit is

$$E[N] = \frac{\tau}{2\,[p+\bar p A(\bar p) - \nu\rho_1 - \bar\nu\rho_2 - \nu\theta - \bar\nu]} - \frac{\theta}{\bar\theta}\rho_1$$

The mean number of customers in the system is

$$E[L] = \frac{\tau}{2\,[p+\bar p A(\bar p) - \nu\rho_1 - \bar\nu\rho_2 - \nu\theta - \bar\nu]} + \rho_1$$

where $\tau = [\nu\,\beta_{1,2} + \bar\nu\beta_{2,2}]\,p^2 + 2\,[\theta\,\nu\,\rho_1 + \bar\nu\,\rho_2] + 2\,\bar p\,[\nu\,\rho_1 + \bar\nu\,\rho_2 + \nu\theta + \bar\nu - p][1 - A(\bar p)]$.

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