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PATH INTEGRALS AND SPINNING PARTICLES IN TWO-DIMENSIONAL SPACES

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Technique based on the method of the motion associated with a loop is applied to quantum particles on the base of path integrals. This formalism has been modified in order to take into account spin. In order to illustrate the details of the technique we issue, we consider thoroughly the case of 2-dimensional sphere. With the use of proper time transformation we show that there is possibility to study also pseudosphere and two-dimensional cosmological models.

Интегралы по путям и частица со спином в двумерных пространствах
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Метод движений, ассоциированных с циклом, применяется для исследования квантовой частицы на основе интегралов по путям. Данный формализм модифицирован для учета спиновых свойств частицы. Детали метода демонстрируются на примере 2-сферы. Вычисление интегралов по путям на псевдосфере и в двумерных космологических моделях производится с использованием преобразования собственного времени.

1. Introduction

The development of quantum field theory causes an increasing interest in geometric methods. One could mention the interpretation of a gauge field and a field intensity tensor as a connection and a curvature. Motion associated with a loop represents one of such geometric approaches [1]. According to the method, motion along a closed path is accompanied by a rotation and a translation in a tangent space [2–4]. In this way one can deal directly with two main covariant quantities: the torsion and curvature tensors. However, application of such methods to quantum object immediately causes some difficulties in the analysis. First, one has to use the methods of quantum theory based on the concepts of a classical path, a trajectory and so on. Second, the spin properties of an object should be also taken into account since a closed path leads to rotation.

The first problem can be solved by means of Feynman path integrals [5]. This way, however, only the Feynman propagator can be obtained (see calculations of other Green functions in [6]). Quantum field theory gives a wide spectrum of methods for extracting physically important information from it [8].

The consideration of a non-scalar field is based on the following observation: in empty curved spaces ($R_{\mu\nu}=0$, but $R^{\alpha}_{\beta\mu\nu}\neq 0$) it is quite sufficient to insert an operator of parallel transport into the integral [9, 10]. In this paper we shall consider the effect of this modification. First, we shall pay attention to the loss of a simple connection between the propagators of scalar and non-scalar fields. It will be shown that this phenomenon is related to motion associated with a loop. In order to avoid mathematical difficulties, we consider two-dimensional spaces. Then the local rotation group is Abelian. However, from the viewpoint of spinor analysis the properties of two-dimensional spaces are very close to those of four-dimensional spaces [16].

2. Description of the method

2.1. Propagator for a spinning particle

We start with the Klein-Gordon equation (hereafter $c=\hbar=1$)

$$\nabla_\mu\nabla^\mu\phi + M^2\phi = 0. \tag{1}$$

Then the equation for the propagator has the form

$$\nabla_\mu\nabla^\mu G(x,x_0) + M^2 G(x,x_0) = \delta(x,x_0). \tag{2}$$

With a Fourier-like transform we pass to the path-integral formulation [7]:

$$G(x,x_0) = \int\limits_0^\infty d\tau\, \mathrm{e}^{-iM^2\tau/2}\left\langle x,\tau|x_0,0\right\rangle, \tag{3}$$

$$i\frac{\partial}{\partial\tau}\left\langle x,\tau|x_0,0\right\rangle = \frac{1}{2}\nabla_\mu\nabla^\mu\left\langle x,\tau|x_0,0\right\rangle \tag{4}$$

so that Eq. (2) is reduced to the Schrödinger-type equation (4). Then

$$K(x,x_0;\tau) = \langle x,\tau|x_0,0\rangle = \int \mathcal{D}x(\tau)\exp\left\{-\frac{i}{2}\int\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{R}{3}\right)d\tau\right\}. \quad (5)$$

The appearance and meaning of the additional term $\sim R/3$ is discussed in [7].

Similarly, for a particle with spin we get

$$(\nabla_\mu\nabla^\mu + M^2)\,\psi^A = 0, \quad (6)$$

$$i\frac{\partial}{\partial\tau}K^A{}_{B_0}(x,x_0) = -\frac{1}{2}\nabla_\mu\nabla^\mu K^A{}_{B_0}(x,x_0) \quad (7)$$

where the Feynman propagator is assumed in the form:

$$G^{A''}{}_{B'}(x'',x') = \int_0^\infty d\tau\, e^{-iM^2\tau/2}\int \mathcal{D}x(\tau)\,\times \exp\left\{-\frac{i}{2}\int\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{R}{3}\right)d\tau\right\}P^{A''}{}_{B'}(x(\tau)) \quad (8)$$

where $P^{A''}{}_{B'}(x(\tau))$ is the parallel transport operator [9, 10], A' is a generalized index related to the point x', A'' is an index related to x'' and so on. The integrals of such a type have a more complicated structure than the path integrals for a scalar particle. This is determined by the complexities of an evaluation of $P^{A''}{}_{B'}(x(\tau))$. Studying the problem in the two-dimensional case, we point at these difficulties and also show a possible way to get them around. Fortunately, the recent development of path integration techniques enables us to solve a wide range of problems [12, 14]. We show, in addition, how to take into account the parallel transport without its direct calculation.

2.2. Basic path integrals

We shall deal with path integrals of the following form:

$$G(x'',x';t) = \int \mathcal{D}p(t)\,\mathcal{D}x(t) \times \exp\left\{i\int\left[p\dot{x} - \frac{p^2}{2} - V(x)\right]dt\right\}. \quad (9)$$

The result can be presented either in the form of an expansion in eigenfunctions:

$$G(x'',x';t) = \sum_{n=0}^{\infty}\Psi_n(x'')\Psi_n(x')\,e^{-iE_nt} \quad \text{(discrete part of spectrum)}$$
$$+ \int dp\Psi_p(x'')\Psi_p^*(x')\,e^{-iE(p)t} \quad \text{(continuous part)}, \quad (10)$$

or in the form

$$G(x'',x';t) = \int f(e)f_1(x_1,E)f_2(x_2,E)\,e^{-iEt}dE$$

where x_1 (x_2) is the smaller (greater) of x'',x' [14].

One of the first exactly calculated integrals, the harmonic oscillator, is presented in [5]. It can be easy evaluated in Cartesian coordinates and transformed then to the polar ones. The similar way path integral solution of radial harmonic oscillator with effective potential

$$V(r) = \frac{\omega^2r^2}{2} + \frac{l^2-1/4}{2r^2} \quad (12)$$

has been obtained [11]. The energy spectrum can be either discrete (real ω) or continuous (imanginary ω). The eigenfunctions are Laguerre polynomials or Whittaker functions, respectively. Any equation of degenerated hypergeometric type can be solved by means of a path integral of this type.

With the use of the methods of group path integration [18], the Pöschl-Teller problem (i.e., the hypergeometric type equation) has been solved. The solution with the Pöschl-Teller potential:

$$V(x) = \frac{1}{2}\left(\frac{\alpha^2-1/4}{\sin^2 x} + \frac{\beta^2-1/4}{\cos^2 x}\right) \quad (13)$$

has the form of an expansion with (we omit the normalization factor [12, 14])

$$\Psi_n = (\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2}P_n^{\alpha,\beta}(\cos 2x),$$
$$E_n = \frac{1}{2}(\alpha+\beta+2n+1)^2 \quad (14)$$

where $P_n^{\alpha,\beta}$ are Jacobi polynomials.

For the modified Pöschl-Teller potential

$$V(x) = \frac{1}{2}\left(\frac{\eta^2-1/4}{\sinh^2 x} - \frac{\nu^2-1/4}{\cosh^2 x}\right), \quad x>0, \quad (15)$$

the solution has been obtained in the form of an expansion in

$$\Psi_p = (\sinh x)^{\eta+1/2}(\cosh x)^{\nu+1/2}\times {}_2F_1\left(\frac{1-ip+\eta+\nu}{2},\frac{1+ip+\eta+\nu}{2};1+\eta;-\sinh^2 x\right),$$
$$E_p = p^2/2; \quad (16)$$

here we give a continuous spectrum only (this potential also can lead to discrete spectrum, see [12]).

2.3. Proper time transformations

The proper time transformation is a powerful computational technique. Let us consider a one-dimensional path integral with the action

$$S = \int\left[\frac{\dot{x}^2}{2}dt - V(x)dt\right]; \quad (17)$$

the coordinate transformation $x = F(\xi)$, $F'(\xi) = f(\xi)$ gives

$$S = \int\left[f^2(\xi)\frac{\dot{\xi}^2}{2}d\tau - V(x)d\tau\right] \quad (18)$$

To keep the kinetic term invariant, one has to apply the local time transformation

$$d\tau \to ds = \frac{d\tau}{f^2(\xi)}$$

$$S = \int \left(\frac{\dot{\xi}^2}{2} ds - f^2(\xi) V(F(\xi)) ds \right). \tag{19}$$

The time interval should also remain unchanged,

$$T = \int d\tau = \int f^2(\xi(s)) ds = \text{const.} \tag{20}$$

To this end, the path integral should have the additional term

$$\delta\left[T - \int f^2(\xi(s))ds\right] = \int \frac{dE}{2\pi i} \exp\left[iE \int f^2(\xi(s))ds - T\right]. \tag{21}$$

Taking into account the change in the integration measure [12]:

$$\mathcal{D}x(t) = f(\xi(t))\mathcal{D}\xi(t) \tag{22}$$

we get finally the following problem (with a new effective potential):

$$W(\xi) = f^2(\xi)\left[V(F(\xi)) - E\right] + \frac{1}{2} f^{1/2} \frac{d^2}{d\xi^2} f^{-1/2}, \tag{23}$$

$$G(x'', x', T) = \int \frac{dE}{2\pi i} e^{-iET} G(\xi'', \xi', E), \tag{24}$$

$$G(\xi'', \xi', E) = \sqrt{f(\xi'')f(\xi')} \int ds\, G_W(\xi'', \xi', s), \tag{25}$$

where $G_W(\xi'', \xi', s)$ is a propagator of a particle moving in the potential field $W(\xi)$. This trick can be considered to be somewhat analogous to the technique which makes use of a coordinates transform in solving the stationary Shrödinger equation [13].

2.4. Generation of an additional term by the spin

Let us start with the problem of parallel transport on sphere. The metric in standard angular variables has the form

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2. \tag{26}$$

It is convenient to pass from the natural basis

$$e_\theta = \frac{\partial}{\partial\theta}, \qquad e_\phi = \frac{\partial}{\partial\phi} \tag{27}$$

to the orthonormal one

$$e_\theta = \frac{\partial}{\partial\theta}, \qquad e_\phi = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \tag{28}$$

and, correspondingly,

From (29) one can easily calculate the connection

$$\omega^\theta{}_\phi = \cos\theta\, \mathbf{d}\phi. \tag{30}$$

Hence the parallel transport reduces to a rotation around the axis; the rotation angle is given by

$$\chi = -\int \cos\theta d\phi. \tag{31}$$

Then, explicitly, a rotation associated with a loop Γ is

$$\chi(\Gamma) = -\oint_\Gamma \cos\theta\, d\phi = \int \sin\theta\, dS \tag{32}$$

where dS is an area element of the surface enveloped by Γ.

In the general case of a two-dimensional manifold we meet the only "degree of freedom": in a tangent Euclidean space $e_i^2 = 1$, $d\,e_{1,2} \sim e_{2,1}\, dx$, so the rotation takes place in a single plane. Thus any geometric object can be expanded into a sum of eigenvectors of the rotation operator:

$$i\hat{\mathbf{L}}\epsilon^s = s\epsilon^s, \quad s = 0, \pm\frac{1}{2}, \pm 1, \ldots \tag{33}$$

In particular, for vectors

$$s = \pm 1, \qquad \epsilon^{\pm 1} = \frac{1}{\sqrt{2}}\left(e_\theta \pm i e_\phi\right), \tag{34}$$

and we come to an isotropic basis [15]. Then the law of parallel transport acquires its simplest form

$$\hat{\mathrm{P}}\epsilon^s = e^{is\chi}\epsilon^s = e^{-is\int\cos\theta d\phi}\epsilon^s. \tag{35}$$

Moreover, the "Schrödinger equation" also has simple form in the isotropic basis:

$$i\frac{\partial\psi^s}{\partial t} = -\frac{1}{2}\nabla_i\nabla^i\psi^s = \frac{1}{2}\left[\frac{1}{\sin^{1/2}\theta} p_\theta \sin\theta\, p_\theta \frac{1}{\sin^{1/2}\theta} + \frac{(p_\phi + s\cos\theta)^2}{\sin^2\theta}\right]\psi^s \tag{36}$$

where $p_\mu = -ig^{-1/4}\partial_\mu g^{1/4}$, $g = |\det g_{\mu\nu}|$ [7].

When the metric is pseudo-Euclidean, the isotropic basis is

$$\epsilon^{\pm 1} = \frac{1}{\sqrt{2}}(\vec{e}_0 \pm \vec{e}_1), \tag{37}$$

and the parallel transport can be written as follows:

$$\hat{\mathrm{P}}\epsilon^s = e^{s\chi}\epsilon^s.$$

Then in both cases

$$e^{iS(x(t))} \quad \to \quad e^{iS(x(t)) + is\chi(x(t))}$$

and an additional term should be incorporated into the action and the Lagrangian:

$$\mathcal{L} \to \mathcal{L}_{\text{eff}} = \mathcal{L} - is\omega_i\dot{x}^i.$$

3. Path integral solutions on a sphere and a pseudosphere

3.1. Solution on a sphere

To begin with, we shall consider a short-time kernel with a fixed final point. It can be written as a product of a scalar kernel and a spin-dependent phase factor:

$$K_s(x'', x'; \epsilon) = K_0(x'', x'; \epsilon)(P(x'', x'))^s \tag{38}$$

where a geodesic curve in the short-time kernel is implied. By a transformation of the basis one can convert $P(x'', x')$ to 1 for any fixed x'':

$$K_s(x'', x'; \epsilon) = K_0(x'', x'; \epsilon) \tag{39}$$

(with the same eigenvalues and eigenfunctions). This is not true in general. It can be easily seen for two-step approximation taken as an example:

$$\begin{aligned} K_s(x''', x'; 2\epsilon) &= \int K_s(x''', x''; \epsilon)\frac{dx''}{2\pi i\epsilon}K_s(x'', x'; \epsilon) \\ &= \int P^s(x''', x'')K_0(x''', x''; \epsilon)\frac{dx''}{2\pi i\epsilon}P^s(x'', x')K_0(x'', x'; \epsilon) \neq P^s(x''', x')K_0(x''', x'; 2\epsilon). \end{aligned} \tag{40}$$

This is explained by the path dependence of the parallel transport — as the simplest manifestation of a motion associated with a loop. In angular coordinates K_s can be written as [19]

$$K_s(x'', x'; t) = \int \mathcal{D}(\theta)\sin\theta(t)\mathcal{D}(\phi)\exp\left\{i\int\left[\frac{\dot\theta^2}{2} + \frac{\dot\phi^2}{2} - is\cos\theta\dot\phi + \frac{1}{2}\left(\frac{1/4}{\sin^2\theta} + \frac{1}{4}\right)\right]dt\right\}. \tag{41}$$

In this case the Lagrangian does not depend on ϕ (but only on $\dot\phi$). Accordingly, the Hamiltonian will depend on the momentum only, and then the integration in the phase space leads to

$$\begin{aligned} \int e^{i\int \mathcal{L}(\dot\phi)dt}\mathcal{D}\phi = \int e^{i\int[p_\phi\dot\phi - H(p_\phi)]dt}\mathcal{D}\phi\mathcal{D}p_\phi &= \int\frac{dp_\phi}{2\pi}\sum_{n=-\infty}^{\infty}\exp\left\{ip_\phi(\Delta\phi + 2\pi n) - itH(p)\right\} \\ &= \sum_{m=-\infty}^{\infty}\exp\left\{im\Delta\phi - itH(m)\right\} \end{aligned} \tag{42}$$

where the periodicity conditions and the Poisson summation formula

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{i2\pi np} = \sum_{m=-\infty}^{\infty}\delta(p - m)$$

have been used [17]. It is then easy to obtain

$$K_s = \sum_{m=-\infty}^{\infty} e^{im(\phi'' - \phi')}\int\mathcal{D}(\theta)\exp\left\{i\int\left(\frac{\dot\theta^2}{2} - \frac{1}{2}\frac{(m + s\cos\theta)^2 - 1/4}{(\sin^2\theta)^2} + \frac{1}{8}\right)dt\right\}. \tag{43}$$

The transformation with $\gamma = \theta/2$, $d\tau = dt/4$ leads to:

$$K_s = \sum_{m=-\infty}^{\infty} e^{im(\phi'' - \phi')}\int\mathcal{D}(\gamma)\exp\left\{i\int\frac{1}{2}\left[\dot\gamma^2 - \frac{(m+s)^2 - 1/4}{(\sin\gamma)^2} - \frac{(m-s)^2 - 1/4}{(\cos\gamma)^2} + s^2 + \frac{1}{4}\right]d\tau\right\} \tag{44}$$

This is the path integral of the form mentioned above (related to the Pöschl-Teller potential). It can be rewritten as an expansion in eigenfunctions [12, 18]:

$$K_s(\theta'', \phi''; \theta', \phi'; t) = \sum_{n,m}\Phi_n^{\alpha,\beta}(\theta'')\, e^{im\phi''}\Phi_n^{\alpha,\beta}(\theta')\, e^{-im\phi'}\, e^{-iE_n t}; \tag{45}$$

$$\Phi_n^{\alpha,\beta} = \left[2^{3/2}(n + \alpha + \beta + 1)\frac{n!\Gamma(n + \alpha + \beta + 1)}{\Gamma(n + \beta + 1)\Gamma(n + \alpha + 1)}\right]^{1/2}(\sin\theta/2)^{\alpha + 1/2}(\cos\theta/2)^{\beta + 1/2}P_n^{\alpha,\beta}(\cos\theta),$$

where

$$\alpha = |m + s| \qquad \beta = |m - s| \qquad E_n = \frac{1}{2}\left\{[n + (\alpha + \beta + 1)/2]^2 - s^2 - \frac{1}{4}\right\}$$

In the case $s = 0$ one has

$$\alpha = \beta = |m|, \qquad (\sin\theta/2)^{\alpha+1/2}(\cos\theta/2)^{\beta+1/2} = \left(\tfrac{1}{2}\sin\theta\right)^{|m|+1/2},$$

$$\Phi_n^{\alpha,\beta} \sim (\sin\theta)^{m/2} P_{n+|m|}^{|m|}(\cos\theta), \qquad E_n = \frac{1}{2}\left[(n+|m|+\frac{1}{2})^2 - \frac{1}{4}\right] \frac{(n+|m|)(n+|m|+1)}{2},$$

we have an expansion in Legendre polynomials $P_l^{|m|}$ with the energy levels $E = l(l+1)/2$, $l = n + |m|$ [19].

3.2. Solution on a pseudosphere

The psuedosphere is a non-compact manifold of constant curvature with the positively-definite metric

$$ds^2 = d\theta^2 + \sinh^2\theta d\phi^2.$$

The way of calculations is very similar to the previous one. Thus, the effective potential is

$$V(\theta) = \frac{1}{2}\left[\left(s^2 - \frac{1}{4}\right)\coth^2\theta + 2ms\frac{\coth\theta}{\sinh\theta} + m^2\frac{1}{\sinh^2\theta}\right] + \frac{1}{4}. \tag{46}$$

After the transformation $\theta = 2\gamma$ we get the modified Pöschl-Teller potential

$$W(\gamma) = \frac{1}{2}\left[\frac{(m+is)^2 - 1/4}{\sinh^2\gamma} - \frac{(m-is)^2 - 1/4}{\cosh^2\gamma}\right] + 2s^2 + \frac{1}{2}, \tag{47}$$

so that the eigenfunctions are

$$\begin{aligned} {}_s\psi_k^m(\theta,\phi) = {} & \mathrm{e}^{im\phi}(\cosh\theta)^{m-is+1/2}(\sinh\theta)^{m+is+1/2}\sqrt{\frac{k\sinh k\pi}{\pi\Gamma(m+is+1)}}\sqrt{\frac{\Gamma((1-ik)/2+m)\Gamma((1+ik)/2+m)}{\Gamma((1-ik)/2+is)\Gamma((1+ik)/2+is)}} \\ & \times {}_2F_1\left(\frac{1-ik+2m}{2}, \frac{1+ik+2m}{2}; 1+m+is; -\sinh^2\theta/2\right). \end{aligned} \tag{48}$$

4. Polar coordinates in flat space

The metric is

$$ds^2 = dr^2 + r^2 d\phi^2$$

and the tetrad connection is simply $\omega = \mathbf{d}\phi$. Then the spin s must be added to the angular momentum m:

$$V(r) = \frac{(m+s)^2 - 1/4}{2r^2}, \tag{49}$$

and the problem reduces to the scalar case. One can see that in flat space the phase factor is integrable,

$$\int \mathbf{d}\phi = \Delta\phi.$$

Then the following result can be obtained: the propagator of a spinning field is a product of the scalar propagator and the identity operator (coincides with the parallel transport operator).

5. Two cosmological models

5.1. Infinitely expanding model and the space-time transformation

In this section we deal with indefinite metric. We will begin with a consideration of one more space of constant curvature,

$$ds^2 = dt^2 - \mathrm{e}^{2t}dx^2,$$

a model with infinite expansion. Note that we do not need an integration in E in (24): an integration in τ leads to $\delta(E - M^2/2)$, so that the required result is $G(x'', x', M^2/2)$. The effective potential ($k = p_x = \mathrm{const}$) is

$$\mathrm{e}^{-2t}k^2 \quad \ldots \quad 4s^2 - 4M^2 + 1 \tag{50}$$

After the space-time transformation $\eta = -2\log t$ one has

$$W(\eta) = 2iks - 2k^2\eta^2 + \frac{16s^2 - 16M^2 + 3}{8\eta^2}, \tag{51}$$

a radial oscillator type potential. Then one can find the propagator:

$$\begin{aligned} G(t'', x''; t', x') = \int dk\ (2ik)^{\sqrt{4s^2-4M^2+1}}\ \frac{e^{ik(x''-x')}}{\pi}\ \frac{\Gamma((2s + \sqrt{4s^2 - 4M^2 + 1} + 1)/2)}{\Gamma(\sqrt{4s^2 - 4M^2 + 1} + 1)} \\ \times\, {}_1F_1\left(\frac{-2s + \sqrt{4s^2 - 4M^2 + 1} + 1}{2};\ \sqrt{4s^2 - 4M^2 + 1} + 1;\ 2i\,e^{-t_1}k\right) \\ \times\, U\left(s + \frac{\sqrt{4s^2 - 4M^2 + 1} + 1}{2};\ \sqrt{4s^2 - 4M^2 + 1} + 1;\ 2i\,e^{-t_2}k\right) \\ \times \exp\Big[-(e^{-t_1} + e^{-t_2})ik + \sqrt{4s^2 - 4M^2 + 1}(t_2 + t_1)/2\Big], \end{aligned} \tag{52}$$

where $U(a; c; x)$ is the second solution of the degenerate hypergeometric equation. Accordingly, the wave function takes the form

$$\begin{aligned} \Psi(t, x) = \exp\left(\tfrac{1}{2}\sqrt{4s^2 - 4M^2 + 1} - i\,e^{-t}k + ikx\right) \\ \times\, {}_1F_1\left((-2s + \sqrt{4s^2 - 4M^2 + 1} + 1)/2);\ \sqrt{4s^2 - 4M^2 + 1} + 1;\ 2i\,e^{-t}k\right). \end{aligned} \tag{53}$$

It is of interest that the same coordinate transformation before the separation of variables gives the slightly different potential

$$-4iks + \frac{3 - 16M^2 + 16s^2}{8\eta^2} - 2k^2\eta^2 \tag{54}$$

due to special transformation properties of the connection. The final result will be the same.

5.2. A model with finite expansion

This is a model of previous type but with finite metric coefficients:

$$ds^2 = dt^2 - \tanh^2 t dx^2, \qquad t > 0.$$

The effective potential is

$$V(t) = \left(isk + \frac{s^2}{2} - \frac{k^2}{2} - \frac{1}{8}\right)\frac{1}{\sinh^2 t} + \left(-\frac{s^2}{2} - \frac{3}{8}\right)\frac{1}{\cosh^2 t} - \frac{k^2}{2}. \tag{55}$$

Then one has

$$G(x'', t''; x', t') = \int dk\ e^{ik(x''-x')} \int G(t'', t', k) \tag{56}$$

where

$$\begin{aligned} G(t'', t', k) = \frac{\Gamma((ik + i\omega + s + \sqrt{s^2 + 1} + 1)/2)}{\Gamma(ik + s + 1)}\ \frac{\Gamma((ik + i\omega + s - \sqrt{s^2 + 1} + 1)/2)}{\Gamma(1 + i\omega)} \\ \times\, (\tanh t_1 \tanh t_2)^{ik+s+1/2}\ {}_2F_1\left(\frac{ik + i\omega + s - \sqrt{s^2 + 1} + 1}{2}, \frac{ik + i\omega + s + \sqrt{s^2 + 1} + 1}{2}; 1 - i\omega; \frac{1}{\cosh^2 t_1}\right) \\ \times\, {}_2F_1\left(\frac{ik + i\omega + s - \sqrt{s^2 + 1} + 1}{2}, \frac{ik + i\omega + s + \sqrt{s^2 + 1} + 1}{2}; ik + s + 1;\ \tanh^2 t_2\right) \end{aligned} \tag{57}$$

and $\omega = \sqrt{k^2 + M^2}$. The corresponding wave functions are

$$\begin{aligned} \Psi(t, x) = e^{ikx}(\cosh t)^{\sqrt{s^2+1}+1/2}(\sinh t)^{ik+s+1/2} \\ \times\, {}_2F_1\left(\frac{ik + i\omega + s - \sqrt{s^2 + 1} + 1}{2}, \frac{ik + i\omega + s + \sqrt{s^2 + 1} + 1}{2}; 1 - i\omega;\ -\sinh^2 t\right). \end{aligned} \tag{58}$$

6. Conclusion

In contrast to the differential equations approach, the path integral formalism deals with a global object, the propagator. The propagator enables one to reveal connections between compact and non-compact spaces, continuous and discrete spectra.

The result obtained for a sphere has a direct application: separation of variables in the spherically-symmetrical problem in four dimensions [20] leads to an equation which is similar to (36). Hence it is possible to consider the result as a first step to path integration in more general cases.

The two-dimensional solutions have similar forms for an arbitrary spin. The situation like this also occurs in some four-dimensional problems. One can mention the Teukolsky equation [21] describing massless particles of spin $0,\ 1/2,\ 1,\ 3/2,\ 2$ in the Kerr geometry.

Our interest to path integration is stimulated as an opportunity to make clear the description of the interaction of a spinning quantum particle with the curvature. We expect that this approach can be useful in obtaining common features of fields of different spin in curved space and in studying the problem of quantization of the gravitational field.

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