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ON QUASI-HOMOGENEOUS PRODUCTION FUNCTIONS WITH CONSTANT ELASTICITY OF FACTORS SUBSTITUTION

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The study of the shape and properties of production functions is a subject of great interest in economic analysis. The class of production functions with constant Hicks elasticity of substitution includes many important production functions in economics; in particular, Cobb – Douglas production function and CES production function. In 2010, L. Losonczi proved that a two-factor homogeneous production function satisfies the constant Hicks elasticity of substitution property if and only if this homogeneous function is either a Cobb – Douglas production function or a CES production function. The similar result for multi-factor homogeneous production functions was proved by B.-Y. Chen in 2012. In this paper we generalize results of L. Losonczi and B.-Y. Chen to the class of *quasi-homogeneous* two-factor production functions with constant Hicks elasticity of substitution. Namely the notion of quasi-homogeneous two-factor production functions is introduced, relation between the condition of homogeneity and the condition of quasi-homogeneity is established, and the class of quasi-homogeneous two-factor production functions with constant elasticity of factors substitution by Hicks is obtained. Moreover, we pointed out the analytical form for quasi-homogeneous two-factor production functions with unit elasticity of substitution. The obtained results can be applied in modeling of production processes.

Key words: quasi-homogeneous production function; constant elasticity of substitution.

О КВАЗИОДНОРОДНЫХ ПРОИЗВОДСТВЕННЫХ ФУНКЦИЯХ С ПОСТОЯННОЙ ЭЛАСТИЧНОСТЬЮ ЗАМЕЩЕНИЯ ФАКТОРОВ

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Основной класс производственных функций, используемых в практическом анализе, – однородные производственные функции с постоянной эластичностью замещения факторов производства по Хиксу. Этот класс включает функцию Кобба – Дугласа и CES-функцию. В 2010 г. Л. Лошонци показал, что двухфакторная однородная произ-

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водственная функция обладает свойством постоянной эластичности замещения факторов производства по Хиксу тогда и только тогда, когда она является или функцией Кобба – Дугласа, или CES-функцией. Для многофакторных однородных производственных функций аналогичный результат доказал Б.-Й. Чен в 2012 г. В данной работе результаты, полученные Л. Лошонци и Б.-Й. Чен, обобщены в класс *квазиоднородных* двухфакторных производственных функций с постоянной эластичностью замещения факторов производства по Хиксу. Введено понятие «квазиоднородность», установлены связи между условиями однородности и квазиоднородности, во множестве двухфакторных производственных функций выделен класс квазиоднородных производственных функций с постоянной эластичностью замещения факторов по Хиксу, указан аналитический вид двухфакторных производственных функций, обладающих свойствами квазиоднородности и единичной эластичностью замещения факторов по Хиксу. Полученные результаты могут быть использованы при моделировании производственных процессов.

Ключевые слова: квазиоднородная производственная функция; постоянная эластичность замещения факторов.

Introduction. Consider a two-factor production function

$$Y:(K,L)\to F(K,L) \text{ for all } (K,L)\in G, \tag{1}$$

where K is the quantity of capital employed; L is the quantity of labour used; Y is the quantity of output, and the nonnegative function F is a twice continuously differentiable function on the domain G from the first quadrant

$$\mathbf{R}_+^2=\{(K,L):\ K\ge 0,\ L\ge 0\}.$$

Almost all economic theories presuppose a production function, either on the firm level or the aggregate level. In this sense, the production function is one of the key concepts of mainstream neoclassical theories. By assuming that the maximum output technologically possible from a given set of inputs is achieved, economists using a production function in analysis are abstracting from the engineering and managerial problems inherently associated with a particular production process.

In 1928, the American scientists Ch. W. Cobb and P. H. Douglas introduced in the paper "A theory of production" [1] the famous two-input production function

$$Y:(K,L)\to AK^{\alpha}L^{\beta} \text{ for all } (K,L)\in\mathbf{R}_+^2,\ A>0,\ \alpha,\beta\in(0;1),\ \alpha+\beta=1, \tag{2}$$

nowadays called Cobb – Douglas production function, in order to describe the distribution of income in the manufacturing industry of the United States of America during the period 1899–1922. In later work [2], P. H. Douglas prompted to allow for the exponents on K and L vary, which resulting in estimates that subsequently proved to be very close to improved measure of productivity developed at that time.

Note also that the Cobb – Douglas production function is especially notable for being the first time an aggregate or economy-wide production function had been developed, estimated, and then presented to the profession for analysis.

In 1961, the economists K. J. Arrow, H. B. Chenery, B. S. Minhas, and R. M. Solow introduced another two-input production function [3]

$$Y:(K,L)\to\left(\alpha K^{\gamma}+\beta L^{\gamma}\right)^{1/\gamma} \text{ for all } (K,L)\in G,\ \alpha,\beta>0,\ \gamma\ne 0,\ 1, \tag{3}$$

nowadays called CES production function (or the ACMC production function).

At the present time the Cobb – Douglas production function and the CES production function are widely used in economics to represent the relationship of an output to inputs. Specifically, these production functions were used to calculation consumer price index for the Belarusian pharmaceutical market [4] and in hybrid models of economic growth [5] for the Eurasian Economic Union's countries to 2050.

Concerning the history of production functions see the paper [6] of S. K. Mishra.

The most common quantitative indices of production factor substitutability are forms of the elasticity of substitution. In 1932 the British economist J. R. Hicks introduced the notion of elasticity of factors substitution in case of two inputs for the purpose of analyzing changes in the income shares of labour and capital [7]:

$$\sigma^{H}(K,L)=\frac{F_KF_L\left(KF_K+LF_L\right)}{KL\left(2F_KF_LF_{KL}-F_K^2F_{LL}-F_L^2F_{KK}\right)} \text{ for all } (K,L)\in G'\subset G,$$

where the subscripts of the function F denote partial derivatives, i. e.

$$\partial_K F=F_K,\ \partial_L F=F_L,\ \partial_{KK}^2F=F_{KK},\ \partial_{LL}^2F=F_{LL},\ \partial_{KL}^2F=F_{KL}.$$

For instance, the production functions (2) and (3) are productions functions with constant Hicks elasticity of factors substitution. The Cobb – Douglas production function (2) has unit elasticity of substitution, i. e. $\sigma^H(K, L) = 1.$ The CES production function (3) has the Hicks elasticity of factors substitution $\sigma^H(K, L) = 1/(1-\gamma).$

Let us remark that the practice of economic-mathematical analysis also uses other definitions of elasticity of substitution (by R. G. Allen [8; 9], by B. N. Mikhalevski [10], by D. McFadden [11], direct and inverse elasticities of substitution [12, p. 63–78]).

Problem statement. In his paper [13], Laszlo Losonczi proved that the homogeneous two-factor production function with constant Hicks elasticity of substitution is either the Cobb – Douglas production function or the CES production function. More precisely, we have the following (see [13] for details).

Theorem 1. *Suppose the two-factor production function (1) is a homogeneous function of degree* $m \neq 0$ *and has constant Hicks elasticity of substitution* $\sigma \neq 0$*. Then the two-factor production function (1) has the form*

$$F:(K, \mathrm{L}) \to \begin{cases} \beta K^{\alpha} L^{m-\alpha}, & \text{if } \sigma = 1; \\ \left(\beta_1 K^{\gamma} + \beta_2 L^{\gamma}\right)^{m/\gamma}, & \text{if } \sigma \neq 1, \end{cases}$$

where α *such that* $\alpha \neq 0$ *and* $\alpha \neq m$*, numbers* β*,* β_1*,* $\beta_2 > 0$ *and* $\gamma = (\sigma - 1)/\sigma$.

This result (theorem 1) complements the main propositions of the classical works [3; 9] and is consistent with known results on the classification of production functions [12, p. 111–113]. Note also that the analogue of theorem 1 for multi-factor production functions was proved by B.-Y. Chen in [14].

In this paper we generalize theorem 1 to the class of *quasi-homogeneous* two-factor production functions with constant Hicks elasticity of substitution.

The two-factor production function (1) is called *quasi-homogeneous* of degree $q \in \mathbf{R}$ with weight vector $g = (g_1, g_2) \in \mathbf{R}^2 \setminus \{(0,0)\},$ if the condition hold [15, p. 39–40]

$$F\left(\lambda^{g_1} K, \lambda^{g_2} L\right) = \lambda^q F(K, L) \text{ for all } (K, L) \in G \text{ and all } \lambda \in (0; +\infty). \tag{4}$$

For example, the two-factor production function

$$F:(K, L) \to 2KL + 3K^3 + \frac{L^2}{K} \text{ for all } (K, L) \in \{(K, L): K > 0, L \geq 0\}$$

is a quasi-homogeneous production function of degree 3 with weight vector $g = (1, 2)$.

Notice that if a quasi-homogeneous production function of degree q has the weight vector $g = (1, 1),$ then this production function is homogeneous of degree q.

The quasi-homogeneous condition (4) describes a condition of change of output when the quantity of capital and labour grow in different number of times.

Main results. The following statement (theorem 2) describes the analytical form of quasi-homogeneous two-factor production functions with unit elasticity of substitution.

Theorem 2. *Suppose the two-factor production function (1) is a quasi-homogeneous function of degree* q *with weight vector* $g = (g_1, g_2) \in \mathbf{R}^2 \setminus \{(0,0)\}$ *and has unit elasticity of substitution. Then this production function has the analytic form*

$$F:(K, L) \to AK^{\alpha} L^{\beta} \text{ for all } (K, L) \in G \subset \mathbf{R}_+^2, \ A > 0,$$

where non-vanishing real numbers α *and* β *such that the following condition hold*

$$\alpha g_1 + \beta g_2 = q.$$

P r o o f. Let $F: G \to \mathbf{R}_+$ be a quasi-homogeneous two-factor production function of degree q with weight vector $g = (g_1, g_2)$. Then, by generalized Euler's theorem to quasi-homogeneous functions [15, p. 40], we have identity

$$g_1 K \cdot F_K(K, L) + g_2 L \cdot F_L(K, L) = qF(K, L) \text{ for all } (K, L) \in G. \tag{5}$$

Since the production function $F: G \to \mathbf{R}_+$ is a twice continuously differentiable on the domain G, we see that the second-order partial derivatives $F_{LK} = F_{KL}$. Now differentiating the identity (5) with respect to the variables K and L we get the system

$$g_1K\cdot F_{KK}(K,L)+g_2L\cdot F_{KL}(K,L)=(q-g_1)F_K(K,L)\ \text{for all}\ (K,L)\in G,$$
$$g_1K\cdot F_{KL}(K,L)+g_2L\cdot F_{LL}(K,L)=(q-g_2)F_L(K,L)\ \text{for all}\ (K,L)\in G. \tag{6}$$

If the production function $F:G\to\mathbf{R}_+$ has unit elasticity of factors substitution, then $\sigma^H(K,L)=1$ for all $(K,L)\in G'\subset G.$ In other words, we have the identity

$$2F_KF_LF_{KL}-F_K^2F_{LL}-F_L^2F_{KK}=\frac{F_KF_L(KF_K+LF_L)}{KL}\ \text{for all}\ (K,L)\in G'. \tag{7}$$

From the identity (7) it follows that the second-order mixed partial derivative

$$F_{KL}(K,L)=\frac{F_KF_L(KF_K+LF_L)+KL\left(F_K^2F_{LL}+F_L^2F_{KK}\right)}{2KLF_KF_L}\ \text{for all}\ (K,L)\in G'. \tag{8}$$

Substituting the expression (8) for the partial derivative F_{KL} in the differential system (6), we obtain the system of partial differential equations

$$\frac{g_1K\cdot F_K+qF}{F_K}F_{KK}+\frac{g_2L\cdot F_K}{F_L}F_{LL}=2(q-g_1)F_K-g_2L\left(\frac{F_K}{L}+\frac{F_L}{K}\right),$$
$$\frac{g_1K\cdot F_L}{F_K}F_{KK}+\frac{g_2L\cdot F_L+gF}{F_L}F_{LL}=2(q-g_2)F_L-g_1K\left(\frac{F_K}{L}+\frac{F_L}{K}\right).$$

From this differential system it follows that the second-order partial derivatives

$$F_{KK}(K,L)=\left(\frac{F_K(K,L)}{F(K,L)}-\frac{1}{K}\right)F_K(K,L),\quad F_{LL}(K,L)=\left(\frac{F_L(K,L)}{F(K,L)}-\frac{1}{L}\right)F_L(K,L). \tag{9}$$

Further, substituting the expression (9) in (8), we get the mixed partial derivative

$$F_{KL}(K,L)=\frac{F_K(K,L)F_L(K,L)}{F(K,L)}. \tag{10}$$

After solving the first equation (the partial differential equation of the second order) from the system (9), we find the solution

$$F(K,L)=C_2(L)K^{C_1(L)}, \tag{11}$$

where C_1 and C_2 are some functions of the variable L.

By substituting (11) into the identity (10), we have $C_1(L)=\alpha=\text{const}.$ Therefore,

$$F(K,L)=C_2(L)K^{\alpha}. \tag{12}$$

Furthermore, substituting the function (12) into the second equation of the partial differential system (9), we obtain the second-order ordinary differential equation

$$\frac{C_2''(L)}{C_2'(L)}=\frac{C_2'(L)}{C_2(L)}-\frac{1}{L}.$$

From this differential equation it follows that the function $C_2(L)=AL^{\beta}.$ Consequently,

$$F(K,L)=AK^{\alpha}L^{\beta}, \tag{13}$$

where A, α and β are arbitrary real numbers.

Since the function (13) is a quasi-homogeneous production function of degree q with weight vector $g=(g_1,\,g_2)$, we see that the parameters A, α, and β satisfies the conditions $A>0$ and $\alpha g_1+\beta g_2=q$. This completes the proof of theorem 2.

Case $\sigma^H(K,L)=\sigma=\text{const}.$ Using methods of theorem 1, we have a quasi-homogeneous two-factor production function $F:G\to\mathbf{R}_+$ of degree q with weight vector $g=(g_1,g_2)$ and constant Hicks elasticity of substitution $\sigma\neq 1$ satisfies the system of partial differential equations

$$F_{KK}=\left(\frac{g_1(g_1-g_2)(\sigma-1)}{\sigma q^2}K\left(\frac{F_K}{F}\right)^2+\frac{\sigma q+(\sigma-1)(g_2-2g_1)}{\sigma q}\left(\frac{F_K}{F}\right)-\frac{1}{\sigma K}\right)F_K,$$

$$F_{LL}=\left(\frac{g_2(g_2-g_1)(\sigma-1)}{\sigma q^2}L\left(\frac{F_L}{F}\right)^2+\frac{\sigma q+(\sigma-1)(g_1-2g_2)}{\sigma q}\left(\frac{F_L}{F}\right)-\frac{1}{\sigma L}\right)F_L, \qquad (14)$$

$$F_{KL}=\left(2-\frac{(g_1+g_2)(\sigma-1)}{\sigma q}+\frac{g_1(g_1-g_2)(\sigma-1)}{\sigma q^2}K\left(\frac{F_K}{F}\right)+\frac{g_2(g_2-g_1)(\sigma-1)}{\sigma q^2}L\left(\frac{F_L}{F}\right)\right)\frac{F_K F_L}{2F}.$$

After solving the first partial differential equation of system (14), we get a quasi-homogeneous two-factor production function $F:G\to\mathbf{R}_+$ of degree q with weight vector $g=(g_1,g_2)$ and constant Hicks elasticity of substitution $\sigma\neq 1$ has the form

$$F:(K,L)\to C_2(L)\exp\int z(K,L)\,dK, \qquad (15)$$

where the function z satisfies the functional identity

$$z^{g_2}\left(Kz-\frac{q}{g_1}\right)^{-g_1}\left(Kz-\frac{q}{g_1-g_2}\right)^{g_1-g_2}=C_1(L)K^{-g_2/\sigma}.$$

By substituting the production function (15) into the second and third equations of the partial system (14), we can find the functions C_1 and C_2 of the variable L.

For example, solving the system (14) under condition $g_1=g_2=1$ (case of homogeneous production functions), we obtain the statement of theorem 1 in the case $\sigma\neq 1$.

Conclusions. In this article the notion of quasi-homogeneous two-factor production functions is introduced (see formula (4)), relation between the condition of homogeneity and the condition of quasi-homogeneity is established, and the class of quasi-homogeneous two-factor production functions with constant elasticity of factors substitution by Hicks is obtained (see formula (15)). Moreover, we pointed out the analytical form for quasi-homogeneous two-factor production functions with unit elasticity of substitution (theorem 2). This strengthens theorem 1 of L. Losonczi [13]. The obtained theoretical results can be used to simulate real production processes.

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