Linear-fractional network problem

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1. Abstract

A direct exact relaxation algorithm is considered for a network problem of the fractional programming. The algorithm includes the proof of an optimization criterion, the development of a formula for the increment of the objective function and building of an usable direction of flow changing.

Linear-fractional problem, network, direction, optimization, flow.

2. A homographic network problem

Consider the extreme problem of the fractional programming in a network form:

$$f(x)=\frac{p(x)}{q(x)}=\frac{\sum\limits_{(i,j)\in U} p_{ij}x_{ij}+\beta}{\sum\limits_{(i,j)\in U} q_{ij}x_{ij}+\gamma}\to \max , \tag{1}$$

$$\sum_{j\in I_i^+(U)} x_{ij}-\sum_{j\in I_i^-(U)} x_{ji}=b_i,\ i\in I , \tag{2}$$

$$d_{*ij}\le x_{ij}\le d_{ij}^*,\ (i,j)\in U , \tag{3}$$

where $S=\{I,U\}$ is a finite oriented connected network without multiple arcs and loops with node set I, $|I|=m$, and arc set U, $|U|=n$, $m\le n$; β,γ are scalars, $x=(x_{ij},(i,j)\in U)$ is a flow of the problem, $x\in X$ (restrictions (2) and (3) are fulfilled on the vector x), X is a convex set of flows (plans); $I_i^+(U)=\{j:(i,j)\in U\}$, $I_i^-(U)=\{j:(j,i)\in U\}$. Restrictions (2) are primal restrictions; (3) are direct restrictions.

We suppose that the denominator $q(x)=\sum\limits_{(i,j)\in U} q_{ij}x_{ij}+\gamma$ of the objective function doesn't change the sign on the plan set. Therefore, without a restriction of the generality, we can suppose that the denominator $q(x)>0$, $\forall x\in X$, otherwise we can consider the following problem: $f(x)=\dfrac{-p(x)}{-q(x)}$.

The arc set U_R of the covering tree of the source network S is the support of the network [5]. Let $U_N=U\setminus U_R$ be a nonsupport arc set of network S. A pair $\{x,U_R\}$ consisting of any given flow x and any given support U_R is referred to as a support flow. Let's name the support flow $\{x,U_R\}$ non-degenerate if it is non-degenerate on the direct restrictions $d_{*ij}<x_{ij}<d_{ij}^*$, $(i,j)\in U_R$.

3. Formula for the increment of the objective function

Let $\{x,U_R\}$ is a support flow of the problem (1) – (3). Designate through $\bar{x}=x+\Delta x$ some other flow of the problem (1) – (3), where Δx is the flow increment. It is known [4, 6] that support (dependent) components of a vector $x=(x_{ij},(i,j)\in U)$ can be represented through the nonsupport components $x_{\tau\rho}$, $(\tau,\rho)\in U_N$ in the following way:

$$x_{ij}=\sum_{(\tau,\rho)\in U_N} x_{\tau\rho}\,sign(i,j)^{L(\tau,\rho)}+\sum_{s\in I\setminus root} b_s\,sign(i,j)^{Ls},\ (i,j)\in U_R , \tag{4}$$

where $root$ is any node $i\in I$; $sign(i,j)^{L(\tau,\rho)}$ is the sign of arc (i,j) in the circle $L(\tau,\rho)$ caused by an arc $(\tau,\rho)\in U_N$ (a circle direction is detected by an arc (τ,ρ)); $sign(i,j)^{Ls}$ is the sign of arc (i,j) in the single chain Ls of the covering tree (this chain connects any node s to the node $root$).

The flow increment Δx satisfies a homogeneous system corresponding to the primary restrictions of problem (2):

$$\sum_{j\in I_i^+(U)}\Delta x_{ij} - \sum_{j\in I_i^-(U)}\Delta x_{ji} = 0,\ i\in I. \quad (5)$$

On account of (4) and (5) the increment Δx on the support arcs $(i,j)\in U_R$ is calculated in the following way:

$$\Delta x_{ij} = \sum_{(\tau,\rho)\in U_N}\Delta x_{\tau\rho}\, sign(i,j)^{L(\tau,\rho)},\ (i,j)\in U_R. \quad (6)$$

Let's calculate the objective function increment Δf:

$$\Delta f = f(\bar{x}) - f(x) = f(x+\Delta x) - f(x) =$$

$$= \frac{p(x+\Delta x)}{q(x+\Delta x)} - \frac{p(x)}{q(x)} = \frac{p(x+\Delta x)q(x) - p(x)q(x+\Delta x)}{q(x)q(x+\Delta x)} =$$

$$= \frac{\sum_{(i,j)\in U} p_{ij}\Delta x_{ij}\left(\sum_{(i,j)\in U} q_{ij}x_{ij} + \gamma\right) - \sum_{(i,j)\in U} q_{ij}\Delta x_{ij}\left(\sum_{(i,j)\in U} p_{ij}x_{ij} + \beta\right)}{\left(\sum_{(i,j)\in U} q_{ij}x_{ij} + \gamma\right)\left(\sum_{(i,j)\in U} q_{ij}(x_{ij}+\Delta x_{ij}) + \gamma\right)}.$$

As far as $\sum_{(i,j)\in U} q_{ij}x_{ij} + \gamma = q(x) > 0$, $\forall x\in X$ we can divide the numerator and the denominator by $q(x)$. As a result:

$$\Delta f = \frac{\sum_{(i,j)\in U} p_{ij}\Delta x_{ij} - f(x)\sum_{(i,j)\in U} q_{ij}\Delta x_{ij}}{\sum_{(i,j)\in U} q_{ij}(x_{ij}+\Delta x_{ij}) + \gamma}.$$

Let's transform the numerator and the denominator of the objective function on the ground of analytical expression of arc flows x_{ij}, $(i,j)\in U_R$ (4):

$$\sum_{(i,j)\in U} p_{ij}x_{ij} = \sum_{(\tau,\rho)\in U_N} p_{\tau\rho}x_{\tau\rho} + \sum_{(i,j)\in U_R} p_{ij}x_{ij} = \sum_{(\tau,\rho)\in U_N} x_{\tau\rho}\Delta_p^{(\tau,\rho)} + P,$$

where $\Delta_p^{(\tau,\rho)} = p_{\tau\rho} + \sum_{(i,j)\in U_R} p_{ij}\, sign(i,j)^{L(\tau,\rho)}$, $(\tau,\rho)\in U_N$, $P = \sum_{(i,j)\in U_R} p_{ij}\sum_{s\in I\backslash root} b_s\, sign(i,j)^{Ls}$.

$$\sum_{(i,j)\in U} q_{ij}x_{ij} = \sum_{(\tau,\rho)\in U_N} x_{\tau\rho}\Delta_q^{(\tau,\rho)} + Q,$$

where $\Delta_q^{(\tau,\rho)} = q_{\tau\rho} + \sum_{(i,j)\in U_R} q_{ij}\, sign(i,j)^{L(\tau,\rho)}$, $Q = \sum_{(i,j)\in U_R} q_{ij}\sum_{s\in I\backslash root} b_s\, sign(i,j)^{Ls}$.

Let's transform the formula for the increment of the objective function. For this purpose we shall consider separately the sums standing in numerator and denominator. Using formula (6), we have:

$$\sum_{(i,j)\in U} p_{ij}\Delta x_{ij} = \sum_{(\tau,\rho)\in U_N} p_{\tau\rho}\Delta x_{\tau\rho} + \sum_{(i,j)\in U_R} p_{ij}\Delta x_{ij} = \sum_{(\tau,\rho)\in U_N}\Delta x_{\tau\rho}\Delta_p^{(\tau,\rho)},$$

$$\sum_{(i,j)\in U} q_{ij}\Delta x_{ij} = \sum_{(\tau,\rho)\in U_N} q_{\tau\rho}\Delta x_{\tau\rho} + \sum_{(i,j)\in U_R} q_{ij}\Delta x_{ij} = \sum_{(\tau,\rho)\in U_N}\Delta x_{\tau\rho}\Delta_q^{(\tau,\rho)}.$$

In view of the received formulas, the increment of the objective function will be transformed as follows:

$$\Delta f = \frac{\sum_{(\tau,\rho)\in U_N}\Delta x_{\tau\rho}\Delta^{(\tau,\rho)}}{\sum_{(\tau,\rho)\in U_N} x_{\tau\rho}\Delta_q^{(\tau,\rho)} + \sum_{(\tau,\rho)\in U_N}\Delta x_{\tau\rho}\Delta_q^{(\tau,\rho)} + Q + \gamma},$$

where $\Delta^{(\tau,\rho)} = \Delta_p^{(\tau,\rho)} - f(x)\Delta_q^{(\tau,\rho)}$, $(\tau,\rho)\in U_N$ is an evaluation vector.

4. An allowable direction of flow changing

The vector $l = l(U)\in R^n$ is called an allowable direction for a flow x, if $\exists\theta^0 > 0$ such that all vectors $x(\theta) = x + \theta l$, $0\le\theta\le\theta^0$ are flows of the problem, i.e. the following relations are carried out:

$$\sum_{j\in I_i^+(U)} x_{ij}(\theta) - \sum_{j\in I_i^-(U)} x_{ji}(\theta) = b_i, i\in I,\ (Ax(\theta) = b) \quad (7)$$

$$d_{*ij} \le x_{ij}(\theta) \le d_{ij}^*, (i,j)\in U\ \ (d_* \le x(\theta) \le d^*),\ \forall\theta\in[0,\theta^0], \quad (8)$$

$$l_R = l(U_R) = (l_{ij}, (i,j)\in U_R),\ l_N = l(U_N) = (l_{ij}, (i,j)\in U_N). \quad (9)$$

It follows from relations (7) that the allowable direction $l = l(U)$ satisfies the system: $\sum_{j\in I_i^+(U)} l_{ij} - \sum_{j\in I_i^-(U)} l_{ji} = 0$, $i\in I$ and, hence [4, 6], $l_{ij} = \sum_{(\tau,\rho)\in U_N} l_{\tau\rho}\, sign(i,j)^{L(\tau,\rho)}$, $(i,j)\in U_R$.

Let's calculate the derivative of the objective function on an allowable direction:

$$\frac{\partial f(x)}{\partial l}=\frac{\sum\limits_{(\tau,\rho)\in U_N} l_{\tau\rho}\Delta^{(\tau,\rho)}}{q(x)}. \tag{10}$$

5. Optimality criterion

Theorem (optimality criterion): For optimality of the support flow $\{x,U_R\}$ it is enough, and in case of non-degeneracy it is also necessary, that the following conditions are carried out:

$$\begin{aligned}&\Delta^{(\tau,\rho)}\geq 0, \text{ under } x_{\tau\rho}=d^*_{\tau\rho},\\&\Delta^{(\tau,\rho)}\leq 0, \text{ under } x_{\tau\rho}=d_{*\tau\rho},\\&\Delta^{(\tau,\rho)}=0, \text{ under } d_{*\tau\rho}<x_{\tau\rho}<d^*_{\tau\rho},\ (\tau,\rho)\in U_N.\end{aligned} \tag{11}$$

Proof. Sufficiency. Suppose that relations (11) are satisfied for the support flow $\{x,U_R\}$. We shall show that x is the optimal flow. Let l is any allowable direction. From (8) and (11), we have that:

$$\Delta^{(\tau,\rho)}\leq 0 \text{ and } l_{\tau\rho}\geq 0, \text{ if } x_{\tau\rho}=d_{*\tau\rho} \text{ and} \tag{12}$$

$$\Delta^{(\tau,\rho)}\geq 0 \text{ and } l_{\tau\rho}\leq 0, \text{ if } x_{\tau\rho}=d^*_{\tau\rho},\ (\tau,\rho)\in U_N. \tag{13}$$

As by hypothesis the denominator of the objective function $q(x)>0$, $\forall x\in X$, then from (10), (12) and (13) it follows that $\frac{\partial f(x)}{\partial l}\leq 0$ for any allowable direction l for the flow x and, hence, x is the optimal flow.

Necessity. Assume for the purpose of contradiction that $\{x,U_R\}$ is a non-degenerate optimal support flow, for which even one condition (11) is not carried out, i.e. there is such an arc $(\tau_0,\rho_0)\in U_N$ that:

$$\text{either } \Delta^{(\tau_0,\rho_0)}<0 \text{ under } x_{\tau_0\rho_0}=d^*_{\tau_0\rho_0}, \tag{14}$$

$$\text{or } \Delta^{(\tau_0,\rho_0)}>0 \text{ under } x_{\tau_0\rho_0}=d_{*\tau_0\rho_0}, \tag{15}$$

$$\text{or } \Delta^{(\tau_0,\rho_0)}\neq 0 \text{ under } d_{*\tau_0\rho_0}<x_{\tau_0\rho_0}<d^*_{\tau_0\rho_0}. \tag{16}$$

Let's construct a special direction $l=l(U)$ for the flow x:

$$\begin{aligned}&l_{\tau_0\rho_0}=\operatorname{sgn}\Delta^{(\tau_0,\rho_0)},\ l_{\tau\rho}=0,\ (\tau,\rho)\in U_N\setminus(\tau_0,\rho_0),\\&l_{ij}=\sum_{(\tau,\rho)\in U_N} l_{\tau\rho}\,sign(i,j)^{L(\tau,\rho)}=l_{\tau_0\rho_0}\,sign(i,j)^{L(\tau_0,\rho_0)},\ (i,j)\in U_R.\end{aligned} \tag{17}$$

Let's prove that the direction constructed using relations (17) is allowable. We'll show that equality (7) follows from (17) for $\forall\theta>0$, where $x_{ij}(\theta)=x_{ij}+\theta l_{ij}$, $(i,j)\in U$, and (7) is equivalent to the of relation:

$$\sum_{j\in I_i^+(U)} l_{ij}-\sum_{j\in I_i^-(U)} l_{ji}=0,\ i\in I. \tag{18}$$

Under the assumption that $U=U_R\cup U_N$ equality (18) can be written as:

$$\sum_{j\in I_i^+(U_R)} l_{ij}-\sum_{j\in I_i^-(U_R)} l_{ji}+\sum_{j\in I_i^+(U_N)} l_{ij}-\sum_{j\in I_i^-(U_N)} l_{ji}=0,\ i\in I. \tag{19}$$

Let's denote $L_R=\sum\limits_{j\in I_i^+(U_R)} l_{ij}-\sum\limits_{j\in I_i^-(U_R)} l_{ji}$; $L_N=\sum\limits_{j\in I_i^+(U_N)} l_{ij}-\sum\limits_{j\in I_i^-(U_N)} l_{ji}$.

Let's transform L_R using (17):

$$L_R=\sum_{j\in I_i^+(U_R)} l_{ij}-\sum_{j\in I_i^-(U_R)} l_{ji}=l_{\tau_0\rho_0}\left(\sum_{j\in I_i^+(U_R)} sign(i,j)^{L(\tau_0,\rho_0)}-\sum_{j\in I_i^-(U_R)} sign(j,i)^{L(\tau_0,\rho_0)}\right).$$

Let $i\in I\setminus\{\tau_0,\rho_0\}$. Substituting the direction (17) in L_N yields $L_N=0$ as L_N doesn't include component $l_{\tau_0\rho_0}$. The circle $L(\tau_0,\rho_0)$ contains two arcs for every node $i\in I\setminus\{\tau_0,\rho_0\}$, let's j_1 and j_2 are the nodes incident to these arches. Three situations are possible:

1) (i,j_1), (i,j_2); 2) (j_1,i), (j_2,i); 3) (i,j_1), (j_2,i), where incidence the mentioned arches.

Without restriction of a generality we suggest that $sign(i,j_1)^{L(\tau_0,\rho_0)}=sign(\tau_0,\rho_0)^{L(\tau_0,\rho_0)}=1$ for 1) and 3) cases and $sign(j_1,i)^{L(\tau_0,\rho_0)}=sign(\tau_0,\rho_0)^{L(\tau_0,\rho_0)}=1$ for 2) case then:

1) $sign(i,j_2)^{L(\tau_0,\rho_0)}=-1$ hence $L_R=l_{\tau_0\rho_0}(1+(-1))=0$;

2) $sign(j_2,i)^{L(\tau_0,\rho_0)}=-1$ hence $L_R=l_{\tau_0\rho_0}(-1-(-1))=0$;

3) $sign(j_2,i)^{L(\tau_0,\rho_0)}=1$ hence $L_R=l_{\tau_0\rho_0}(1-1)=0$.

Thereby, $L_R+L_N=0$, $i\in I\setminus\{\tau_0,\rho_0\}$.

Let's consider a case when $i=\tau_0$ then node $\rho_0\in I^+_{\tau_0}(U_N)$ and hence $L_N=l_{\tau_0\rho_0}$. When calculating L_R there may be two cases:

1) arc $(\tau_0,j_1)\in L(\tau_0,\rho_0)$, $j_1\in I^+_{\tau_0}(U_N)$ then $sign(\tau_0,j_1)^{L(\tau_0,\rho_0)}=-1$;

2) arc $(j_1,\tau_0)\in L(\tau_0,\rho_0)$, $j_1\in I^-_{\tau_0}(U_N)$ then $sign(j_1,\tau_0)^{L(\tau_0,\rho_0)}=1$;

In both cases $L_R=l_{\tau_0\rho_0}(-1)=-l_{\tau_0\rho_0}$. Thereby $L_R+L_N=0$.

If $i=\rho_0$ then node $\tau_0\in I^-_{\rho_0}(U_N)$ and hence $L_N=-l_{\tau_0\rho_0}$. When calculating L_R there may be two cases too:

1) arc $(\rho_0,j_1)\in L(\tau_0,\rho_0)$, $j_1\in I^+_{\rho_0}(U_N)$ then $sign(\rho_0,j_1)^{L(\tau_0,\rho_0)}=1$ and $L_N=l_{\tau_0\rho_0}$;

2) arc $(j_1,\rho_0)\in L(\tau_0,\rho_0)$, $j_1\in I^-_{\rho_0}(U_N)$ then $sign(j_1,\rho_0)^{L(\tau_0,\rho_0)}=-1$ and $L_N=l_{\tau_0\rho_0}(-(-1))=l_{\tau_0\rho_0}$.

Therefore, in this case $L_R+L_N=0$ too.

Thus, we have proven the relation (7).

Let's prove the validity of relation (8) for the constructed direction (17). Since the flow x is a non-degenerate flow then $d_{*ij}<x_{ij}<d^*_{ij}$, $(i,j)\in U_R$ and hence there is a small enough number $\theta^1>0$ such that $d_{*ij}\le x_{ij}+\theta l_{ij}\le d^*_{ij}$, $(i,j)\in U_R$, $\forall\theta\in[0,\theta^1]$. In view of (14) – (16) and (17) there is a small enough number $\theta^2>0$ that $d_{*\tau\rho}\le x_{\tau\rho}+\theta l_{\tau\rho}\le d^*_{\tau\rho}$, $(\tau,\rho)\in U_N$, $\forall\theta\in[0,\theta^2]$. Thus, assuming $\theta^0=\min\{\theta^1,\theta^2\}$, we have that direction constructed by equalities (17) satisfies relations (7) and (8) and hence it is allowable.

Therefore, according to the formula for the derivative of the objective function on allowable direction (10) and in view of $q(x)>0$, $\forall x\in X$, we have $\dfrac{\partial f(x)}{\partial l}=\dfrac{l_{\tau_0\rho_0}\Delta^{(\tau_0,\rho_0)}}{q(x)}=\dfrac{\Delta^{(\tau_0,\rho_0)}\operatorname{sgn}\Delta^{(\tau_0,\rho_0)}}{q(x)}=\dfrac{\left|\Delta^{(\tau_0,\rho_0)}\right|}{q(x)}>0$. That means that x is a nonoptimal flow. *This contradiction completes the proof.*

6. Construction of the iteration

Let relations (11) are not carried out on the support flow $\{x,U_R\}$ then the new support flow $\{\bar{x},U_R\}$ is under construction: $\bar{x}=x+\theta l$, $\bar{x}-x=\Delta x=\theta l$, where θ is a step, l is an allowable direction.

According to the principle underlying the method, it is necessary to choose direction l so that the objective function is not decreasing on the iterations, i.e. $\Delta f=f(\bar{x})-f(x)\ge 0$. Therefore:

$$\Delta f=\frac{\theta\sum\limits_{(\tau,\rho)\in U_N}l_{\tau\rho}\Delta^{(\tau,\rho)}}{\sum\limits_{(\tau,\rho)\in U_N}x_{\tau\rho}\Delta_q^{(\tau,\rho)}+\sum\limits_{(\tau,\rho)\in U_N}\Delta x_{\tau\rho}\Delta_q^{(\tau,\rho)}+Q+\gamma}\ge 0.$$

Thus, since the step $\theta>0$ and by hypothesis $q(x+\Delta x)>0$ direction l must satisfy the inequality:

$$\sum_{(\tau,\rho)\in U_N}l_{\tau\rho}\Delta^{(\tau,\rho)}\ge 0. \tag{20}$$

Direction $l=l(U)$, on which the condition (20) is satisfied, refers to as a direction of objective function increase.

Thus, at construction of iteration a direction l should be both allowable, and a direction of the objective function increase. Such direction is called usable direction.

As an usable direction l for a flow x a vector is selected, along which the derivative of the objective function on an allowable direction reaches its maximum value under simplex normalization:

$$\sum_{(i,j)\in U_N}\left|l_{ij}\right|=1. \tag{21}$$

From condition (8) for flow $\bar{x}$: $d_{*ij}\le x_{ij}+\theta l_{ij}\le d^*_{ij}$, $(i,j)\in U$, $\forall\theta\in[0,\theta^0]$, we receive $l_{ij}\ge 0$ if $x_{ij}=d_{*ij}$ and $l_{ij}\le 0$ if $x_{ij}=d^*_{ij}$, $(i,j)\in U$. The maximum value of the derivative of the objective function on an allowable direction under the normalizing condition (21)

$$\max\frac{\partial f(x)}{\partial l}=\max\frac{\sum\limits_{(\tau,\rho)\in U_N}l_{\tau\rho}\Delta^{(\tau,\rho)}}{q(x)}$$

is reached on the arc $(\tau_0,\rho_0)\in U_N$, which is found from the condition:

$$\left|\Delta^{(\tau_0,\rho_0)}\right| = \max_{(\tau,\rho)\in U_N} \left\{ \max_{x_{\tau\rho}=d_{*\tau\rho}} \Delta^{(\tau,\rho)}, \max_{x_{\tau\rho}=d^*_{\tau\rho}} -\Delta^{(\tau,\rho)}, \max_{d_{*\tau\rho}<x_{\tau\rho}<d^*_{\tau\rho}} \left|\Delta^{(\tau,\rho)}\right| \right\}. \quad (22)$$

In view of (22) the components of the usable direction, which provides the maximum of derivative $\frac{\partial f(x)}{\partial l}$ under the normalizing condition (21), are :

$$l_{\tau_0\rho_0} = \operatorname{sgn} \Delta^{(\tau_0,\rho_0)},\ l_{\tau\rho} = 0,\ (\tau,\rho)\in U_N \setminus (\tau_0,\rho_0),$$
$$l_{ij} = \sum_{(\tau,\rho)\in U_N} l_{\tau\rho}\, sign(i,j)^{L(\tau,\rho)} = l_{\tau_0\rho_0}\, sign(i,j)^{L(\tau_0,\rho_0)},\ (i,j)\in U_R. \quad (23)$$

In addition the equality is valid:

$$\frac{\partial f(x)}{\partial l} = \frac{l_{\tau_0\rho_0}\Delta^{(\tau_0,\rho_0)}}{q(x)} = \frac{\Delta^{(\tau_0,\rho_0)}\operatorname{sgn}\Delta^{(\tau_0,\rho_0)}}{q(x)} = \frac{\left|\Delta^{(\tau_0,\rho_0)}\right|}{q(x)}.$$

As by hypothesis $q(x) > 0$, $\forall x \in X$ then $\frac{\partial f(x)}{\partial l} > 0$.

Any homographic function has the property: if along a selected direction the denominator $q(x)$ of a homographic function $f(x)$ does not change the sign then the function $f(x)$ along this direction varies monotonically. From this property we have that along the constructed direction (23) we shall move until we reach the border of flow set X. For this purpose let's find the maximum permissible step θ^0, at which the vector $\bar{x} = x + \theta^0 l = \{x_{ij} + \theta^0 l_{ij}, (i,j)\in U\}$ is a flow, i.e. conditions (7) and (8) are carried out.

It follows from relations (23) that condition (7) is carried out for any $\theta^0 > 0$. From (23) and (8) we have:

$\theta^0 = \min\{\theta_{\tau_0\rho_0}, \theta_{\tau_1\rho_1}\}$, where $\theta_{\tau_1\rho_1} = \min\{\theta_{ij}, (i,j)\in U_R\}$;

$\theta_{ij} = \frac{d^*_{ij} - x_{ij}}{l_{ij}}$ under $l_{ij} > 0$;

$\theta_{ij} = \frac{d_{*ij} - x_{ij}}{l_{ij}}$ under $l_{ij} < 0$;

$\theta_{ij} = \infty$ under $l_{ij} = 0$, $(i,j)\in U_R \cup (\tau_0,\rho_0)$.

If $\theta^0 = \theta_{\tau_0\rho_0}$ then the support will not change, i.e. $\overline{U}_R = U_R$; if $\theta^0 = \theta_{\tau_1\rho_1}$ then the support will change $\overline{U}_R = (U_R \setminus (\tau_1,\rho_1)) \cup (\tau_0,\rho_0)$.

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