# METHOD OF INVERSE DIFFERENTIAL OPERATORS APPLIED TO CERTAIN CLASSES OF NONHOMOGENEOUS PDES AND ODES

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As demonstrated for various types of nonhomogenuities $\phi$ the Mathematica implementation of MIDO can be applied to a wide class of (linear) PDEs for which in most cases the built-in Mathematica procedure `DSolve` cannot provide a solution; thus `DESolve` really is an extension for solving PDEs with Mathematica. However, there are some *restrictions* inherent to this method :

(i) the (pseudo) differential operator polynomial $\chi$ is subject to decomposition into *linear factors* such that $(\alpha\, \mathcal{D}_t + \beta \mathcal{D}_x + \ldots + \gamma)^\kappa$ where $\kappa$ denotes the multiplicity.

(ii) the *nonhomogenuity* $\phi$ is restricted to a limited class of functions with *linear arguments* such as e. g. $(ax + by + \ldots)$ allowed only but not, for example, arguments which contain higher powers of the variables $(ax^2 + by^3 + \ldots)$. The functions admitted for $\phi$ are *exponential*, *trigonometric* (sin or cos) and *hyperbolic* functions (sinh or cosh) and *products* resp. *sums* of these functions. Other trigonometric or hyperbolic functions such as $\{\tan, \cot\}$ resp. $\{\tanh, \coth\}$ are excluded simply because for them the replacement rules which are essential for MIDO do not hold. Only the subgroup of functions $\{\sin, \cos\}$ resp. $\{\sinh, \cosh\}$ is closed under differentiation.

(iii) if the *nonhomogenuity* $\phi$ is a monomial $M(x, y, z, \ldots) = \sum_{i=1}^{k} \alpha_1 x^{k_i} y^{m_i} z^{n_i} \cdot \ldots$ (which may even contain an additional rational term i. e. $y^{-m_i}$ or a logarithmic factor e. g. $\log y$ ) then any combination of power products with respect to the reference variables in `VList` is allowed.

Although due to limitation of space not demonstrated here MIDO covers *homogeneous* and *inhomogeneous* ODEs too; adaptation of the procedures to ODEs required only some minor modifications with respect to pattern recognition.

**Introduction.** The paper deals with the *Method of Inverse Differential Operators* (MIDO) which is already well established for ordinary differential equations (ODE) but has never been thoroughly applied to nonhomogeneous partial differential equations (PDE). P.K. Kythe, P. Puri and M.R. Schaeferkotter [1] have extended MIDO stepwise to PDEs but the full implementation into the CAS Mathematica is new.

General restriction for the differential equations (DE), PDEs or ODEs, under consideration is that $\mathcal{L}_{x_1,x_2,\ldots} = \sum_{i_1,i_2,\ldots=0}^{n_1,n_2,\ldots} a_{i,j,\ldots} D_{x_1}^{i_1} D_{x_2}^{i_2} \ldots$ is a linear partial differential operator polynomial $\chi\,(D_{x_1}, D_{x_2}, \ldots)$ with *constant* coefficients $a, b, \alpha, \beta, \ldots, -2, 3, \ldots$. Here, $D_{x_1}^m = \partial_{x_1}^m$, $D_{x_2}^n = \partial_{x_2}^n$ etc. represent partial derivatives of order $m, n, \ldots$. In order to facilitate the algebraic manipulation of the differential operator polynomial $\chi$ an intermediate representation in terms of pseudo differential operators $\mathcal{D}_{x_i}$ is introduced which are fully convertable into each other, e. g. $\mathcal{D}_x^3\mathcal{D}_y^2 u(x,y) \Leftrightarrow \partial_{x,x,x}\partial_{y,y}$ $u(x,y)$. Thus, $\chi\left(\mathcal{D}_{x_1},\mathcal{D}_{x_2},\ldots\right)u\left(x_1,x_2,\ldots\right)$ constitutes the lhs of the PDE. The rhs of the DE is either $\phi=0$ in case of a *homogeneous* equation or $\phi\left(x_1,x_2,\ldots\right)$ for a *nonhomogeneous* one. In principle, the order of the PDE in terms of (pseudo) differential operators may be quite general.

As to *homogeneous* PDEs (where $\phi=0$ the general solution is denoted as $u_h$. For the method used it is essential that the differential operator polynomial $\chi\left(\mathcal{D}_{x_1},\mathcal{D}_{x_2},\ldots\right)$ can be *factorized* into linear factors of type $\mathcal{L}^{\kappa}_{x_1,x_2,\ldots,x_n}=\left(\alpha_1\mathcal{D}_{x_1}+\alpha_2\mathcal{D}_{x_2}+\ldots+\alpha_n\mathcal{D}_{x_n}+\gamma\right)^{\kappa}$ with *multiplicity* $\kappa$ for a subset of $n$ independent variables $\{x_1,x_2,\ldots,x_n\}\in\{t,x,y,z,\xi,\eta,\zeta\}$. Hence, each linear factor $\kappa=1$ gives rise to the following type of solution:

$$\left(\alpha_1\mathcal{D}_{x_1}+\alpha_2\mathcal{D}_{x_2}+\ldots+\alpha_n\mathcal{D}_{x_n}+\gamma\right)\Rightarrow u_h(x_1,x_2,\ldots,x_n)=$$
$$=f_1\left(\frac{\alpha_1x_2-\alpha_2x_1}{\alpha_1},\frac{\alpha_1x_3-\alpha_3x_1}{\alpha_1},\ldots,\frac{\alpha_1x_n-\alpha_nx_1}{\alpha_1}\right)\cdot e^{-\gamma x_1/\alpha_1}.$$

If a linear factor, e. g. $\left(\alpha_1\mathcal{D}_{x_1}+\alpha_2\mathcal{D}_{x_2}+\gamma\right)^{\kappa}$, has multiplicity $\kappa>1$ then the corresponding solution is :

$$\left(\alpha_1\mathcal{D}_{x_1}+\alpha_2\mathcal{D}_{x_2}+\gamma\right)^{\kappa}\Rightarrow u_h\left(x_1,x_2\right)=\sum_{k=0}^{\kappa-1}x_1^kf_k\left(\alpha_1x_2-\alpha_2x_1\right)\cdot e^{-\gamma x_1/\alpha_1}.$$

As to *nonhomogeneous* PDEs the functional form of the nonhomogenuity $\phi\left(x_1,x_2,\ldots\right)\neq0$ is subject to certain restrictions which are essential to the applicability of MIDO. In this respect the nonhomogenuity $\phi$ can either be:

(i) an *exponential function* $\phi_1=e^{ax+by+cz+\ldots}$,

(ii) a *trigonometric functions* $\phi_2=\sin\,|\cos(ax+by+cz+\ldots)$,

(iii) a *hyperbolic functions* $\phi_2=\sinh\,|\cosh(ax+by+cz+\ldots)$ or

(iv) any *multiplicative* combination of an exponential $\phi_1$ with trigonometric or hyperbolic functions $\phi_2$ such that $\phi=\phi_1{\cdot}\phi_2=e^{ax+by+cz+\ldots}{\cdot}\sin\,|\cos\,|$ $\sinh\,|\cosh(\ldots)$ resp.,

(v) any *additive* combination $\phi_2\sum_{i=1}^{k}\phi_{2i}$ with terms $\phi_{2i}=e^{ax+by+cz+\ldots}\cdot$ $\sin\,|\cos\,|\sinh\,|\cosh(\ldots)$ resp. ;

(vi) an arbitrary pure *monomial* $\phi_3=M(x,y,z,\ldots)=\sum_{i=1}^{k}\alpha_ix^{k_i}y^{m_i}z^{n_i}\cdot$ $\ldots$,

(vii) a monomial with *rational* part $\phi_3=\sum_{i=1}^{k}\alpha_ix^{k_i}y^{m_i}z^{\mp n_i}\cdot\ldots$,

(iix) a monomial multiplied with *logarithmic* term $\phi_3=\sum_{i=1}^{k}\alpha_ix^{k_i}y^{m_i}z^{\mp n_i}\cdot$ $\log(y)\ldots$ or

(ix) a monomial multiplied with *exponential function* $\phi_1 \cdot \phi_3 = = e^{ax+by+cz+\ldots} \cdot M(x,y,z,\ldots)$,

(x) a monomial multiplied with *trigonometric function* $\phi_2 \cdot \phi_3 = = \sin|\cos(ax+by+cz+\ldots) \cdot M(x,y,z,\ldots)$,

(xi) a monomial multiplied with *hyperbolic function* $\phi_2 \cdot \phi_3 = \sinh|\cosh(ax+by+cz+\ldots) \cdot M(x,y,z,\ldots)$ or

(xii) any sum$\phi = \sum_{i=1}^{k} \phi_i$ with *multiplicative* combinations of $\phi_1, \phi_2$ and $\phi_3$ (as given above).

The nonhomogenuity $\phi$ is restricted to classes of exponential, trigonometric sin or cos and hyperbolic (sinh or cosh) functions for which only *linear* combinations of an arbitrary selection of reference variables from `VList` are allowed as arguments, e. g. $(ax+\beta y-2z+\ldots$ but not $(ax^2+\beta y^3+\ldots)$. However, for monomials $\phi_3$ any combination of power products $x^{k_i} y^{m_i} z^{\mp n_i} \ldots$ of the reference variables is admitted. As regards to trigonometric or hyperbolic functions any terms containing $\tan, \cot$ resp. $\tanh, \coth$ are excluded simply because only the subgroups $\sin, \cos$ or $\sinh, \cosh$ are closed under differentiation. This behavior is reflected in certain *substitution rules* within MIDO.

**The Method.** For short, in order to calculate the particular solution of a DE the essential idea of MIDO is to move the inverse of the differential polynomial $\chi$ from the lhs to the rhs of the DE and apply it on the nonhomogenuity $\phi$:

$$\chi\left(\mathcal{D}_{x_1}, \mathcal{D}_{x_2}, \ldots\right) u\left(x_1, x_2, \ldots\right) = \phi\left(x_1, x_2, \ldots\right) \Longrightarrow u\left(x_1, x_2, \ldots\right) = \\ = \chi\left(\mathcal{D}_{x_1}, \mathcal{D}_{x_2}, \ldots\right)^{-1} \phi\left(x_1, x_2, \ldots\right).$$

For subsequent computation several replacement rules play an important role :

(**1**) *Inversion* of differential operator polynomial $\chi \cdot \chi^{-1}\phi = \mathbf{1}\phi$:

$$\chi\left(D_x, D_y\right)\left[\frac{1}{\chi\left(D_x, D_y\right)}\, \phi(x,y)\right] = \phi(x,y).$$

(**2**) *Factorization* of $\chi = \chi_1 \cdot \chi_2 \cdot \ldots$:

$$\frac{1}{\chi_1\left(D_x, D_y\right) \cdot \chi_2\left(D_x, D_y\right)}\, \phi(x,y) = \frac{1}{\chi_1(D_x, D_y)}\left(\frac{1}{\chi_2(D_x, D_y)}\, \phi(x,y)\right) = \\ = \frac{1}{\chi_2(D_x, D_y)}\left(\frac{1}{\chi_1(D_x, D_y)}\, \phi(x,y)\right).$$

(**3**) *Exponential* nonhomogenuity $\phi_1 = e^{ax+by+cz+\dots}$:

$$\frac{1}{\chi\left(D_x, D_y\right)}\, e^{ax+by} = \frac{1}{\chi(a,b)}\, e^{ax+by}$$

with $\chi(a,b) \neq 0$ and $\phi_1 \equiv e^{ax+by}$.

(**4**) *Trigonometric* nonhomogenuity $\phi = \phi_2 = \sin | \cos(\dots)$:

$$X\left(D_x^2, D_y^2\right)\cos(ax+by) = X\left(-a^2, -b^2\right)\cos(ax+by),$$

$$X\left(D_x^2, D_y^2\right)\sin(ax+by) = X\left(-a^2, -b^2\right)\sin(ax+by).$$

(**5**) *Hyperbolic* nonhomogenuity $\phi = \phi_2 = \sinh | \cosh(\dots)$:

$$X\left(D_x^2, D_y^2\right)\cosh(ax+by) = X\left(a^2, b^2\right)\cosh(ax+by),$$

$$X\left(D_x^2, D_y^2\right)\sinh(ax+by) = X\left(a^2, b^2\right)\sinh(ax+by).$$

(**6**) *Multiplicative* nonhomogenuity $\phi = \phi_1 \cdot \phi_2 = e^{(\dots)} \cdot \sin | \cos |$ $\sinh | \cosh(\dots)$:

$$\frac{1}{\chi\left(D_x, D_y\right)}\, e^{ax+by}\phi_2(x,y) = e^{ax+by}\frac{1}{\chi\left(D_x+a, D_y+b\right)}\,\phi_2(x,y) =$$

$$= e^{ax}\frac{1}{\chi\left(D_x+a, D_y\right)}e^{by}\phi_2(x,y) = e^{by}\frac{1}{\chi\left(D_x, D_y+b\right)}e^{ax}\phi_2(x,y);$$

$$X\left(D_x, D_y\right)e^{ax+by}\phi_2(x,y) = e^{ax+by}X\left(D_x+a, D_y+b\right)\phi_2(x,y),$$

where $X = (\chi^{-1})$ is the inverse of the differential polynomial which is such that the denominator *denom* $(X)$ is *free* of $D_x, D_y, \dots$.

(**7**) *Monomial* nonhomogenuity $\phi_3 = M(x,y,z,\dots) = \sum_{i=1}^{k} x^{k_i}y^{m_i}z^{n_i}\dots$. Moreover, it turns out that monomials with *rational* terms $x^k y^{-m} z^n$ and/or with a *logarithmic* factor $x^k y^{\pm m} z^n \log(z)$ are covered by the algorithm as well.

$$\frac{1}{\chi\left(D_x, D_y, D_z\dots\right)}M(x,y,z,\dots) = \frac{1}{a_0 D_x^n\left(1 - R\left(D_y, D_z, \dots\right)\right)}\times$$

$$\times M(x,y,z,\dots) = \frac{1}{a_0}D_x^{-n}\sum_{j=0}^{N_{\max}} R\left(D_y, D_z, \dots\right)^j M(x,y,z,\dots),$$

where $\chi^{-1}$ is expanded into a *geometric series* with respect to the residual expression $R\left(D_y, D_z, \dots\right)$ and is applied to a monomial $M$ of finite order so that the series of differential operators $D_y, D_z, \ldots$ is truncated at some order $N_{\max}$.

(**8**) *Additive* nonhomogenuity $\phi = \sum_{i=1}^{k} \phi_i$, where terms $\phi_i = M(\ldots) \left| e^{(\ldots)} \right|$ $\sin | \cos | \sinh | \cosh(\ldots)$.

$$\frac{1}{\chi(D_x, D_y)} (\rho_1 \phi_1(x,y) + \rho_2 \phi_2(x,y)) =$$

$$= \rho_1 \frac{1}{\chi(D_x, D_y)} \phi_1(x,y) + \rho_2 \frac{1}{\chi(D_x, D_y)} \phi_2(x,y).$$

**A simple example.** For better understanding of the mechanism how the replacement rules previously given are applied consider the following simple example of a $2^{nd}$ order PDE with variables $\{x, y\}$:

$$\left(3\mathcal{D}_x^2 - 2\mathcal{D}_x\mathcal{D}_y - 5\mathcal{D}_y^2\right) u(x,y) = e^{x-y} + (3x + y).$$

From inspection of the lhs of the PDE the differential operator polynomial $\chi$ turns out to be $\chi = (3{\mathcal{D}_x}^2 - 2\mathcal{D}_x\mathcal{D}_y - 5{\mathcal{D}_y}^2) = (3\mathcal{D}_x - 5\mathcal{D}_y) \cdot (\mathcal{D}_x + \mathcal{D}_y)$, whereas the nonhomogenuity $\phi$ is the sum of an exponential function $\phi_1 = e^{x-y}$ and a (simple) monomial $\phi_3 = (3x + y)$.

As can be seen from the decomposition of the differential polynomial $\chi$ into the linear factors $(3\mathcal{D}_x - 5\mathcal{D}_y)(\mathcal{D}_x + \mathcal{D}_y)$ it is straightforward that the solution of the *homogeneous* PDE is $u_h = f_{1,0}\left(\frac{1}{3}(5x + 3y)\right) + f_{2,0}(y - x)$. According to the second term $f_{2,0}(y - x)$ it is obvious that the function $e^{x-y}$ is only a special instance of $f_{2,0}$ and will satisfy the homogeneous PDE.

In order to calculate the particular solution $u_{p1}$ for the monomial nonhomogenuity $\phi_3 = (3x+y)$ a series expansion of $\chi^{-1}$ into a (truncated) geometric series is done.

$$u_{p1}(x,y) = \chi^{-1}[3x + y] = \frac{1}{3\mathcal{D}_x^2 - 2\mathcal{D}_x\mathcal{D}_y - 5\mathcal{D}_y^2}[3x + y] =$$

$$= \frac{1}{3}\mathcal{D}_x^{-2} \frac{1}{1 - \left(\frac{2}{3}\frac{\mathcal{D}_y}{\mathcal{D}_x} + \frac{5}{3}\left(\frac{\mathcal{D}_y}{\mathcal{D}_x}\right)^2\right)}[3x + y] =$$

$$= \frac{1}{3}{\mathcal{D}_x}^{-2}\left(1 + \left(\frac{2}{3}\frac{\mathcal{D}_y}{\mathcal{D}_x} + \frac{5}{3}\left(\frac{\mathcal{D}_y}{\mathcal{D}_x}\right)^2\right) \mp \ldots\right)[3x + y] =$$

$$= \frac{1}{3}\mathcal{D}_x^{-2}\left[(3x + y) + \frac{2}{3}\mathcal{D}_x^{-1} 1\right] = \mathcal{D}_x^{-2}[x] + \frac{y}{3}\mathcal{D}_x^{-2}[1] + \frac{2}{9}\mathcal{D}_x^{-3}[1] =$$

$$= \frac{x^3}{6} + \frac{y}{3}\frac{x^2}{2} + \frac{2}{9}\frac{x^3}{6} = \left(\frac{1}{6}\,x^2 y + \frac{11}{54}\,x^3\right).$$

It should be noted that the inverse differential operator $\mathcal{D}_x^{-1}$ is the *antiderivative*; thus $\mathcal{D}_x^{-n}$ is a $n$-th order nested (indefinite) integral with respect to $x$.

As a consequence of the discussion of the homogeneous solution it turns out that as regards to $\phi_1 = e^{x-y}$ (after application of replacement rule (6) to $\chi^{-1}$) the expression $\chi^{-1}(\mathcal{D}_x, \mathcal{D}_y)\, e^{x-y} = e^{x-y} \cdot \chi^{-1}(\mathcal{D}_x \to +1, \mathcal{D}_y \to -1)$ [1] is *singular.* Therefore the naive ansatz for $u_{p2}$ does not suffice the nonhomogeneous PDE: due to the fact that the perturbing function $\phi_1 = e^{x-y}$ is already included in the homogeneous solution $u_h$ it is essential to multiply the ansatz with an extra term $x^\kappa$ where $\kappa$ is the multiplicity of the root of the linear factor $(\mathcal{D}_x + \mathcal{D}_y)$ (here $\kappa = 1$). Thus, the general ansatz $u_{p2} = (a_0 + a_1 x + a_2 x^2)\, e^{x-y}$ has to be made with unknown coefficients $\{a_0, a_1, a_2\}$ and substituted into the PDE. Then it turns out that $u_{p2}(x, y)$ is a particular solution of the PDE for all $x$ only if the coefficients are chosen to be $a_1 = \frac{1}{8}$, $a_2 = 0$ with arbitrary value for $a_0$ (because any $e^{x-y}$ already satisfies the homogeneous PDE, therefore $a_0 = 0$ may be chosen). Due to replacement rule (2) the particular solution of the PDE is given as $u_p = u_{p1} + u_{p2} = \left(\frac{1}{6}x^2 y + \frac{11}{54}x^3\right) + \frac{1}{8}x \cdot e^{x-y}$.

**Implementation of MIDO.** The central procedure for the solution of the DE is `DESolve[`$\chi, \phi$`, onoff, opt]`. It works as a kind of "black box": the only imput required is the differential operator polynomial $\chi(\mathcal{D}_{x_1}, \mathcal{D}_{x_2}, \ldots)$ given in terms of pseudo differential operators $\mathcal{D}_{x_i}$ and the nonhomogenuity $\phi(x_1, x_2, \ldots)$. There is no loss of generality if the selection of *differential operators* is restricted to the list $\mathcal{D}$`List`$= \{\mathcal{D}_t, \mathcal{D}_x, \mathcal{D}_y, \mathcal{D}_z, \mathcal{D}_\xi, \mathcal{D}_\eta, \mathcal{D}_\zeta\}$ corresponding to the *independent variables* from list `VList`$= \{t, x, y, z, \xi, \eta, \zeta\}$. Both sets, $\mathcal{D}$`List` and `VList`, may be changed if necessary.

The type of DE (ODE or PDE) is determined as regards to the number of variables $\{x_1, x_2, \ldots\}$ being used. The variables are counted from analyzing the number of distinct differential operators $\mathcal{D}_{x_i}$ used in $\chi$ with reference to `VList`. Thus, if there is only a single variable $x_1$ involved an ODE is given whereas if several variables $(x_1, x_2, \ldots)$ occur a PDE has to be treated. It should be pointed out that the coefficients occuring in $\chi$ and $\phi$ can either be numbers, e. g. $\{1, -3, \sqrt{2}, \ldots\}$ or symbols, e. g. $\{a, b, \ldots, \alpha, \beta, \ldots\}$, or a mixture of both types which is a non-trivial problem to distinguish them from the variables in `VList`.

The algorithm of MIDO takes the following *steps*:

(**1**) Whether the nonhomogenuity $\phi$ is zero or nonzero decided which of the procedures `homogeneousDEsolutions` (for $\phi = 0$) or `nonhomogeneousDEsolutions` (for $\phi \neq 0$) is called and thus either the *homogeneous* solution $u_h$ or the *particular* solution $u_p$ will be calculated. If $\phi = 0$ either the procedure `homogeneousODEsolutions` or `homogeneousPDEsolutions` is called depending on the number of variables and the *homogeneous* solution $u_h$ is evaluated. For $\phi \neq 0$ the most general form of the nonhomogenuity is $\phi = \phi_3 + \phi_3 \cdot \phi_{1|2}$, where $\phi_3$ denotes a monomial (resp. polynomial for a single variable), $\phi_{1|2}$ is a non-monomial which is either an exponential $\phi_1 = \exp(\ldots)$, a trigonometric $\phi_2 = \sin|\cos(\ldots)$ or hyperbolic $\phi_2 = \sinh|\cosh(\ldots)$ function. Moreover, $\phi_{1|2}$ may be multiplied with an additional monomial prefactor $\phi_3$. The routine monomialTest[$\phi$, onoff] which investigates $\phi$ returns the separated components of $\phi$ ( *monomial, non-monomial, monomial prefactor*) for further treatment. Three different flags (type$\phi$1, type$\phi$2, type$\phi$3) are set with values $T$ or $F$ according to which the algorithm makes a distinction of cases between five different combinations.

In the case $(F, T, F)$ where only a *non-monomial* nonhomogenuity $\phi_{1|2}$ is present its type is further analyzed by means of the routine analyze$\phi$[$\phi$, onoff] which returns a parameter type$\phi$. According to its value there will be a switch between different cases: `Exp` (for *exponential* functions), `Trig` (for trigonometric functions $\phi_2 = \sin|\cos$), `Hyp` (for *hyperbolic* functions $\phi_2 = \sinh|\cosh$), `Times` (for products $\phi_1 \cdot \phi_2$) and `Plus` (for $\phi_1 + \phi_2$).

In the case $(T, F, F)$ the nonhomogenuity is a *monomial* $\phi_3 = = M(x, y, z, \ldots) = \sum_{i=1}^{k} x^{k_i} y^{m_i} z^{n_i} \ldots$. The algorithm covers, in addition, as well monomials with *rational* terms $x^k y^{-m} z^n$ and/or with a *logarithmic* factor $x^k y^{\pm m} z^n \log(z)$. Due to five different combinations for the flags (type$\phi$1, type$\phi$2, type$\phi$3) to be considered the computation of $u_p$ branches into one of the procedures listed below :

(**a**) $(T, F, F)$ for pure monomial
$\phi_3 \Rightarrow$ uPMonomial$\otimes$Rational$\phi$1 [$\chi, \phi_3$, onoff];
(**b**) $(F, T, F)$ for non-monomial
$\phi_{1|2} \Rightarrow$ uPExp$\phi$3 [$\chi, \phi_1$, onoff ] or uPTrig$\phi$2 [$\chi, \phi_2$, onoff ] or uPHyp$\phi$2 [$\chi, \phi_2$, onoff] or
combinations uPExp$\phi$1$\otimes$ TrigHyp$\phi$2 [$\chi, \phi_1 \cdot \phi_2$, onoff] resp. uPnonMonomial$\phi$21 $\oplus$ $\phi$22 [$\chi, \phi_1 + \phi_{21} + \phi_{22}, \ldots$, onoff];
( **c**) $(F, T, T)$ for monomial $\cdot$ non-monomial
$\phi_3 \cdot \phi_2 \Rightarrow$ uPMonom$\otimes$ nonMonomial[$\chi, \phi_3 \cdot \phi_2$, onoff];
$(T, T, F)$ for monomial + non-monomial
$\phi_3 + \phi_2 \Rightarrow$ uPMonom$\oplus$ nonMonomial[$\chi, \phi_3 + \phi_2$, onoff];
$(T, T, T)$ for monomial + monomial $\cdot$ non-monomial

$\phi_3 + \phi_3 \cdot \phi_2 \Rightarrow$ `uPMonom`$\otimes$`nonMonomial`$[\chi, \phi_3 + \phi_3 \cdot \phi_2,$ `onoff`$]$.

**(2)** The *coefficients* of the nonhomogenuity $\phi$ are extracted by means of auxiliary routines:

`coeffExp`$\phi[\phi]$, `coeffTrig`$\phi[\phi]$, `coeffHyp`$\phi[\phi]$,

`coeffExpTrigOrHyp`$\phi[\phi]$, `coeffnonMonomial`$\phi[\phi]$,

`coeffMonomial`$\phi[\phi]$, `coeffMonomialLog`$\phi[\phi]$,

`coeffMonom`$\oplus$`nonMonom`$\phi[\phi]$ or `coeffMonom`$\otimes$`nonMonom`$\phi[\phi]$ and passed through to one of the special routines:

`uPExp`$\phi 1|$`uPTrig`$\phi 2|$`uPHyp`$\phi 2[\chi$`, coeffs, `$\phi$`, `$\rho$`, onoff]`,

`uPExp`$\phi 1\otimes$`TrigHyp`$\phi 2[\chi$`, coeffs, `$\phi$`, `$\rho$`, onoff]`,

`uPnonMonomial`$\phi 21 \oplus \phi 22[\chi, \phi$`, onoff]`,

`uPMonom`$\otimes$`Rational`$\phi 1[\chi, \phi$`, onoff]`, `uPMonomial`$\phi 1[\chi, \phi$`, onoff]`,

`uPMonom`$\oplus$`nonMonom`$\phi[\chi, \phi$`, onoff]`,

`uPMonom`$\otimes$`nonMonom`$\phi[\chi, \phi$`, onoff]` which finally determine the solution $u_p$.

**(3)** The *replacement rules* which are important to deal with functions constituting $\phi$ are generated by the subsequent procedures:

`sub`$\mathcal{D}$`2Coeffs`: $\{\mathcal{D}_i \to c_i\}$

**rule** 3: $\frac{1}{\chi(D_x,D_y)}\, e^{ax+by} = \frac{1}{\chi(a,b)}\, e^{ax+by}$,

`sub`$\mathcal{D}\mathcal{D}$`2CoeffTrig`: $\left\{ \mathcal{D}_i{}^n :\to \underbrace{\left(-c_i{}^2\right)^{n/2}}_{n\,\text{even}} \mid \underbrace{\left(-c_i{}^2\right)^{\lfloor n/2 \rfloor}}_{n\,\text{odd}} \mathcal{D}_i \right\}$;

**rule** 4: $X\left(D_x^2, D_y^2\right) \begin{cases} \cos \\ \sin \end{cases} (ax+by) = X\left(-a^2, -b^2\right) \begin{cases} \cos \\ \sin \end{cases} (ax+by)$,

`sub`$\mathcal{D}\mathcal{D}$`2CoeffHyp`: $\left\{ \mathcal{D}_i{}^n :\to \underbrace{\left(+c_i^2\right)^{n/2}}_{n\text{even}} \mid \underbrace{\left(+c_i{}^2\right)^{\lfloor n/2 \rfloor}}_{n\text{odd}} \mathcal{D}_i \right\}$

**rule** 5: $X\left(D_x^2, D_y^2\right) \begin{cases} \cosh \\ \sinh \end{cases} (ax+by) = X\left(a^2, b^2\right) \begin{cases} \cosh \\ \sinh \end{cases} (ax+by)$,

`sub`$\mathcal{D}$`2`$\mathcal{D}$`plusCoeff`$\phi 1$: $\{\mathcal{D}_i \to (\mathcal{D}_i + c_i)\}$

**rule** 6: $X\left(D_x, D_y\right) \phi_2(x,y) e^{ax+by} = e^{ax+by} X\left(D_x + a, D_y + b\right) \phi_2(x,y)$.

**(4)** Another crucial routine is `rationalizeX[X, subDD, onoff]` which *'rationalizes'* the denominator of the inverse differential operator polynomial $\chi^{-1}\left(\mathcal{D}_{x_1}, \mathcal{D}_{x_2}, \ldots\right)$. In analogy to *complex conjugation* $\overline{z}$ whereby the denominator of a complex number $\frac{1}{z} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$ becomes real, an expression which contains terms *linear* in (pseudo) differential operators $\mathcal{D}_i$, for example $\frac{1}{a+\mathcal{D}_i} = \frac{a-\mathcal{D}_i}{a^2-\mathcal{D}_i{}^2} \overset{\text{subDD}}{\Rightarrow} \frac{a-\mathcal{D}_i}{a^2 \pm c_i^2}$, is simplified by application of replacement rules 4 and 5. However, if *products* such as $\mathcal{D}_i \cdot \mathcal{D}_j$ appear in the denominator then the rationalizing process has to be repeated until all (pseudo) differential operators $\mathcal{D}_i, \mathcal{D}_j, \ldots$ become squared and thus can be replaced by $(\pm c_i^2), (\pm {c_j}^2), \ldots$.

**(5)** In order to verify the correctness of the solutions, whether $u_h$ and/or $u_p$ will satisfy the original PDE $\chi(\mathcal{D}_{x_1}, \mathcal{D}_{x_2}, \ldots) u(x_1, x_2, \ldots) = = \phi(x_1, x_2, \ldots)$, there is another essential central procedure given: `testDE[`$\chi, \phi, u$`, PDEtype, onoff, optSimplify]` which deduces from the number of variables resp. differential operators the type of DE to be either an ODE or PDE. For $\phi = 0$ which implies a homogeneous DE, the suitable test procedure to be applicable is `testHomDE[`$\chi$`, uh, PDEtype, onoff, optSimplify]`. (For parameter `PDEtype` the default value is `"Off"`, for `optSimplify` it is Identity. The pseudo differential operators $\mathcal{D}_{x_i}$ (which are used only to faciliate algebraic manipulations of the differential operator polynomial $\chi$) are converted into standard differential operators $\partial^n_{x_i}$. The execution is done with the procedure `Convert`$\chi$`2PDE[`$\chi, \phi$`, onoff]`. Auxiliary routines used for the conversion of the pseudo differential operators $\mathcal{D}^n_x$ into (proper) differential operators $\partial_{\{x,n\}}\#$ are

`rule`$\mathcal{D}$`2D`, $\mathcal{D}$`2DMultiRule1`, `concat`$\mathcal{D}$`Rule1`|2|3; `rule`$\mathcal{J}$`2Int`, `fold`$\mathcal{J}$`2Int`, `fold`$\mathcal{J}$`2Int`$\Sigma$; `uvPairs`, `uvwTriples`, `uvwqQuadruples`, `varsNtuples`; $\mathcal{D}$`uv2`$\mathcal{D}$`xy`, $\mathcal{D}$`uvw2`$\mathcal{D}$`xyz`, $\mathcal{D}$`uvwq2`$\mathcal{D}$`txyz`; `pairs`$\mathcal{D}$`i`$\mathcal{D}$`j1`, `triples` $\mathcal{D}$`i`$\mathcal{D}$`j`$\mathcal{D}$`k1` and `quadruples`$\mathcal{D}$`i`$\mathcal{D}$`j`$\mathcal{D}$`k`$\mathcal{D}$`l1`.

In this way the familiar representation of a DE is reconstructed from $\chi$ as, for example, required by the built-in procedure `DSolve` from Mathematica.

**(6)** Moreover, in addition to the implemented solver `DESolve` both procedures `testDE` and `testHomDE` investigate whether `DSolve[PDEeqn, upVar, `$\chi$`Var]` is capable finding a solution for the given DE. There is a time constraint of 180 CPU seconds given; if it is exceeded the computation will be aborted. It turns out, however, that only in rare cases `DSolve` can provide a solution named `upDS`. Thus, the current implementation of MIDO within Mathematica provides solutions for a wide range of PDEs which are not generally covered by the built-in `DSolve`.

**(7)** For `PDEtype="On"` the type of a $2^{nd}$ order PDE (to be *hyperbolic*| *parabolic*|*elliptic*) is determined from the coefficients of the differential operator polynomial $\chi$ (in analogy to the type of a quadric surface). As regards to the value of the *discriminant* $\Delta = \begin{cases} > 0 & \text{hyperbolic,} \\ = 0 & \text{PDE is parabolic,} \\ < 0 & \text{elliptic,} \end{cases}$

otherwise the discriminant and hence the type of PDE is *indeterminate*.

The procedure which determines the type of a $2^{nd}$ order PDE is `Equation Type` and is switched On|Off by the parameter `PDEtype` from `testDE`.

**Interface for switching between different representations of a DE.** In order to change between different representations for PDEs within `DESolve` and `DSolve` three useful *conversion procedures* are provided :

(**1**) `Convert`$\chi$`2PDEeqn[`$\chi$`, `$\Phi$`, onoff]` casts a differential polynomial $\chi$ with pseudo differential operators $\mathcal{D}_{x_i}$ and nonhomogenuity $\phi = \Phi$ into a PDE in standard form required as input for `DSolve[PDEeqn, u,` $\{x_1, x_2, \ldots\}$], for example:

$$\chi = a_0 + a_1\mathcal{D}_\zeta + a_2\mathcal{D}_x\mathcal{D}_y^3 + a_3\mathcal{D}_x^2\mathcal{D}_y^2 + a_4\mathcal{D}_x^4, \quad \phi = \Phi \rightarrow$$

$$\texttt{PDEeqn} = a_0u[x, y, \zeta] + a_1u^{(0,0,1)}[x, y, \zeta]+$$
$$+a_2u^{(1,3,0)}[x, y, \zeta] + a_3u^{(2,2,0)}[x, y, \zeta] + a_4u^{(4,0,0)}[x, y, \zeta] = \Phi[x, y, \zeta].$$

(**2**) `ConvertPDEeqn2`$\chi$`[PDEeqn, vars, onoff]` converts a PDE given in the standard form for `DSolve[PDEeqn,` $u, \{x_1, x_2, \ldots\}$] into a differential polynomial $\chi$ with pseudo differential operators $\mathcal{D}_{x_i}$ and $\phi = \Phi$, for example:

$$\texttt{PDEeqn} = a_0u[x, y, \zeta] + a_1u^{(0,0,1)}[x, y, \zeta]+$$

$$+a_2u^{(1,3,0)}[x, y, \zeta] + a_3u^{(2,2,0)}[x, y, \zeta] + a_4u^{(4,0,0)}[x, y, \zeta] = \Phi[x, y, \zeta] \rightarrow$$
$$\chi = a_0 + a_1\mathcal{D}_\zeta + a_2\mathcal{D}_x\mathcal{D}_y^3 + a_3\mathcal{D}_x^2\mathcal{D}_y^2 + a_4\mathcal{D}_x^4, \phi = \Phi.$$

(**3**) `ConvertPDEops2PDEeqn[PDEops, onoff]` converts (the lhs of) a PDE given in *pure function* representation into a PDE in standard form with (dummy) nonhomogenuity $\phi = \Phi$ required as input for `DSolve`, for example:

$$\texttt{PDEops} = (a_0\# + a_1\partial_y\# + a_2\partial_{\{x,1\}\{y,3\}}\# + a_3\partial_{\{x,2\},\{z,2\}}\# + a_4\partial_{\{y,4\}}\#)\& \rightarrow$$

$$\texttt{PDEeqn} = a_0u[x, y, z] + a_1u^{(0,1,0)}[x, y, z] + a_2u^{(1,3,0)}[x, y, z]+$$
$$+a_3u^{(2,0,2)}[x, y, z] + a_4u^{(0,4,0)}[x, y, z] = \Phi[x, y, z].$$

Thus, these three conversion procedures will facilitate the switching between different representations of a DE.

**Application of MIDO to 12 classes of nonhomogeneous PDEs.**

**(1)** *exponential* nonhomogenuity $\phi = \phi_1 = \exp(\ldots)$.

(**i**) The $4^{th}$ order PDE $(5 + \mathcal{D}_x^2)\,(1 + \mathcal{D}_y)\,(2 + \mathcal{D}_z)\,u(x, y, z) = 4e^{-5x-y+z}$.
$\chi = (5 + \mathcal{D}_x)^2\,(1 + \mathcal{D}_y)\,(2 + \mathcal{D}_z)\,; \quad \phi = 4e^{-5x-y+z}$;
`DESolve[`$\chi, \phi$`,"Off"];`

It may be noted that in this case `DESolve[`$\chi, \phi$`, onoff]` does not return most general particular solution for the nonhomogeneous PDE

$$(5+\mathcal{D}_x)^2\,(1+\mathcal{D}_y)\,(2+\mathcal{D}_z)\,u(x,y,z) = 4e^{-5x-y+z},$$

but $u_p = \frac{2}{3}x^2ye^{-5x-y+z}$ only which comprises a monomial part $\frac{1}{2}xy^2$ and an exponential function $e^{x+2y}$. However, $u_p$ may rather be supplemented by additional lower order monomial terms

$$u_{\text{supp}} = e^{-5x-y+z}\left(\alpha_1 + x\alpha_2 + x^2\alpha_3 + y\alpha_4 + xy\alpha_5\right)$$

which satisfy the PDE too. This is achieved by the procedure $u_{\text{supp}} =$ $=$ `uPsupplement` $[u_p,$ `onoff`$]$ which requires as input only the existing particular solution $u_p$.

$u_{\text{supp}} =$ `uPsupplement` $[u_p,$ `Off`$]$;

Thus, the supplemented particular solution turns out to be:

$$u_p + u_{\text{supp}} = \frac{2}{3}e^{-5x-y+z}x^2y + e^{-5x-y+z}\left(\alpha_1 + x\alpha_2 + x^2\alpha_3 + y\alpha_4 + xy\alpha_5\right).$$

Testing the resulting solution with `testDE`$[\chi, \phi, u_p + u_{\text{supp}},$ `Off` $]$ gives rise to the subsequent typical output:

(ii) Another example shows how the *degenerate* solution occuring for: $\left(\mathcal{D}_x^2 - 2\mathcal{D}_x\mathcal{D}_y - 5\mathcal{D}_y^2\right)u(x,y) = e^{x-y}$ is handled.

$\chi = (3\mathcal{D}_x{}^2 - 2\mathcal{D}_x\mathcal{D}_y - 5\mathcal{D}_y{}^2)$; $\quad \phi := e^{x-y}$;

$u_h =$ `DESolve`$[\chi, 0, \text{Off}]$;

The PDE $(3\mathcal{D}_x - 5\mathcal{D}_y)\,(\mathcal{D}_x + \mathcal{D}_y)\,U(x,y) = 0$ already possesses the homogeneous solution $u_h = f_{1,0}\left[\frac{1}{3}(5x+3y)\right] + f_{2,0}(-x+y)$

$u_p =$ `DESolve`$[\chi, \phi, \text{Off}]$;

Due to the factorization of $\chi = (3\mathcal{D}_x - 5\mathcal{D}_y)\,(\mathcal{D}_x + \mathcal{D}_y)$ the replacement rule $\{\mathcal{D}_x \to 1, \mathcal{D}_y \to -1\}$ due the second factor (which results from interchange of $e^{x-y}$ with $\chi^{-1}$) causes the inverse differential polynomial $\chi^{-1}$ to becomes singular. In order to cope with this degeneracy of the particular solution $u_p \sim e^{x-y}$ with one of the homogeneous solutions $u_{h1} = f_{2,0}(-x+$ $+y)$ the procedures `uPmodifySingular [up, onoff]` and `optimizeSoluti on[`$\chi, \phi$`, up, onoff]` give rise to the following ansatz $(a_1x + \ldots + a_\kappa x^\kappa)\,\times$ $\times e^{x-y}$ with multiplicity $\kappa = 1$; thus the resulting solution turns out to be $u_p = \frac{1}{8}xe^{x-y}$. Testing the resulting solution with `testDE`$[\chi, \phi, u_h + u_p,$ `Off`$]$ verifies the correctness.

**(2)** *trigonometric* nonhomogenuity $\phi = \phi_2 = \sin|\cos(\ldots)$.

This is a parabolic $2^{nd}$ order PDE

$$\left(3\mathcal{D}_x^2 - \mathcal{D}_y + 4\mathcal{D}_z\right)u(x,y) = \sin\left(ax + by + cz\right).$$

$\chi = (3\mathcal{D}_x{}^2 - \mathcal{D}_y + 4\mathcal{D}_z)\,; \quad \phi = \sin[ax + by + cz];$
`DESolve[`$\chi, \phi$`, "Off"];`
The particular solution is

$$u_p = \frac{(b-4c)\cos(ax+by+cz) - 3a^2\sin(ax+by+cz)}{9a^4 + (b-4c)^2}$$

is again verified by `testDE`.

(**3**) *hyperbolic* nonhomogenuity $\phi = \phi_2 = \sinh \,|\, \cosh(\ldots)$.
For the $4^{th}$ order PDE :$(\mathcal{D}_x{}^4 + 3\mathcal{D}_x{}^2\mathcal{D}_y + 2\mathcal{D}_t)\, u(x,y,t) = \sinh(y)$
$\chi = (\mathcal{D}_x{}^4 + 3\mathcal{D}_x{}^2\mathcal{D}_y + 2\mathcal{D}_t); \quad \phi = \sinh[y];$
`DESolve[`$\chi, \phi$`, "Off"];`
there occurs an *exceptional* case: in the process of 'rationalizing' $\chi^{-1}$ the denominator reduces to $\mathcal{D}_t$ and will vanish after applying the appropriate replacement rules $\{\mathcal{D}_t{}^n \to 0, \mathcal{D}_x{}^n \to 0,\ \mathcal{D}_y{}^n \to \begin{Bmatrix} 1 & n = \text{even} \\ \mathcal{D}_y & n = \text{odd} \end{Bmatrix}$. However, the routine `dTermsException` copes with the situation $\chi_R = \infty$ and instead handles the antiderivative $\mathcal{D}_t^{-1} \Rightarrow \mathcal{J}[t]\ \ \sinh y]$ which gives rise to the correct result $\frac{t}{2}\sinh(y)$ which is supplemented by $\alpha_1 \sinh(y)$ which is verified by `testDE`. The particular solution is $u_p = \frac{t}{2}\sinh(y) + +\alpha_1 \sinh(y)$ which is verified by `testDE`:

(**4**) *multiplicative* nonhomogenuity $\phi = \phi_1 \cdot \phi_2 = e^{(\ldots)} \cdot \sin \,|\, \cos \,|\, \sinh \,|\, \cosh(\ldots)$.

For the $4^{th}$ order PDE:

$$\left(3D_x^4 - D_y + D_z^2\right) u(x,y,z) = e^{\alpha x + \beta z}\sinh(bx + ay).$$

$\chi = (3\mathcal{D}_x^4 - \mathcal{D}_y + \mathcal{D}_z^2)\,; \quad \phi = e^{\alpha x + \beta z}\sinh[bx + ay];$
`DESolve[`$\chi, \phi$`, "Off"];`
there is $\{\mathcal{D}_x, \mathcal{D}_y, \mathcal{D}_z\}$ whereas the coefficient list from $\phi_2$ gives only $b, a$ (originating from $\phi_2 = \sin(bx + ay)$.

Hence, in this specific case the coefficient list must be extended with the help of `makeListsEqualLength[var`$\mathcal{D}$`,`$\phi$`, onoff]` to $\{b, a, 0\}$ so that the correct replacement list will be instead $\{\mathcal{D}_x \to b, \mathcal{D}_y \to a, \mathcal{D}_z \to 0\}$. The correct particular solution is

$$u_p = \left(e^{x\alpha + z\beta}\left(3a\left(b^4 + 6b^2\alpha^2 + \alpha^4\right)\cosh\left[bx + ay\right] + \right.\right.$$
$$\left.\left. + \left(-12ab\alpha\left(b^2 + \alpha^2\right) + \beta^2\right)\sinh\left[bx + ay\right]\right)\right) /$$
$$\left(-9a^2\left(b^2 - \alpha^2\right)^4 - 24ab\alpha\left(b^2 + \alpha^2\right)\beta^2 + \beta^4\right)$$

verified by `testDE`.

(**5**) *additive (non-monomial)* nonhomogenuity $\phi_2 = \sum_{i=1}^{k} \phi_{2i}$ with $\phi_{2i} =$ $= \exp \,|\, \sin \,|\, \cos \,|\, \sinh \,|\, \cosh$.

The example given makes essentially use of `uPnonMonomial`$\phi 21 \oplus \phi 22$ to deal with the sum of *non-monomial* terms $\phi_{21} + \phi_{22} + \phi_{23} + \ldots$.

For the $4^{th}$ order PDE : $(D_x^4 + 3D_x^2 D_y + 2D_t)\, u(x, y) = \phi$ were the nonhomogenuity $\phi = \alpha\ e^{x-y} + \sin(x+y) + \cos(t-x) + 2\sinh(x-2y+$ $+3t) + 3\cosh(x+2y-3t)$ is a mixed sum of exponential, trigonometric and hyperbolic functions

$\chi = (\mathcal{D}_x^4 + 3\mathcal{D}_x^2\mathcal{D}_y + 2\mathcal{D}_t)\,;$

$\phi = \gamma e^{x-y} + \sin[x+y] + \cos[t-x] + 2\sinh[x-2y+3t] + 3\cosh[x+$ $+2y-3t]$;

`DESolve[`$\chi, \phi$`, "Off"]`;

the particular solution is

$$u_p = -\frac{1}{2}\gamma e^{x-y} + \frac{1}{10}(\sin(x+y) + 3\cos(x+y)) +$$

$$+\frac{1}{5}(2\sin(t-x) + \cos(t-x)) + 2\sinh(3t + x - 2y) + 3\cosh(3t - x - 2y)$$

and is verified by `testDE`.

(**6**) *monomial* nonhomogenuity

$\phi = \phi_3 = M(x, y, z, \ldots) = \sum_{i=1}^{k} \alpha_i x^{k_i} y^{m_i} z^{n_i} \cdot \ldots$.

Pure *monomial* nonhomogenuity $\phi_1 = M(x, y, z, \ldots) = \phi_{11} + \phi_{12} +$ $+\phi_{13} + \ldots = \alpha t^{k_1} x^{m_1} y^{n_1} \cdot \ldots + \beta t^{k_2} x^{m_2} y^{n_2} \ldots + \ldots$ gives rise to an expansion of $\chi^{-1}$ into a truncated *geometric* series. The order `iMax` of the truncated geometric series expansion is determined in a *heuristical* way as sum of leading exponents $n_i$ of the monomial variables in $\phi_1$, i. e. `iMax`$=$ $= k_1 + m_1 + n_1 + \ldots$. (In case of a rational term $x_i^{-m}$ the *minimum* exponent is chosen for `iMax=` $|m|$ ). However, this approach sometimes leads to huge expansion order which has to be corrected by a global positive variable `$jMax` (where its default value is 0) to diminish the expansion order. `iMax` serves as input to the routine `truncatedSeries[`$\chi$`, lead`$\mathcal{D}$`, iMax-$jMax, onoff]`. E. g. $\chi = (\mathcal{D}_t + 2\mathcal{D}_x + 3\mathcal{D}_y - 7\mathcal{D}_\zeta)$ with `lead`$\mathcal{D} = \mathcal{D}_t$ as leading term gives rise to

$$\chi^{-1} = \mathcal{D}_t^{-1} \cdot \frac{1}{1 - \underbrace{\left(-\frac{2\mathcal{D}_x}{\mathcal{D}_t} - \frac{3\mathcal{D}_y}{\mathcal{D}_t} + \frac{7\mathcal{D}_\zeta}{\mathcal{D}_t}\right)}_{\rho\mathcal{D}}} =$$

$$= \mathcal{D}_t^{-1} \cdot \frac{1}{1 - \rho/\mathcal{D}_t} = \sum_{i=0}^{iMax} \mathcal{D}_t^{-(i+1)} \cdot (\rho)^i .$$

Only in cases where the differential operator polynomial $\chi(\mathcal{D}_t, \mathcal{D}_x, \ldots)$ is not too complicated and the nonhomogenuity $\phi$ is only a *monomial* then the built-in procedure `DSolve` is (sometimes) able to calculate a solution.

The $1^{st}$ order PDE in the variables $\{t, x, y, z\}$:

$$(\mathcal{D}_t + 2\mathcal{D}_x + 3\mathcal{D}_y - 4\mathcal{D}_z)\, u(t, x, y, z) = (3x + y) + 5x^4y^5t^6 + \alpha x y^2 z^3 .$$

$\chi = (\mathcal{D}_t + 2\mathcal{D}_x + 3\mathcal{D}_y - 4\mathcal{D}_z)$; $\phi = (3x + y) + 5x^4y^5t^6 + \alpha x y^2 z^3$ ;
`$jMax` $= 14$;
`DESolve[`$\chi, \phi$`, "Off"]`;

has the solution $u_p$ where the expansion order is reduced from `iMax`$=$ $= 6 + 4 + 5 + 3 = 18$ down to order 4 by subtraction of `$jMax=14`.

$$u_p = \frac{25920}{77} t^{11} y - \frac{2160}{7} t^{10} x y^2 + \frac{600}{7} t^9 x^2 y^3 - \frac{75}{7} t^8 x^3 y^4 + \frac{5}{7} t^7 x^4 y^5 - \frac{768}{5} t^5 y \alpha +$$

$$+ t^4 \left(16 x y^2 \alpha - 144 y z \alpha\right) + t^3 \left(16 x y^2 z \alpha - 48 y z^2 \alpha\right) +$$

$$+ t^2 \left(6 x y^2 z^2 \alpha - 6 y z^3 \alpha\right) + t \left(x y^2 z^3 \alpha + 3x + y\right) .$$

In order to test the *efficiency* of the implementation of MIDO the following examples have been investigated:

( **i**) $\chi = (\mathcal{D}_x + \mathcal{D}_y - \mathcal{D}_z)$ with $\phi = (x + y + z)^n$, $n = 1, 2, 15, 20, 30, 50, 70, 100)$;

( **ii**) $\chi = \left(\mathcal{D}_x^k + \mathcal{D}_y^k - \mathcal{D}_z{}^k\right)$ with $\phi = (x + y + z)^{10}$, $k = 1, 2, 3, 5, 10)$.

For case ( **i**) with $n = 100$ one has to choose `$jMax`$= 3n - 1 = 299$ (!) to reduce the expansion order to `iMax`$= 1$ so that there results an antiderivative $\mathcal{J}[x]^m$ with respect to the leading term $\mathcal{D}_x$ up to order $m = 2$ , i. e. $\mathbf{1}\mathcal{J}[x] - \mathcal{D}_y\mathcal{J}[x]^2 + \mathcal{D}_z\mathcal{J}[x]^2$. The particular solution $u_p$ consists of 5253 terms,

$$u_p = \frac{x^{101}}{101} + \frac{\ll 1 \gg}{101} + \ll 200 \gg +$$

$$+ x \left(y^{100} + 100 y^{99} z + 4950 y^{98} z^2 + 161700 y^{97} z^3 + 3921225 y^{96} z^4 + \right.$$

$$+ 75287520 y^{95} z^5 + \ll 90 \gg + 3921225 y^4 z^{96} + 161700 y^3 z^{97} +$$

$$\left. + 4950 y^2 z^{98} + 100 y z^{99} + z^{100}\right) .$$

For case ( **ii**) with $k = 5$ it requires $\$$`jMax` $= 3n - 1 = 29$ to reduce the expansion order. There results an antiderivative $\mathcal{J}[x]^m$ with respect to the leading term $\mathcal{D}_x$ up to order $m = 2k = 10$. The particular solution is :

$$u_p = \frac{x^{15}}{360360} + \frac{yx^{14}}{24024} + \frac{zx^{14}}{24024} + \frac{y^2x^{13}}{3432} + \frac{z^2x^{13}}{3432} + \frac{yzx^{13}}{1716} +$$

$$+ \frac{y^{10}z^5}{120} + \frac{y^{11}z^4}{264} + \frac{y^{12}z^3}{792} + \frac{y^{13}z^2}{3432} + \frac{y^{14}z}{24024}.$$

( **7**) *monomial* nonhomogenuity with *rational part* $\phi = \phi_3 = x^{\pm m}y^{\pm n}z^{\pm k} \cdot \ldots$.

This is an elliptic $2^{nd}$ order PDE $\left(\mathcal{D}_x^2 + \mathcal{D}_y^2 + \mathcal{D}_z^2\right) u(x, y, z) = \beta x^{-5}y^4z$ with a monomial containing a rational term $x^{-5}$.

$\chi = \left(\mathcal{D}_x^2 + \mathcal{D}_y^2 + \mathcal{D}_z^2\right)$; $\phi = \beta x^{-5}y^4z$; $\$$`jMax`$= 3$;

`DESolve[`$\chi$`, `$\phi$`,"Off"];`

The particular solution is

$$u_p = \frac{z\beta}{12x^3}\left(-12x^4 - 6x^2y^2 + y^4 + 12x^4\log(x)\right).$$

( **8**) *monomial* $\cdot$ log nonhomogenuity $\phi = \phi_3 = x^{\pm m}y^{\pm n}z^{\pm k} \cdot \ldots \log(y)$.

This is an elliptic $2^{nd}$ order PDE $\left(\mathcal{D}_x^2 + \mathcal{D}_y^2 + \mathcal{D}_z^2\right) u(x, y, z) = \beta x^{-5}y^4z$ with a monomial containing in addition a *rational* term $z^{-4}$ and a *logarthmic* term $\log(z)$.

$\chi = \left(\mathcal{D}_x^2 + \mathcal{D}_y^2 + \mathcal{D}_z^2\right)$; $\phi = x^2y^3z^{-4}\log[z]$; $\$$`jMax`$= 0$;

`DESolve[`$\chi, \phi$`, "Off", Simplify];`

The particular solution is

$$u_p = \frac{y}{36z^2}\left(5x^2y^2 + 72z^4 + \left(22y^2z^2 - 24z^4 + 6x^2\left(y^2 + 11z^2\right)\right)\log(z) + \right.$$

$$\left. + 6z^2\left(3x^2 + y^2 - 6z^2\right)\log(z)^2\right).$$

( **9**) *monomial* $\cdot$ *exponential* nonhomogenuity $\phi = \phi_3 \cdot \exp(\ldots)$.

This is a $3^{rd}$ order PDE $({\mathcal{D}_x}^3 + 3{\mathcal{D}_y}^2 + \mathcal{D}_z - 4)\, u(x) = (ax + by^2 + cz^3) \cdot \cdot (\alpha + x + y^4)\, e^z$ with the product of two *monomials* multiplied with an exponential function.

$\chi = (\mathcal{D}_x^3 + 3{\mathcal{D}_y}^2 + \mathcal{D}_z - 4)$; $\phi = (ax + by^2 + cz^3)(\alpha + x + y^4)\, e^z$;

`DESolve[`$\chi, \phi$`, "Off", polyForm];`

The particular solution is

$$u_p = e^z\left(-\frac{by^6}{3} - \frac{1}{3}cz^3y^4 - \frac{1}{3}cz^2y^4 - 10by^4 - \frac{2cy^4}{27} - \frac{1}{3}axy^4 - \frac{2}{9}czy^4 - \right.$$

$$-4cz^3y^2 - 8cz^2y^2 - 120by^2 - \frac{32cy^2}{9} - 4axy^2 - \frac{1}{3}bxy^2 - 8czy^2 -$$

$$-\frac{1}{3}b\alpha y^2 - 8cz^3 - \frac{1}{3}cxz^3 - -\frac{ax^2}{3} - 24cz^2 - \frac{1}{3}cxz^2 -$$

$$-240b - \frac{160c}{9} - 8ax - \frac{2bx}{3} - \frac{2cx}{27} - 32cz -$$

$$-\frac{2cxz}{9} - \frac{1}{3}cz^3\alpha - \frac{1}{3}cz^2\alpha - \frac{2b\alpha}{3} - \frac{2c\alpha}{27} - \frac{ax\alpha}{3} - \frac{2cz\alpha}{9}\Big).$$

( **10**) *monomial* · *trigonometric* nonhomogenuity $\phi = \phi_3 \cdot \sin|\cos(\ldots)$.

This is a $3^{rd}$ order PDE $\left(\mathcal{D}_x^3 + 3\mathcal{D}_y^2 - 4\right) u(x,y) = xy^2 \cdot \cos(5x+3y)$ with a *monomial* $xy^2$ multiplied with $\cos(5x+3y)$.

$\chi = \left(\mathcal{D}_x^3 + 3\mathcal{D}_y^2 - 4\right)$; $\quad \phi = xy^2 \cos[5x+3y]$;

`DESolve[`$\chi, \phi$`, "Off", Simplify]`;

The particular solution is

$$u_p = \frac{1}{9459684612549602}\left(-\left(8293x\left(-134245548+\right.\right.\right.$$

$$\left.+1156873500y + 2131989319y^2\right) +$$

$$+450\left(12178155469 + 39626027250y + 84041643478y^2\right)\Big)\cos(5x+3y)+$$

$$+\left(-8293x\left(450837750 + 2188953936y + 8596731125y^2\right) +\right.$$

$$+75\left(15605928750 + 212466759516y + 266498664875y^2\right)\Big)\sin(5x+3y)\Big).$$

( **11**) *monomial* · *hyperbolic* nonhomogenuity $\phi = \phi_3 \cdot \sinh|\cosh(\ldots)$.

This is a $3^{rd}$ order PDE $\left(\mathcal{D}_x{}^3 + 3\mathcal{D}_y^2 - 4\right) u(x,y) = y \cdot \cosh(5x)$ with a simple *monomial* $y$ multiplied with $\cosh(5x)$.

$\chi = \left(\mathcal{D}_x^3 + 3\mathcal{D}_y^2 - 4\right)$; $\quad \phi = y\cosh[5x]$;

`DESolve[`$\chi, \phi$`, "Off"]`;

The particular solution is

$$u_p = \frac{4y\cosh(5x)}{15609} + \frac{125y\sinh[(5x)}{15609}.$$

( **12**) *additive* nonhomogenuity

$\phi = \sum_{i=1}^{k} \phi_i$ with $\phi_i = M(\ldots)|\exp|\sin|\cos|\sinh|\cosh(\ldots)$.

This is a $4^{th}$ order PDE $(\mathcal{D}_t^2 + 2\mathcal{D}_x^4 + 3\mathcal{D}_x^2\mathcal{D}_y - 4\mathcal{D}_z)\, u(t,x,y,z) = \phi$ with a sum of a *monomial* part $\phi_1$ and *non-monomial* part $\phi_2$ : $\phi_1 = (3x+y) + \alpha x^4y^5t^6 + \beta xy^2z^3$; $\phi_2 = \alpha e^{x-y} + \sin(x+y) + \cos(t-x) + 2\sinh(x-2y+3t) + 3\cosh(x+2y-3t)$.

$\chi = \left(\mathcal{D}_t^2 + 2\mathcal{D}_x^4 + 3\mathcal{D}_x^2\mathcal{D}_y - 4\mathcal{D}_z\right);$ `$jMax` $= 15;$

$\phi_1 = (3x + y) + \alpha x^4 y^5 t^6 + \beta x y^2 z^3;$

$\phi_2 = \alpha e^{x-y} + \sin[x + y] + \cos[t - x] + 2\sinh[x - 2y + 3z] + 3\cosh[x+$ $+2y - 3z];$

`DESolve`$[\chi, \phi_1 + \phi_2,$ `"Off"`$];$

The particular solution $u_p$ is (collected with respect to powers of $t$:

$$u_{p1} = \frac{1}{154}t^{12}y^3\alpha + t^{10}\left(-\frac{1}{28}x^2y^4\alpha - \frac{y^5\alpha}{105}\right) +$$

$$+t^8\left(\frac{1}{56}x^4y^5\alpha + \frac{1}{105}xy^2\beta\right) + \frac{2}{15}t^6xy^2z\beta +$$

$$+\frac{1}{2}t^4xy^2z^2\beta + t^2\left(\frac{1}{2}xy^2z^3\beta + \frac{3x}{2} + \frac{y}{2}\right)$$

$$u_{p2} = -\alpha e^{x-y} - \frac{3}{160}\left(\cosh(x + 2y - 3z) - 9\sinh(x + 2y - 3z)\right) +$$

$$+\frac{1}{80}\left(-\sinh(x - 2y + 3z) - 9\cosh(x - 2y + 3z)\right) +$$

$$+\cos(t - x) + \frac{1}{13}\left(2\sin(x + y) + 3\cos(x + y)\right).$$

**Список литературы**


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